# Linear resolution of power of Monomial ideal

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# Preliminaries

- **Simplicial Complexes**
- **Minimal free resolution and betti numbers**
- **Reduced Homology and Hochster's formula**

# Simplicial Complexes

$Notations:$ $[n]$ denotes set $\{1, \ldots, n\}$

$S = \mathbb{K}[x_1, \ldots, x_n]$ and $\mathbb{K}$ is a field.

Definition: $\Delta \subseteq P([n])$ is simplicial complex. if: $F \in \Delta$ and $G \subset F$ then $G \in \Delta$

$F \in \Delta$ is called **Face** and maximal faces called **Facet**.

$\dim(F) = |F| - 1$ and $\dim(\Delta) = max\{\dim(F) \mid F \in \Delta\}$

$\mathcal{F}(\Delta)$ is a set of all Facets of $\Delta$

$\mathcal{N}(\Delta)$ is a set of all minimal non-face of $\Delta$

## Simplicial Complexes

Stanley-Reisner ideal and Facet ideal:

$I_\Delta =< x_F ; F \in \mathcal{N}(\Delta) >$ is called **Stanley-Reisner ideal**

$I(\Delta) =< x_F ; F \in \mathcal{F}(\Delta) >$ is called **facet ideal**

Ring $\mathbb{K}(\Delta) = \dfrac{S}{I_\Delta}$ is called **Stanley-Reisner ring** of $\Delta$

## Simplicial Complexes

One may ask if there exist $\Delta'$ such that $I(\Delta') = I_\Delta$

$\Delta^\vee$ is Alexander dual of $\Delta$ and $\mathcal{F}(\Delta^\vee) = \{[n] \setminus F; F \in \mathcal{N}(\Delta)\}$

$\Delta^c$ is a complement of $\Delta$ and $\mathcal{F}(\Delta^c) = \{F^c \mid F \in \mathcal{F}(\Delta)\}$

$$I_{\Delta^\vee} = I(\Delta^c)$$

# Simplicial Complexes associated to a graph

Simplicial Complexes associated to a graph

Independent Complex

$Ind(G)$
is called independent complex of $G$ and facets of this simplicial complexare maximal independet sets of $G$

Clique Complex

$\Delta_C(G)$
is called clique complex of $G$ and facets of this simplicial complex are maximal clique sets of $G$

$$\Delta_C(G^c) = Ind(G)$$

# Edge ideal

Definition:(Edge Ideal) If $G$ is a graph. $I(G)$ denotes edge ideal of $G$ if:

$$\{i, j\} \in E(G) \Leftrightarrow x_i x_j \in I(G)$$

Lemma:

$$I_{Ind(G)} = I(G)$$

# Minimal free resolution and betti numbers

We know that every module of $S$ have minimal free resolution like:

$$0 \longrightarrow F_k \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where each $F_i = \bigoplus S(-a_{i,j}) = \bigoplus S(-j)^{\beta_{i,j}(M)}$ that is, $F_i$ requires $\beta_{i,j}$ minimal generator of degree $j$

$$\beta_{i,j}(M) \text{ is called } i, j^{th} \text{ graded betti numbers}$$

One can investigate that $\beta_{i,j}(M) = \dim Tor_i^s(\mathbb{K}, M)_j$

# Minimal free resolution and betti numbers

Graded betti numbers are important because other homological invariant like **projective dimension** and **regularity** can be computed by them.

## Projective dimension

$$Pd(M) = max\{i \mid \beta_{i,j}(M) \neq 0\}$$

## Regularity

$$Reg(M) = max\{j - i \mid \beta_{i,j}(M) \neq 0\}$$

## Minimal free resolution and betti numbers

**Definition(Linear resolution)**: We called that a module $M$ has a linear resolution if the following condition holds for $M$:

$$\exists d; \quad \forall i, \forall j \neq i+d \quad \beta_{i,j}(M) = 0$$

From definition we can conclude that every generator of $M$ has a degree $d$ and $Reg(M) = d$

Example: $I = (x, y, z)$ in $S = \mathbb{K}[x, y, z]$ then minimal free resolution of $\frac{S}{I}$ is:

$$0 \longrightarrow S^1(-3) \longrightarrow S^3(-2) \longrightarrow S^3(-1) \longrightarrow I \longrightarrow 0$$

# Redued homology

Suppose $\Delta$ be a $d-1$ dimensional simplicial complex on $[n]$ with the set of vertex $\{v_0,\ldots,v_n\}$. Set $C_i(\Delta) \cong \mathbb{K}^{f_i(\Delta)}$.

