An Algebraic Introduction to Vector Bundles-*2*

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Let Y be scheme. A vector bundle on Y is a morphism $f: X \longrightarrow Y$ of schemes with an open covering $\{U_i\}$ of Y and an isomorph.

$$\psi_i : f^{-1}(U_i) \longrightarrow \mathbb{A}^n_{U_i} = \mathrm{spec}\big(\mathcal{O}(U_i)[x_1, \dots, x_n]\big)$$

and for any affine open subset $V \subseteq U_i \cap U_j$, $V = \mathrm{spec}(A)$

transion functor

$$\begin{tikzcd}
\mathrm{spec}(A[\underline{x_i}]) =: \mathbb{A}^n_V \arrow[rr, "\theta_{ij}"] \arrow[dr, "\psi_i^{-1}"'] & & \mathbb{A}^n_V := \mathrm{spec}(A[\underline{x_i}]) \\
& f^{-1}(V) \arrow[ur, "\psi_j"'] &
\end{tikzcd}$$

θ_{ij} are linear in the following sense:

θ_{ij} corresponds to $\tilde{\theta}_{ij} : A[\underline{x_i}] \longrightarrow A[\underline{x_i}]$

$$\tilde{\theta}_{ij}\big|_A = \mathrm{id}\big|_A$$

$$x_i \xmapsto{\tilde{\theta}_{ij}} \sum_j a_{ij} x_j$$

__Example__ Let $x \in \mathbb{P}^n_{\mathbb{C}}$. Then

$$x = [x_0 : x_1 : \cdots : x_n]$$

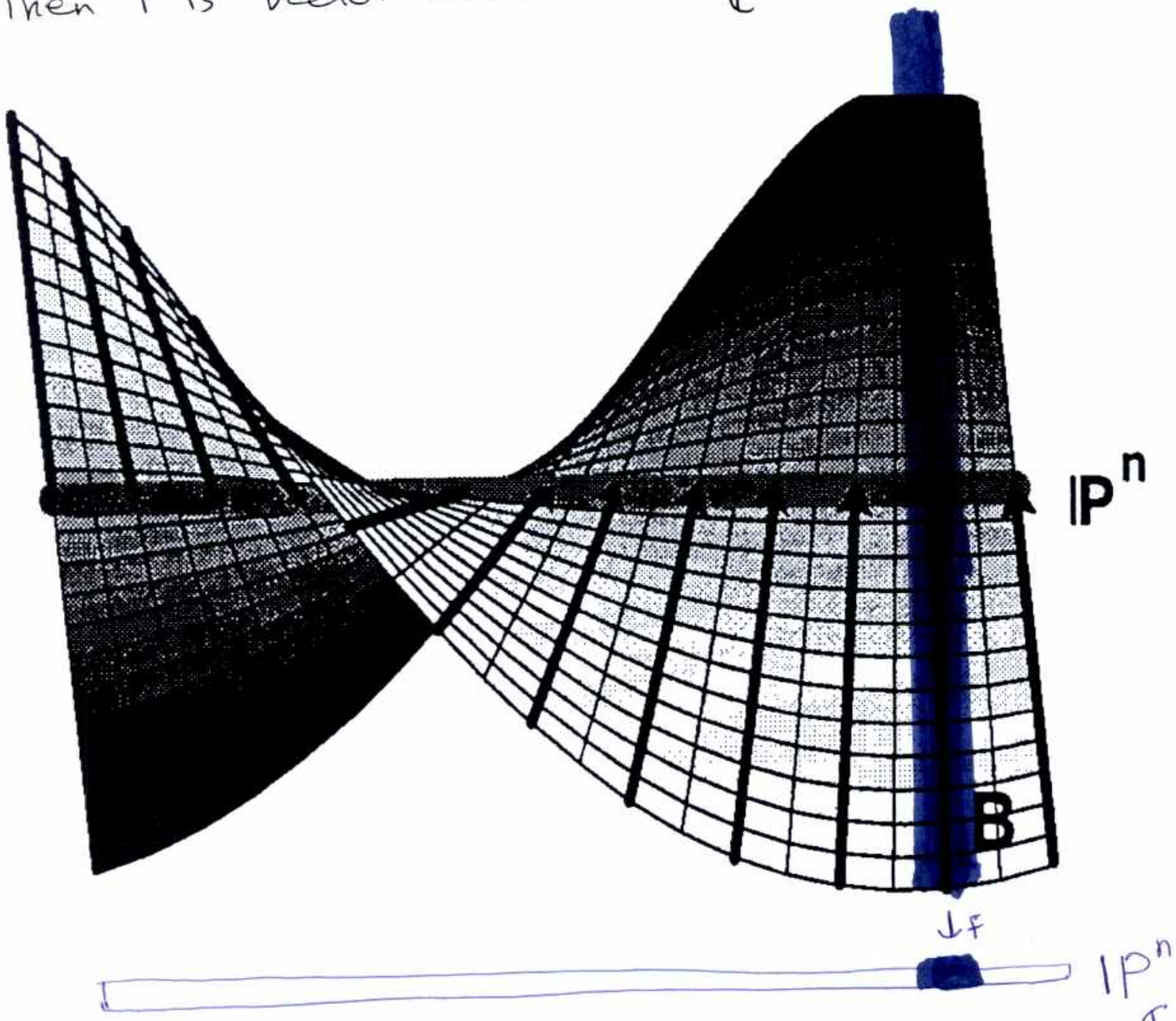
correspondce to a line

$$\ell_x := \{(tx_0, tx_1, \dots, tx_n) \mid t \in \mathbb{C}\} \subseteq \mathbb{C}^{n+1}$$

$B \subseteq \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$ defined as follows

$$\begin{tikzcd} B = \{(x, \ell_x) \mid x \in \ell_x\} \subseteq \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} \arrow[d] \arrow[d, bend right, "f"'] \\ \mathbb{P}^n_{\mathbb{C}} \end{tikzcd}$$

Then f is veetor bundle on $\mathbb{P}^n_{\mathbb{C}}$. Look at the following:



Aim

