

Projective dimension, graph domination parameters, and independence complex homology

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Question

Let x_1, \dots, x_n correspond to the vertices of G and $I \subseteq S = \mathbf{k}[x_1, \dots, x_n]$ be the edge ideal associated to G .

- What are non-trivial bounds on the biggest integer n such that $\tilde{H}_i(\text{ind}(G), \mathbf{k}) = 0$ for $0 \leq i \leq n$?
- What are (combinatorially constructed) bounds on the projective dimension of S/I ?



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- 1 **Graph theory**
- 2 **Algebraic background**

Graph theory

For a graph G , let $V(G)$ denote its vertex set. We write (v, w) to denote an edge of G between v and w . If v is a vertex of G , we let $N(v)$ denote the set of its neighbors.

If G is a graph and $W \subseteq V(G)$, the *induced subgraph* $G[W]$ is the subgraph of G with vertex set W , where (v, w) is an edge of $G[W]$ if and only if it is an edge of G and $v, w \in W$. For $v \in V(G)$, the star of v , written $st(v)$, is the induced subgraph $G[N(v) \cup \{v\}]$.



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Preliminaries and background

We also write G^c for the complement of G , the graph on the same vertex set as G where (v, w) is an edge of G^c whenever it is not an edge of G . We also write $Is(G)$ to denote the set of isolated vertices of G , and we let $\bar{G} = G - Is(G)$.

We also write $K_{m,n}$ to denote the complete bipartite graph with m vertices on one side and n on the other. Recall that $K_{1,3}$ is known as the claw, and graphs with no induced subgraph isomorphic to $K_{1,3}$ are called claw-free.



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Definition

Let C be a class of graphs such that $G - x$ is in C whenever $G \in C$ and x is a vertex of G . We call such a class hereditary.

Note that, by definition, hereditary classes of graphs are closed under the removal of induced subgraphs (such as stars of vertices). Most widely-studied classes of graphs arising in graph theory are hereditary (such as claw-free graphs, perfect graphs, planar graphs).



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Algebraic background

Fix a field \mathbf{k} , and let $S = \mathbf{k}[x_1, x_2, \dots, x_n]$ (as \mathbf{k} is fixed throughout, we suppress it from the notation). If G is a graph with vertex set $V(G) = \{x_1, x_2, \dots, x_n\}$, the edge ideal of G is the monomial ideal $I(G) \subseteq S$ given by $I(G) = \langle x_i x_j : (x_i, x_j) \text{ is an edge of } G \rangle$.

We say a subset $W \subseteq V(G)$ is independent if no two vertices in W are adjacent (equivalently, $G[W]$ has no edges). Closely related to the edge ideal $I(G)$ of G is its independence complex, $ind(G)$, which is the simplicial complex on vertex set $V(G)$ whose faces are the independent sets of G .



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Algebraic background

For any ideal $I \subseteq S$ generated by squarefree monomials, the *Stanley-Reisner complex* of I is the simplicial complex that contains the face $F = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$ whenever $x_{i_1}x_{i_2}\dots x_{i_k} \notin I$. Thus the independence complex of a graph is the Stanley-Reisner complex of the associated edge ideal. The *projective dimension* of $S/I(G)$ is defined as the shortest length of a projective resolution of $S/I(G)$.



For a graph G , we write $pd(G)$ as shorthand for $pd(S/I(G))$.

Theorem (Hochsters Formula)

Let Δ be the Stanley-Reisner complex of a squarefree monomial ideal $I \subseteq S$. For any multigraded Betti number $\beta_{i,m}$ where m is a squarefree monomial of degree $\geq i - 1$, we have

$$\beta_{i,m} = \dim_{\mathbf{k}}(\tilde{H}_{degm-i-1}(\Delta[m]), \mathbf{k}),$$

where $\Delta[m]$ is the subcomplex of Δ consisting of those faces whose vertices correspond to variables occurring in m .



Preliminaries and background

Here and throughout, if Δ is a complex, we write $\tilde{H}_k(\Delta) = 0$ to mean that the associated homology group has rank zero.

Corollary

Let G be a graph with vertex set V . Then $pd(G)$ is the least integer i such that $\tilde{H}_{|W|-i-j-1}(ind(G[W])) = 0$ for all $j > 0$ and $W \subseteq V$.