Consider the following complex:

$$0 \longrightarrow C_{d-1}(\Delta) \xrightarrow{\partial_{d-1}} C_{d-2}(\Delta) \xrightarrow{\partial_{d-2}} \cdots \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) \xrightarrow{\partial_{-1}} 0$$

such that every $\partial_i$ defined by:

$$\begin{cases} \partial_i : C_i(\Delta) \longrightarrow C_{i-1}(\Delta) \\ F \longmapsto \sum_{0\leq j\leq i-1}(-1)^j(F \setminus \{v_j\} \end{cases}$$

We have $\quad \partial_{i-1} \circ \partial_i = 0$

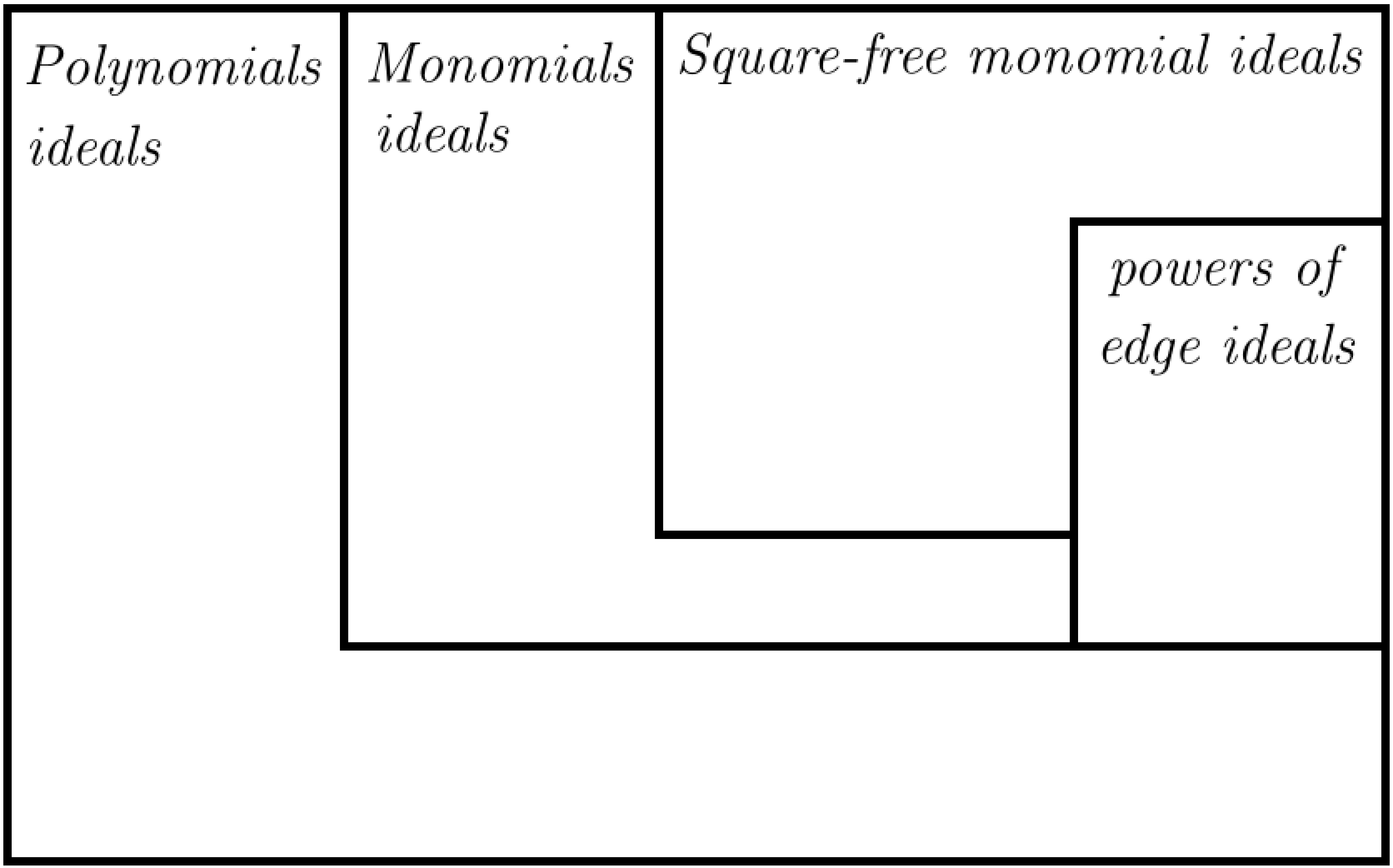
Therefore we can define $\frac{Ker(\partial_i)}{Im(\partial_{i+1})}$ as a $i^{th}$ reduced homology of $\Delta$ and we denote it by $\tilde{H}_i(\Delta)$