$$\left\{\begin{array}{c}\text{vector bundle}\\ \text{on } Y\end{array}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{array}{c}\text{locally free}\\ \text{sheaves}\\ \text{on } Y\end{array}\right\}$$

Idea: I) Let $X \xrightarrow{f} Y$ be V.B. on Y and $U \subseteq Y$ open

Defin $\phi(X)(U) = \{ s: U \mapsto X \mid f\circ s = id|_U \}$

• $s_1, s_2 \in \phi(X)(U)$

$s_i^\#: \mathcal{O}_X \longrightarrow \mathcal{O}_X(f^{-1}(U))$

Define $(s_1+s_2)^\# := s_1^\# + s_2^\#$

• $\alpha \in \mathcal{O}(U)$. Note that

$$s_1^\#: \underbrace{\varinjlim_{V \supseteq s(U)} \mathcal{O}_X(V)}_{\substack{\text{call this inverse}\\ \text{image of } \mathcal{O}_X}} \longrightarrow \mathcal{O}(U)$$

It is equal with $\mathcal{O}(U)[\underline{X}_i] \xrightarrow{s_1^\#} \mathcal{O}(U)$

Now we can defin $(\alpha s_1)^\# = \alpha s_1^\#$

• $s_1^\#$ define completely by $\left(s_1^\#(x_1), \dots, s_1^\#(x_n)\right)$

so $\phi(X)(U)$ is locally free.

II) Let F be any locally free sheaf on Y

Define the following presheaf

$$U \longmapsto sym(F(U)) = \frac{\bigoplus F(U)^{\otimes n}}{\langle x\otimes y - y\otimes x \mid x, y \in F(U) \rangle}$$

Denote the associated sheaf by $Sym(\mathcal{F})$

<u>Fact</u> $\exists!$ an scheme X and

$$X \xrightarrow{f} Y$$

s.t. $f^{-1}(U) = Spec(\, sym(\mathcal{F})\,(U))$

and if $U \hookrightarrow V$ open + affine, then

$$f^{-1}(U) \hookrightarrow f^{-1}(V)$$

is the restricion map correspondce to

$$sym(F)(V) \longrightarrow sym(F)(U).$$

In particular, X is a vector bundle

Denote it by $\mathcal{S}pec(\, sym(\mathcal{F})\,)$.