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Theorem (Schweig and Dao, 2013)

Let C be a hereditary class of graphs and let $f : C \mapsto R$ be a function satisfying the following conditions:

(1) $f(G) \leq |V(G)|$ when G is a collection of isolated vertices.

Furthermore, for any $G \in C$ with at least one edge there exists a nonempty set of vertices v_1, v_2, \dots, v_k such that if we set

$G_i = G - v_1 - v_2 - \dots - v_i$ for $0 \leq i \leq k$ (where $G_0 = G$), then:

(2) $f(G_i - st_{G_i} v_{i+1}) + 1 \geq f(G)$ for $0 \leq i \leq k - 1$.

(3) $f(\bar{G}_k) + |Is(G_k)| \geq f(G)$. Then for any graph $G \in C$

$pd(G) \leq |V(G)| - f(G)$.



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Domination parameters and projective dimension

We first give a catalog of basic domination parameters. Let G be a graph, and recall that a subset $A \subseteq V(G)$ is dominating if every vertex of $V(G) - A$ is a neighbor of some vertex in A (that is, $N(A) \cup A = V(G)$).

(1) $\gamma(G) = \min\{|A| : A \subseteq V(G) \text{ is a dominating set of } G\}$.

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For any subset $A \subseteq V(G)$, we let $\gamma_0(A, G)$ denote the minimum cardinality of a subset $X \subseteq V(G)$ such that $A \subseteq N(X)$ (note that we allow $A \cap X \neq \emptyset$).

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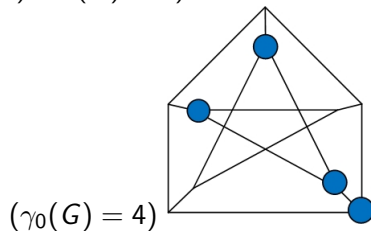
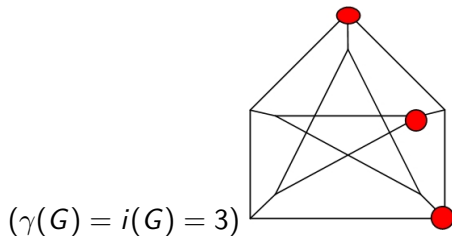
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Domination parameters and projective dimension



Domination parameters and projective dimension

(4) $\tau(G) = \max\{\gamma_0(A, G) : A \subseteq V(G) \text{ is independent}\}$. We also introduce a new graph domination parameter, which we call edgewise domination. Note that this differs from the existing notion of edge-domination, which is not often discussed in the literature, as it is equivalent to domination in the associated line graph.

(5) If $E(G)$ is the set of edges of G , we say a subset $E \subseteq E(G)$ is edgewise dominant if any non-isolated vertex $v \in G$ is adjacent to an endpoint of some edge $e \in E$. We define $\epsilon(G) = \min\{|E| : E \subseteq E(G) \text{ is edgewise dominant}\}$.



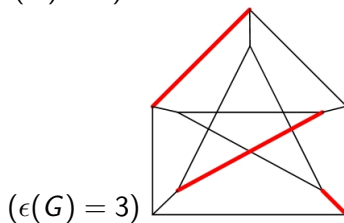
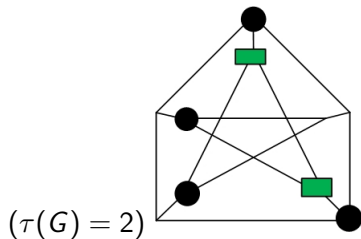
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Domination parameters and projective dimension



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Proposition

For any G , $\gamma(G) \leq i(G)$ and $\tau(G) \leq \gamma(G)$. Furthermore $\epsilon(G) \geq \frac{\gamma_0(G)}{2}$.

Proof

The first inequality is obvious. Let $X \subseteq V$ be a dominating set of G of minimum cardinality, and let $A \subseteq V$ be an independent set with $\tau(G) = \gamma_0(A, G)$. Then $A \subseteq (N(X) \cup X)$, by definition. If $x \in A \cap X$, then $N(x) \cap A = \emptyset$ (otherwise A would not be independent). For each $x \in X \cap A$, replace x with one of its neighbors (which is possible since $x \in V(\bar{G})$), and call the resulting set X' . Then $A \subseteq N(X')$. Since $|X| \geq |X'| \geq \gamma_0(A, G)$, we have $\gamma(G) = |X| \geq |X'| \geq \tau(G)$. We now prove the last inequality. Let $E(G)$ be the edge set of G , and let $E \subseteq E(G)$ be an edgewise dominant set of G . If we let A be the set of vertices in edges of E , then A is easily seen to be strongly dominant, meaning $|E| \geq \frac{|A|}{2} \geq \frac{\gamma_0(G)}{2}$.