# Hochster's Formula

**Hochster's Formula(1977)**: Let $\Delta$ be a simplicial complex on $[n]$. Betti numbers of $I_\Delta$ can be computed by this formula:

$$\beta_{i,j}(I_\Delta) = \sum_{|W|=j} \dim \tilde{H}_{j-i-2}(\Delta_W)$$

$$S = \mathbb{K}[x_1, \ldots, x_n]$$



# Linear resolution of monomial ideals

- Graded betti numbers of edge ideals
- Linear resolution of all powers of monomial ideals
- Monomial ideals with 2-linear resolution

## Graded betti numbers of edge ideals

Suppose minimal free resolution of edge ideal $I(G)$ has a form:

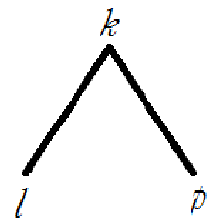
$$0 \longrightarrow F_k \xrightarrow{\partial_k} \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} I(G) \longrightarrow 0$$

By construction of minimal free resolution we know that generators of $F_0$ correspond to the generators of $I(G)$, therefore all of the generators of $F_0$ have a degree 2 and cardinal of them is the same as cardinal of generators of $I(G)$, that are edges of $G$. So we have:

$$\beta_{0,2} = \mid E(G) \mid , \ \beta_{0,j} = 0 \ \forall j \neq 2.$$

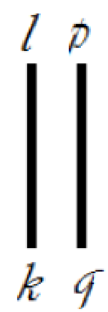
# Graded betti numbers of edge ideals

First syzygies have a relation to the shape of the edge of $G$ i.e. if we have




type of edges in graph we can conclude $\beta_{1,3} \neq 0$

And if we have




type of edges we can conclude that $\beta_{1,4} \neq 0$

## Graded betti numbers of edge ideals

Definition: A graph $G$ is called **chordal** if every cycle has a chord.

Definition: A simplicial complex $\Delta$ on $[n]$ is called **flag** if every minimal nonface of $\Delta$ is a 2-elements subset of $[n]$

Lemma: A simplicial complex $\Delta$ is flag if and only if $\Delta$ is the clique complex of a finite graph.

Corrolary: Every flag complex $\Delta$ is the clique complex of its 1-skeleton. Also $I_\Delta = I(G^c)$.

## Graded betti numbers of edge ideals

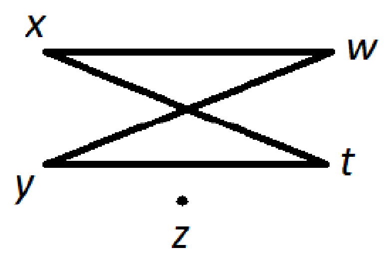
**Fröberg's theorem**: The edge ideal $I(G)$ has a linear resolution if and only if the graph $G^c$ is chordal.

