Binomial D-module

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Workshop on Commutative Algebra, Ipm, 2013, Tehran, Iran

Introduction

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- Let $\hat{x}_1, \ldots, \hat{x}_n$ be the operators of $k[X]$ which are defined on a polynomial $f \in k[X]$ by the formulate $\hat{x}_i.f = x_i f$. Similarly, $\partial_1, \ldots, \partial_n$ are the operators defined by $\partial_i(f) = \frac{\partial f}{\partial x_i}$.

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- The n-th Weyl algebra D_n is the $k-$subalgebra of $\mathbf{End}_k(k[X])$ generated by the operators $\hat{x}_1, \ldots, \hat{x}_n, \partial_1, \ldots, \partial_n$.

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- Note that we have:

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Definition

If an element of D_n is written as a linear combination of this basis then we say that it is in canonical form.

Degree of an operator

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- Let $P \in D_n$. The degree of P is the largest length of the multi-indices (α, β) *in* $\mathbb{N}^n \times \mathbb{N}^n$ for which $x^\alpha \partial^\beta$ appears with non-zero coefficient in the canonical form of P.

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Theorem

The degree satisfies the following properties; for $P, Q \in D_n$*:*

- $deg(P + Q) \leq max\{deg(P), deg(Q)\}.$
- $deg(PQ) = deg(P) + deg(Q)$
- $deg[P, Q] \leq deg(P) + deg(Q) - 2$

Definition

- Denote by B_k the set of all operators of D_n of degree $\leq k$. $B = \{B_k\}_{k \in \mathbb{N}}$ is called *Bernstein filtration*.
- Let M be a left D_n-module and Γ a filtration of M with respect to the Bernstein filtration B.When $gr^{\Gamma} M$ is finitely generated over $k[X, \xi]$ we say that Γ is a good filtration of M.

Remark

In this presentation when we talk about dim M we mean the degree of the Hilbert polynomial of the associated graded algebra of M with a good filtration respect to *Bernstein filtration*.

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Let R be a ring.

- An R-module M is irreducible or simple, if it has no proper submodules.
- Let M be a left R-module. An element $u \in M$ is a torsion element if $\mathbf{ann}_R(u)$ is a nonzero left ideal. if every element of M is torsion, then M is called a torsion module.

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Theorem

k[X] is an irreducible, torsion D-module. Besides this,

$$k[X] \simeq \frac{D}{\sum_1^n D\partial_i}$$

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- Let U be an open subset of $\mathbb{C}^n$. The set $H(U)$ of holomorphic functions defined on U is a left D_n-module.

Differential equations

- Let P be an operator in D_n. It may be represented in the form $\sum_\alpha g_\alpha \partial^\alpha$ where $\alpha \in \mathbb{N}^n$ and $g_\alpha \in k[x_1, \ldots, x_n] = k[X]$. This differential operator gives rise to the equation

$$P(f) = \sum_\alpha g_\alpha \partial^\alpha (f) = 0,$$

where f may be a polynomial or, if $k = \mathbb{R}$, a $\mathcal{C}^\infty$ function on the variables $x_1, \ldots, x_n$.

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  where f may be a polynomial or, if $k = \mathbb{R}$, a $\mathcal{C}^\infty$ function on the variables $x_1, \ldots, x_n$.
- More generally, if $P_1, \ldots, P_m$ are differential operations in D, then we have a system of differential equations

$$P_1(f) = \cdots = P_m(f) = 0. \tag{1}$$

Differential equations

- A polynomial solution of (1) is a polynomial $f \in k[X]$ which satisfies $P_i(f) = 0$, for $i = 1, \ldots, m$. The set of all polynomial solutions of (1) forms a k-vector space.

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- The D-module associated to the system of differential equations (1) is $\frac{D}{\sum_1^m DP_i}$.

Differential equations

Theorem

Let M be the D-module associated with the system (1). the vector space of polynomial solutions of the system (1) is isomorphic to $\mathbf{Hom}_D(M, k[X])$.

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- Let M be a finitely generated D_n-module. Then $\mathbf{dim}(M) \leq 2n$.
- If M is a finitely generated non-zero left D_n-module, then $\mathbf{dim}(M) \geq n$.

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- *finite sums of holonomic D-modules are holonomic.*
- *Holonomic D-modules are torsion modules.*
- *Holonomic D-modules are artinian.*
- *Holonomic D-modules are cyclic.*

Definition

- Let $P = \sum c_{\alpha\beta} X^{\alpha} \partial^{\beta} \in D_n$. We define initial form $\mathbf{in}(P)$ as follows:

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where ξ are commutative variables.

- Let I be an ideal of D_n. Then the k-vector space

$$\textbf{in}(I) := k.\{\textbf{in}(P) : P \in I\}$$

is a left ideal of the polynomial ring $k[X, \xi]$ and called characteristic ideal of I.

Definition

Let I be a non-zero ideal of D_n. We call I holonomic if its characteristic ideal has dimension n. The holonomic rank of I is the following vector space dimension over the field $k(X) = k(x_1, \ldots, x_n)$:

$$\mathbf{rank}(I) := \mathbf{dim}_{k(X)}\left(\frac{k(X)[\xi]}{k(X)[\xi].\mathbf{in}(I)}\right).$$

Theorem

Consider an ordinary differential equation of order m,

$$[a_m(x)\frac{d^m}{dx^m} + \cdots + a_0(x)] \bullet f = 0, \qquad a_0, \ldots, a_m \in \mathbb{C}[x].$$

Let U be a simply connected domain contained in $\{P \in \mathbb{C} : a_m(P) \neq 0\}$. Then, the dimension of the space of holomorphic solutions on U is equal to m.