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Corollary

For any graph G

$$|V(G)| - i(G) \leq pd(G) \leq |V(G)| - \max\{\epsilon(G), \tau(G)\}$$

Hedetniemi et al. in 1981 proved the next theorem.

Theorem

Let G be a graph. Then $i(G) + \gamma_0(G) \leq |V(G)|$.



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With two relations $|V(G)| - i(G) \leq pd(G)$ and $i(G) + \gamma_0(G) \leq |V(G)|$, the result is clear.



Domination parameters and projective dimension

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Theorem(Schweig; 2013)

Let $f(G)$ be either $\gamma(G)$ or $i(G)$. Suppose \mathcal{C} is a hereditary class of graphs such that whenever $G \in \mathcal{C}$ has at least one edge, there is some vertex v of G with $f(G) \leq f(G - v)$. Then for any $G \in \mathcal{C}$,
 $pd(G) \leq |V(G)| - f(G)$.

If $f(G)$ is $i(G)$, then for any $G \in \mathcal{C}$, we have the equality
 $pd(G) = |V(G)| - i(G)$.



Lemma

Suppose $N(v) - w \subseteq N(w)$ and (v, w) is an edge. Then $\gamma(G) \leq \gamma(G - w)$ and $i(G) \leq i(G - w)$.

Proof

Let X be a (independent) dominating set of $G - w$; It suffices to show that X dominates G . Since $v \in G - w$, X either contains v or a neighbor of v . In the first case, since (v, w) is an edge of G , X still dominates G . For the second case, note that any neighbor of v is a neighbor of w (since $N(v) - w \subseteq N(w)$), and thus X must be a (independent) dominating set of G as well.



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Theorem(Dirac)

Let G be a chordal graph with at least one edge. Then there exists a vertex v of G so that $N(v) \neq \emptyset$ and $G[N(v)]$ is complete. (This vertex is called simplicial vertex).

Using Dirac's Theorem and Schweig's Theorem, we can recover a formula for the projective dimension of a chordal graph.



Corollary

The class of chordal graphs satisfies the conditions of last theorem, and so $pd(G) = |V(G)| - i(G)$, for any chordal graph G .

Proof

Let v be as in Dirac's Theorem, and let w be any neighbor of v . Since $G[N(v)]$ is complete, we have $N(v) - w \subseteq N(w)$, and so last Lemma applies.



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Remark

In looking for possible generalizations of last corollary, one may be tempted to ask if the same equality holds for perfect graphs (as all chordal graphs are perfect). However, this is easily seen to be false for the 4-cycle C_4 , as $pd(C_4) = 3$, but $n - i(G) = 4 - 2 = 2$.



Chordal graphs

It makes sense to ask when a graph G satisfies $i(G) = \gamma(G)$. A graph G is called domination perfect if this equality holds for G and all its induced subgraphs.

Corollary

Suppose \mathcal{C} is a hereditary class of domination perfect graphs such that whenever $G \in \mathcal{C}$ has at least one edge, there is some vertex v of G with $\gamma(G) \leq \gamma(G - v)$. Then for any $G \in \mathcal{C}$,

$$pd(G) = |V(G)| - \gamma(G) = |V(G)| - i(G).$$


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Corollary

Let G be a graph. Then $\tilde{H}_k(\text{ind}(G)) = 0$ whenever $k < |V(G)| - \text{pd}(G) - 1$.

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Theorem(Aharoni and Berger; 2002)

If G is chordal then $\tilde{H}_k(\text{ind}(G)) = 0$ for $k < \gamma(G) - 1$.

Proof

Let $e = uv$ be an edge of G and let $V_0 = V - (N[u] \cup N[v])$. Consider the graphs $G - e = (V; E - e)$ and $G_0 = G[V_0]$.