Sketch of proof: Since $I(G) = I_{\Delta_C(G^c)}$, what we must prove is that the stanley-reisner ideal of the clique complex $\Delta_C(G^c)$ has the linear resolution if and only if $G^c$ is chordal.

We know that if $G$ is a chordal graph then $\tilde{H}_i(\Delta_C(G)) = 0$ for all $i \neq 0$. By using this fact and Hochster's formula we can prove the theorem.

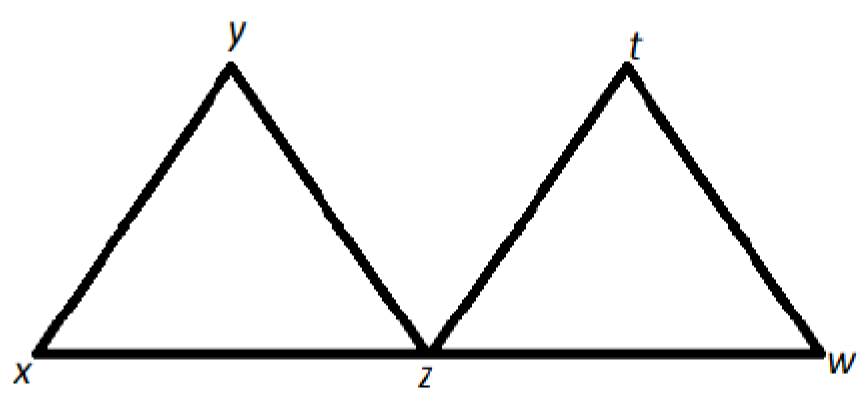
## Graded betti numbers of edge ideals

Example: Let $I = (xw, xt, yw, yt)$ in $S = \mathbb{K}[x, y, z, t, w]$. Then $I = I(G)$ for the graph:




Note that we should consider isolate vertex corresponds to the variable $z$

Now complement of $G$ is:



# Graded betti numbers of edge ideals

We see that Hochster's formula is very usefull for computing graded betti numbers of edge ideals. But it is also useful in more general case( i.e square free monomial ideals), for example in the case of quasi-tree and quasi forest simplicial complexes we can compute all of the graded betti numbers of $I_\Delta$ and $I_{\Delta^\vee}$. This result was stated by S.Faridi in 2013 for simplicial trees and we extended it for quasi forest simplicial complexes.

## Linear resolution of all powers of monomial ideals

In this section we ask when such an ideal has the property that all of its powers have a linear resolution.

In general, powers of ideals with linear resolution need not to have linear resolutions.

The first example of such an ideal was given by Terai. He showed that over a base field of characteristic $\neq 2$ the Stanley Reisner ideal

$$I = (abd, abf, ace, adc, aef, bde, bcf, bce, cdf, def)$$

of the minimal triangulation of the projective plane has a linear resolution, while $I^2$ has no linear resolution.

The example depends on the characteristic of the base field. If the base field has characteristic 2, then $I$ itself has no linear resolution.

## Linear resolution of all powers of monomial ideals

Another example, namely $I = (def, cef, cdf, cde, bef, bcd, acf, ade)$ is given by Sturmfels. Again I has a linear resolution, while $I^2$ has no linear resolution.

1. it does not depend on the characteristic of the base field
2. it is a linear quotient ideal.

Definition: An equigenerated ideal $I$ is said to have **linear quotients** if there exists an order $f_1, \ldots, f_m$ of the generators of $I$ such that for all $i = 1, \ldots, m$ the colon ideals $(f_1, \ldots, f_{i-1}) : f_i$ are generated by linear forms.

# Linear resolution of all powers of monomial ideals

It is known that the regularity of powers $I^n$ of a graded ideal $I$ is bounded by a linear function $an + b$, and is a linear function for large $n$.

For ideals I whose generators are all of degree $d$ Römer shows that one has the bound $Reg(I^n) \leq nd + Reg_x(R(I))$.