And define $\Upsilon(F) := \mathcal{S}pec(\, sym(F)\,)$

III) The presheaf $U \longmapsto Hom(\mathcal{F}(U), \mathcal{O}(U))$

is infact sheaf. Denot it by $\mathcal{H}om(\mathcal{F}, \mathcal{O})$

or by $\mathcal{F}^{\vee}$

IV) $\Upsilon\,\phi(X) = ?$ and $\phi\,\Upsilon(\mathcal{F}) = \mathcal{F}^{\vee}$

Concerning Dr. Sabayesh' remark:

<u>Remark</u> There is one point where these are NOT behave same

- If $W \subsetneq V$ is subbundle, $\underbrace{W_{k(x)}}_{\substack{\shortparallel \\ \text{fiber at } x}} \subsetneq \underbrace{V_{k(x)}}_{\substack{\shortparallel \\ \text{fiber at } x}}$ is linear subspace

But subsheaf $F_1 \subseteq F_2$ does not imply that $\underbrace{(F_1)_x}_{\substack{\text{fiber} \\ \text{at } x}} \subsetneq (F_2)_x$ defined by $F_1 \otimes_{\mathcal{O}_X} \frac{\mathcal{O}_{X,x}}{m_{X,x}}$

- $F_1 \subseteq F_2$ correspondce to a subbundle iff F_2/F_1 is also free!

Now, we want give an intersting thm of Brenner which uses the material of our talks. First, we need to defin plus closure and tight closure:

<u>Def</u> Let R be an integral domain and $I = (f_1, \ldots, f_n)$ be an ideal of R.

1) $I^+ = \{ f \in R \mid \exists\, \exists R \hookrightarrow S \text{ finite map s.t. } f \in IS \}$

2) If char $R = p > 0$

$$I^* = \{ f \in R \mid \exists_{\neq 0}\, c \in R \text{ s.t } cf^{p^e} \in (f_1^{p^e}, \ldots, f_n^{p^e})R \quad e \geq 0 \}.$$

<u>Thm</u> (Brenner)

Let $K := \overline{\mathbb{F}_p}$ and R be an 2-dim. graded integral domain of finite type over K. Let I be a homogeneous ideal of R. Then $I^* = I^+$

Idea of proof (This is NOT proof, just an idea!)

step 1 WLOG I is R_+-primary

step 2 Translate things to Alg. Geo.

Let $X = \mathrm{Proj}(R)$. Then X is smooth projective curve (since $\dim R = 2$ and $R_{(p)}$ is regular for $p \in X$)

suppose $I := (f_1, \ldots, f_n)$ where $f_i \in R_{d_i}$

and take $f \in R_m$.

Look at

$$0 \to K \to \bigoplus R(-d_i) \xrightarrow{(f_1 \ldots f_n)} R \to R/I \to 0$$

apply $(\tilde{\ })$ to it and tensor with $\mathcal{O}(m)$:

$$0 \to \tilde{K} \to \bigoplus \mathcal{O}_X(m-d_i) \to \mathcal{O}_X(m) \to 0 \quad (*)$$

Apply the functor $\Gamma : \{\text{sheaves on } X\} \to \{\text{Abelian groups}\}$ to $(*)$:

$$\mathcal{F} \mapsto \mathcal{F}(X)$$

$$\begin{array}{ccccccc} 0 \to \Gamma(\tilde{K}) \to & \Gamma(\bigoplus \mathcal{O}_X(m-d_i)) & \to & \Gamma(\mathcal{O}_X(m)) & \xrightarrow{\delta} H^1(\tilde{K}(m)) \\ & \| & & \| & \\ & \bigoplus_{i=1}^{n} R_{m-d_i} & \xrightarrow{(f_1, \ldots, f_n)} & R_m & \end{array}$$

Step 3 Shadow of Torsor

$$c := \delta(f) \in H^1(X, \widetilde{K}(m))$$
$$\| $$
$$\mathrm{Ext}^1(\mathcal{O}_X, \widetilde{K}(m))$$
$$\downarrow\uparrow$$
$$\{\, 0 \to \widetilde{K}(m) \to \mathcal{S} \to \mathcal{O}_X \to 0 \mid \text{which is exact} \,\}$$

Set $T := \mathrm{Proj}(\mathrm{sym}(\mathcal{S}^\vee)) \setminus \mathrm{Proj}(\widetilde{K}(m))$

Step 4 $f \in I^* \equiv$ Non affiness of T

Step 5 $f \in I^+ \equiv \exists$ projective curve inside T

Step 6 Nonaffiness of T $\equiv \exists$ projective curve inside T . $\square$