This sequence is exact:

$$\rightarrow \tilde{H}_{i-1}(\text{ind}(G_0)) \rightarrow \tilde{H}_i(\text{ind}(G)) \rightarrow \tilde{H}_i(\text{ind}(G - e)) \rightarrow \tilde{H}_{i-2}(\text{ind}(G_0)) \rightarrow$$



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Homology and connectivity of independence complexes

We prove by induction on k and the number of edges, $|E|$.

Let u be a simplicial vertex of G and let $e = uv$ be any edge incident with u . Clearly $G - e$ is chordal and $\gamma(G - e) \geq \gamma(G) > k + 1$ hence by induction $\tilde{H}_k(\text{ind}(G - e)) = 0$. If $S \subseteq V_0$ is a dominating set of G_0 then $S \cup \{v\}$ is a dominating set of G hence $\gamma(G_0) \geq \gamma(G) - 1 > k$. G_0 is an induced subgraph of a chordal graph, hence chordal itself. It follows by induction that $\tilde{H}_{k-1}(\text{ind}(G_0)) = 0$. Thus $\tilde{H}_k(\text{ind}(G)) = 0$ follows from the above exact sequence.



Corollary

If G is chordal, $\tilde{H}_k(\text{ind}(G)) = 0$ for $k < i(G) - 1$.

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For any graph G , $\tilde{H}_k(\text{ind}(G)) = 0$ for $k < \tau(G) - 1$.



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For any graph G , $\tilde{H}_k(\text{ind}(G)) = 0$ whenever $k < \frac{\gamma_0(G)}{2} - 1$.

Proof

First proof (Schweig; 2013) :

The result follows from $\epsilon(G) \geq \frac{\gamma_0(G)}{2}$ and the fact that $\tilde{H}_k(\text{ind}(G)) = 0$ whenever $k < \epsilon(G) - 1$.



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Second proof (Aharoni, Chudnovsky, Meshulam; 2000) :

This sequence is exact:

$$\rightarrow \tilde{H}_{i-1}(\text{ind}(G_0)) \rightarrow \tilde{H}_i(\text{ind}(G)) \rightarrow \tilde{H}_i(\text{ind}(G - e)) \rightarrow \tilde{H}_{i-2}(\text{ind}(G_0)) \rightarrow$$

We prove by induction on k and the number of edges, $|E|$.

Let $e = uv$ be any edge of G . Clearly

$$\gamma_0(G - e) \geq \gamma_0(G) > 2k + 1$$

hence by induction $\tilde{H}_k(\text{ind}(G - e)) = 0$. If $S \subseteq V_0$ satisfies $N(S) = V_0$ then $N(S \cup \{u, v\}) = V$. Therefore $\gamma_0(G_0) \geq \gamma_0(G) - 2 > 2(k - 1) + 1$.

hence by induction $\tilde{H}_{k-1}(\text{ind}(G_0)) = 0$. Thus $\tilde{H}_k(\text{ind}(G)) = 0$ follows from the above exact sequence with $i = k$.



Corollary

Let G be a connected graph and let $A \subseteq V(G)$ be such that the distance between any two members of A is at least 3. Then $\tilde{H}_k(\text{ind}(G)) = 0$ for $k \leq |A| - 2$.

Proof

The set A is independent since none of its members are distance 1 from each other. Now let X be a set of vertices realizing $\gamma_0(A, G)$. If $x \in X$ is adjacent to two vertices in A , then these two vertices would be distance 2 from one another. Thus, $\tau(G) \geq |X| = |A|$, and so the relation $pd(G) \leq |V(G)| - \tau(G)$ completes the proof.



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- ① H. Dao, J. Schweig, Projective dimension, graph domination parameters, and independence complex homology, *Journal of Combinatorial Theory, Series A* 120 (2013) 453-469
- ② R. Meshulam, Domination numbers and homology, *Journal of Combinatorial Theory, Series A* 102 (2003) 321-330.
- ③ R. Aharoni, E. Berger, R. Ziv, A tree version of König's theorem, *Combinatorica* 22 (2002) 335-343.
- ④ R. Allan, R. Laskar, S. Hedetniemi, A note on total domination, *Discrete Math.* 49 (1) (1984) 713.
- ⑤ M. Chudnovski, Systems of disjoint representatives, M.Sc. Thesis. Technion. Haifa. 2000.