Here $R(I)$ is Rees ring and $Reg_x(R(I))$ is $x-regularity$ of $R(I)$ that defined as follow:

## Linear resolution of all powers of monomial ideals

Let $S = \mathbb{K}[x_1, \dots, x_n]$, $I \subset S$ an equigenerated graded ideal with set of generator $\{f_1, \dots, f_m\}$. Then the Rees ring

$$R(I) = \bigoplus_{j \geq 0} I^j t^j = S[f_1 t, \dots, f_m t] \subset S[t]$$

is naturally bigraded with $deg(x_i) = (1, 0)$ for $i = 1, \dots, n$ and $deg(f_i t) = (0, 1)$ for $i = 1, \dots, m$

Let $T = S[y_1, \dots, y_m]$ be the polynomial ring over $S$ in the variables $y_1, \dots, y_m$. We define a bigrading on $T$ by setting $deg(x_i) = (1, 0)$ for $i = 1, \dots, n$, and $deg(y_j) = (0, 1)$ for $j = 1, \dots, m$ Then there is a natural surjective homomorphism of bigraded K-algebras $\phi : T \longrightarrow R(I)$ with $\phi(x_i) = x_i$ for $i = 1, \dots, n$ and $\phi(y_j) = f_j t$ for $j = 1, \dots, m$.

## Linear resolution of all powers of monomial ideals

Let

$$F.: 0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R(I) \longrightarrow 0$$

be the bigraded minimal free $T$–resolution of $R(I)$. Here $F_i = \bigoplus_j T(-a_{ij}, -b_{ij})$ for $i = 0, \ldots, p$. The $x - regularity$ of $R(I)$ is defined to be the number

$$Reg_x(R(I)) = max_{i,j}\{a_{i,j} - i\}$$

**Theorem**:
$Reg(I^n) \leq nd + Reg_x(R(I))$. In particular, if $Reg_x(R(I)) = 0$, then each power of $I$ admits a linear resolution.

# Monomial ideals with 2-linear resolution

**Theorem(HHZ-2003)**: Let $I$ be a monomial ideal generated in degree 2. Then following conditions are equivalent:

(a) $I$ has a linear resolution;

(b) $I$ has linear quotients;

(c) Each powers of $I$ has a linear resolution.

# $C_4$–free edge ideals

- **LCM lattice**

- **Why not $C_4$?**

- **Conjecture of Peeva and Nevo**

## LCM lattice

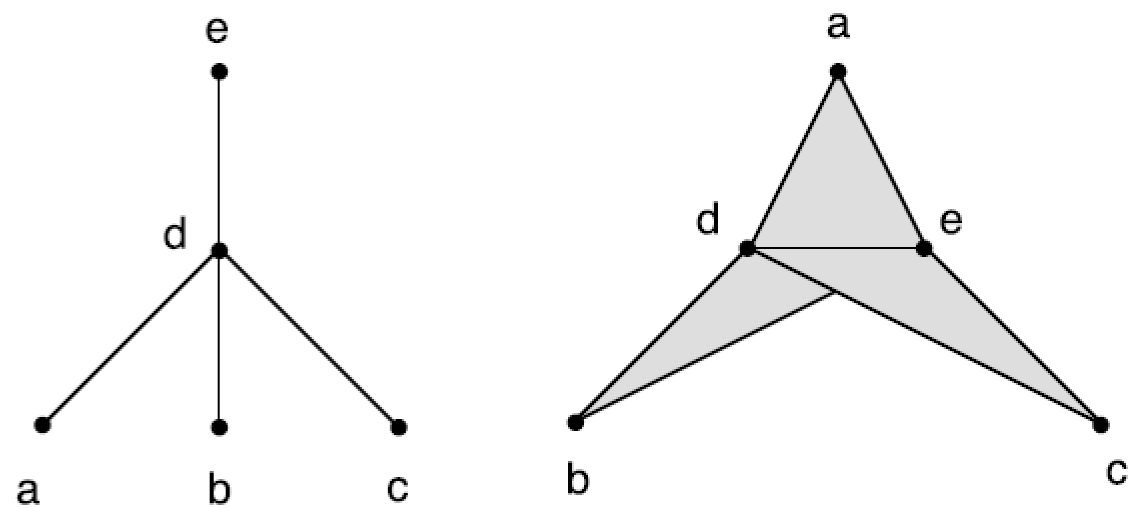
In this section we mention LCM lattice method that introduce by Gasharov, Peeva and Welker in 1999. It is a usefull method to computing graded betti numbers.