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Let U be a simply connected domain contained in $\{P \in \mathbb{C} : a_m(P) \neq 0\}$. *Then, the dimension of the space of holomorphic solutions on U is equal to m.*

Example

Gauss hypergeometric equation:

$$[x(1-x)\frac{d^2}{dx^2} + (c - x(a+b+1))\frac{d}{dx} - ab] \bullet f = 0.$$

Definition

Let I be an ideal of D_n. **Sing**(I) is the zero set of the following ideal:

$$(\mathbf{in}(I) :< \xi_1, \ldots, \xi_n >^{\infty}) \bigcap \mathbb{C}[x_1, \ldots, x_n]$$

Theorem

Let I be a holonomic ideal and U a simply connected domain in $\frac{\mathbb{C}^n}{\mathbf{Sing}(I)}$*. Consider the system of differential equations* $I \bullet f = 0$ *that is,*

$$p \bullet f = 0, p \in I, \text{for holomorphic functions } f \text{ on } U.$$

The dimension of the complex vector space of solutions is equal to **rank**(I)*.*

Convention 1

$A = (a_{ij}) \in \mathbb{Z}^{d\times n}$ denotes an integer $d \times n$ matrix of rank d whose columns $a_1, \ldots, a_n$ all lie in a single open linear half-space of $\mathbb{R}^d$; equivalently, the cone generated by the columns of A is pointed (contains no lines), and all of the a_i are nonzero. We also assume that $\mathbb{Z}A = \mathbb{Z}^d$; that is, the columns of A span $\mathbb{Z}^d$ as a lattice.

Convention 2

Let $B = (b_{jk}) \in Z^{n\times m}$ be an integer matrix of full rank $m \leq n$. Assume that every nonzero element of the column-span of B over the integers $\mathbb{Z}$ is mixed, meaning that it has at least one positive and one negative entry; in particular, the columns of B are mixed. We write $b_1, \cdots, b_n$ for the rows of B. Having chosen B, we set $d = n - m$ and pick a matrix $A \in Z^{d\times n}$ whose columns span $\mathbb{Z}^d$ as a lattice, such that AB = 0.

Definition

- For each $i \in \{1, ..., d\}$, the ith Euler operator is $E_i = a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n$. Given a vector $\beta \in \mathbb{C}^d$, we write $E - \beta$ for the sequence $E_1 - \beta_1, ..., E_d - \beta_d$.

Definition

- For each $i \in \{1, ..., d\}$, the ith Euler operator is $E_i = a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n$. Given a vector $\beta \in \mathbb{C}^d$, we write $E - \beta$ for the sequence $E_1 - \beta_1, ..., E_d - \beta_d$.
- For an A-graded binomial ideal I of $\mathbb{C}[\partial]$, we denote by $H_A(I, \beta)$ the left ideal $I + < E - \beta >$ in the Weyl algebra D. The binomial D-module associated to I is $\frac{D}{H_A(I,\beta)}$.

Question

When is $\frac{D}{H_A(I,\beta)}$ a holonomic D-module?

Definition

Given an integer matrix M with q rows, we call $I(M) \subseteq k[t_1, \cdots, t_q] = k[\mathbb{N}^q]$ the binomial ideal

$$I(M) =< t^u - t^v | u - v \ \ is \ \ a \ \ column \ \ of \ \ M, \ \ u, v \in \mathbb{N}^q >$$

$$=< t^{w_+} - t^{w_-} | w = w_+ - w_- \ \ is \ \ a \ \ column \ \ of \ \ M >$$

Here for an integer vector $w \in \mathbb{Z}^q$, the vector w_+ has ith coordinate w_i if $w_i \geq 0$ and 0 otherwise. The vector $w_- \in \mathbb{N}^q$ is defined by $w_+ - w_- = w$, or equivalently, $w_- = (-w)_+$.

Definition and Proposition

The lattice basis ideal corresponding to the lattice $\mathbb{Z}B = \{Bz : z \in \mathbb{Z}^m\}$ is defined by $I(B) =< \partial^{u_+} - \partial^{u_-} : u = u_+ - u_- \text{ is a column of } B >\subseteq \mathbb{C}[\partial_1, \cdots, \partial_n]$.
Each of the minimal primes of I(B) arises, after row and column permutations, from a block decomposition of B of the form

$$\begin{pmatrix} N & B_J \\ M & 0 \end{pmatrix}$$

where M is a mixed submatrix of B of size $q \times p$ for some $0 \le q \le p \le m$.

Definition

Any integer matrix M with q rows defines an undirected graph $\Gamma(M)$ having vertex set $\mathbb{N}^q$ and an edge from u to v if u - v or v - u is a column of M. An M-subgraph of $\mathbb{N}^q$ is a connected component of $\Gamma(M)$. An M-subgraph is bounded if it has finitely many vertices, and unbounded otherwise.

Lemma

Let M be a $q \times q$ mixed invertible integer matrix, and assume that $q \geq 0$. Fix $\gamma \in \mathbb{N}^{\bar{J}}$, and denote by Γ the M-subgraph containing Γ.

- The system I(M) of differential equations has a unique formal power series solution of the form $G_\Gamma = \sum_{u \in \Gamma} \lambda_u x^u$ in which $\lambda_\Gamma = 1$.
- The other coefficients λ_u of G_Γ for $u \in \Gamma$ are all nonzero.

Theorem

- *The set* $\{G_\Gamma :$ Γ *runs over a set of representatives for the* $M-$ *subgraphs of* $\mathbb{N}^J\}$ *is a basis for the space of all formal power series solutions of I(M).*
- *The set* $\{G_\Gamma : \Gamma \in S(M)\}$ *is a basis for the space of polynomial solutions of I(M).*

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