Definition: A lattice is a kind of poset that each two elements have meet and joint.

Definition: The **Hasse diagram** of $\mathcal{P}$ is the graph with vertices the elements in $\mathcal{P}$ so that if $y$ covers $x$ then $y$ is placed higher than $x$ and they are connected with an edge.

# LCM lattice

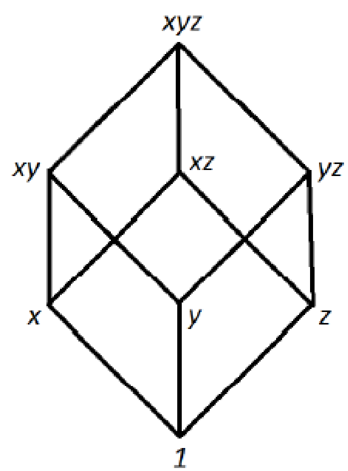
Definition: The **order complex** $O(\mathcal{P})$ of $\mathcal{P}$ is the simplicial complex whose vertices are the elements of $\mathcal{P}$ and whose faces are the chains in the poset. We implicitly think of a poset $\mathcal{P}$ as a topological space by considering its order complex $O(\mathcal{P})$.



# LCM lattice

**LCM lattice**:
Construction: Let $I = (m_1, \ldots, m_r)$, we denote by $\mathcal{L}_I$ the lattice with elements the least common multiples of subsets of $m_1, \ldots, m_r$ ordered by divisibility. We call $\mathcal{L}_I$ the lcm-lattice of $I$. For $m \in \mathcal{L}_I$we denote by $(1, m)_{\mathcal{L}_I}$ the open interval in $\mathcal{L}_I$ below $m$; it consists of all non-unit monomials in $\mathcal{L}_I$ that strictly divide $m$.

Example: Let $I = (x, y, z)$ in $S = \mathbb{K}[x, y, z]$. LCM lattice $I$ is:



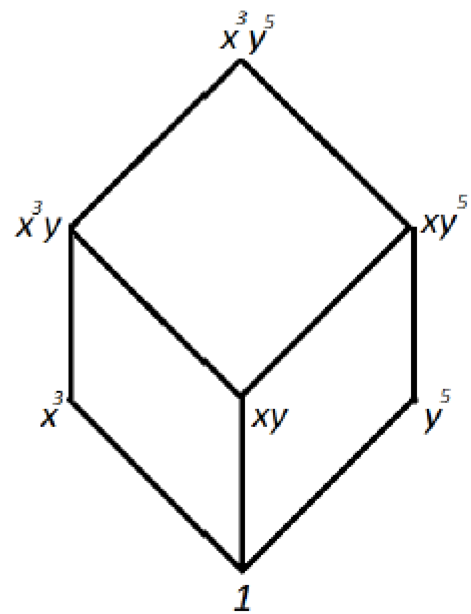
## LCM lattice

**Theorem(GPW)**: For $i \geq 1$ we have:

$$\beta_{i,m}(I) = \begin{cases} \dim \tilde{H}_{i-1}(O((1,m)_{\mathcal{L}_I})) & 1 \neq m \in \mathcal{L}_I \\ 0 & m \notin \mathcal{L}_I \end{cases}$$

# LCM lattice

Example: Let $I = (x^3, xy, y^5)$. LCM lattice of $I$ is

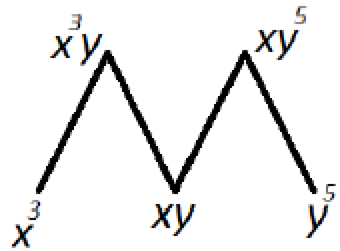



By theorem we only need to find $O((1, m)_{\mathcal{L}_I})$ for $m = x^3y, xy^5, x^3y^5$.

# LCM lattice

$m = x^3y$ then geometric realization of $O((1, x^3y)_{\mathcal{L}_I})$ is


Clearly $\tilde{H}_0(O((1, x^3y)_{\mathcal{L}_I})) = 1$, therefor $\beta_{1,x^3y} = 1$. For $m = xy^5$ it is the same. But for $m = x^3y^5$ geometric realization of $O((1, x^3y^5)_{\mathcal{L}_I})$ is




We have $\tilde{H}_i(O((1, x^3y)_{\mathcal{L}_I})) = 0$, Then obviously minimal free resolution is:

$$0 \to S(-4) \oplus S(-6) \xrightarrow{\partial_2} S(-3) \oplus S(-2) \oplus S(-5) \xrightarrow{\partial_1} I \to 0$$

$I(G)$ has a linear resolution

Fröberg (1990)

$G^c$ is a chordal graph

?

Herzög Hibi Zheng(2003)

For all $n \geq 2$ ideal $I(G)^n$ has a linear resolution

What we can say about reverse?

# Why not $C_4$?

*Observation* :$G^c$ has no induced 4-cycles if and only if the Betti numbers $\beta_{1,j}(I(G))$ vanish for $j \neq 3$.

The condition that $G^c$ has no induced 4-cycles has strong implications for the LCM-lattice.

Theorem: If $G^c$ has no induced 4-cycle, then for any $s \geq 1$ we have

$$Reg(I(G)^s) = 2s + max_{m \in \mathcal{L}_{\mathcal{I}(\mathcal{G})}, m \neq 1}\left\{\alpha\Big(O\big((1,m)_{\mathcal{L}_{I(G)}}\big)\Big)\right\}$$

where for a simplcial complex $\Delta$, $\alpha(\Delta)$ denotes the largest codimension of a non-vanishing reduced homology.

## Why not $C_4$?

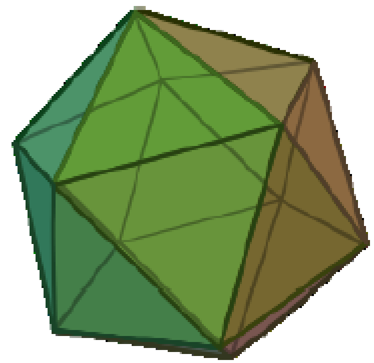
Francisco, Há, and Van Tuyl proved the following result;

Theorem(Francisco-Há-Van Tuyl; non-published): If $I(G)^s$ has a linear resolution for some $s \geq 1$, then $G^c$ has no induced 4-cycles.

Question: Is it true that if $G^c$ has no induced 4-cycles then $I(G)^s$ has a linear resolution for all $s \geq 2$?

# Why not $C_4$?

Counterexample: If $G^c$ is this graph:



$Reg(I(G)) = 4$

$Reg(I(G)^2) = 5$

$Reg(I(G)^3) = 6$

$Reg(I(G)^4) = 8$

## Conjecture of Peeva and Nevo

Conjecture(2013):

(1) The main question is if it is true that $G^c$ is $C_4$–free if and only if $I(G)^s$ has a linear minimal free resolution for every $s \gg 0$?

(2) The question is meant to be a tool for the study of the first question. Suppose that $G^c$ has no induced 4-cycles. Is it true that for $s \geq 1$, we have that

$$Reg(I(G))^{s+1} \leqslant max\{2s+2, Reg(I(G))^s + 1\}?$$

# Conjecture of Peeva and Nevo

A positive answer to (2) will imply that the following conditions are equivalent:

(1) $I(G)^s$ has a linear resolution for some $s \geq 2$.

(2) $I(G)^s$ has a linear resolution for every $s \geq Reg(I(G)) - 1$.

(3) $I(G)$ has only linear minimal first syzygies, that is, $\beta_{1,j}(I(G)) = 0$ for $j \neq 3$.

(4) $G^c$ has no induced 4-cycles.

# Conjecture of Peeva and Nevo

It is worth to mentioned that question (2) is open even in the case $Reg(I(G)) = 3$:

Is it true that $I(G)^s$ has a linear resolution for all $s \geq 2$ if $G^c$ has no induced 4-cycles and $Reg(I(G)) = 3$?

Thank you for your attention