# On the weakly polymatroidal property of the edge ideals

Rahim Rahmati-Asghar

University of Marageh

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# History

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  ⇝ H. Whitney, *On the abstract properties of linear dependence*, American Journal of Mathematics, **57** (3) 509-533.
- It was also independently discovered by Takeo Nakasawa whose work was forgotten for many years (Nishimura, Kuroda).
  ⇝ H. Nishimura, S. Kuroda. *A lost mathematician, Takeo Nakasawa. The forgotten father of matroid theory*, Basel: Birkhäuser Verlag (2009).
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# Why matroid theory?

By today, matroids are among the most intensively studied objects in combinatorics. They play an important role in different fields such as optimization, enumeration, algebraic combinatorics, applications to electrical networks, statistics, quantum computation, quantum cryptography, etc.

- Matroids have become well-known in combinatorial optimization because of their close relationship to greedy algorithm. This algorithm is an abstraction of the algorithm of Kruskal (1956) for finding a minimum weighted spanning tree in a graph.

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# Definitions

Defn. A matroid is a pair $(E, \mathcal{M})$ consisting of a finite set $E$ and a collection $\mathcal{M}$ of subsets of $E$ that satisfy the following conditions:

(i) $\emptyset \in \mathcal{M}$;

(ii) if $X \in \mathcal{M}$ and $Y \subseteq X$, then $Y \in \mathcal{M}$;

(iii) for $X, Y \in \mathcal{M}$ with $|X| = |Y| + 1$ there exists an $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{M}$.

The sets in $\mathcal{M}$ are typically called **independent** sets. An independent set is called a **basis** if it is not a proper subset of another independent set. Denote by $\mathcal{B}$ the set of bases.
The rank of a subset $X$, denoted by $\mathrm{r}(X)$, of the ground set $E$ is the size of the largest independent subset of $X$. The rank of $\mathcal{M}$ is defined $\mathrm{r}(\mathcal{M}) = \max\{\mathrm{r}(X) : X \in \mathcal{M}\}$.

# Examples

Exmp. Let $E = \{1, 2, 3, 4, 5\}$. Then $\mathcal{M}$, defined the following, is a matroid on $E$.

$$\mathcal{M} = \{123,\ 124,\ 245,\ 235\}.$$

**Exmp.** (Linear matroid) Let $F$ be a field, $A \in F^{m\times n}$ an $m \times n$ matrix over $F$, $I = \{1, \ldots, n\}$ be index set of columns of $A$. Then $U \subseteq I$ is independent if the corresponding columns are linearly independent.

Exmp. (Graphic matroid) Let $E$ be the set of edges of a graph $G$ and let $\mathcal{M}$ be consists of those subsets that contain no circuit.

Thm. A non-empty collection $\mathcal{B}$ of subsets of $E$ is the set of bases of a matroid iff it satisfies the following condition:
(Exchange property)

For $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ there exists a $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

Proof. ($\Rightarrow$): All bases have the same cardinality and $B_1 \backslash x \in \mathcal{M}$.
Therefore there exists $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

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($\Leftarrow$): Let $B$ satisfies the exchange property.
Then all members of $\mathcal{B}$ have the same cardinality. If $B_1 B_2 \in \mathcal{B}$ with $|B_1| < |B_2|$, we can successively exchange the elements in $B_1 \backslash B_2$ with elements from $B_2 \backslash B_1$. We arrive at a set $B_3 \in \mathcal{B}$, where $B_3$ is a proper subset of $B_2$, a contradiction.
Clearly, $\mathcal{M}$ satisfies (i) and (ii). Let $X, Y \in \mathcal{M}$ with $|X| = |Y| + 1$ and let $X \subseteq B_1$ and $Y \subseteq B_2$. For all $b \in (B_2 \backslash B_1) \backslash Y$, there is $a \in B_1 \backslash B_2$, such that $(B_2 \backslash b) \cup a \in \mathcal{B}$.
If $a \in X$, then $Y \cup a \subseteq (B_2 \backslash b) \cup a$ and $Y \cup a \in \mathcal{M}$, as desired.
Otherwise, since that $|(B_1 \backslash B_2) \backslash X| < |(B_2 \backslash B_1) \backslash Y|$, after finitely many steps we obtain $a' \in X$ with $(B_2 \backslash b') \cup a' \in \mathcal{B}$ for some $b'$.
Therefore $Y \cup a' \in \mathcal{M}$.

Let $\mathcal{M}$ be a matroid. Set $\mathcal{B}^* = \{B^c : B \in \mathcal{B}(\mathcal{M})\}$.

Rem. $\mathcal{B}^*$ is the set of bases of a matroid, called **dual matroid** of $\mathcal{M}$.

Proof. We use the fact that for $x \in B_1 \setminus B_2$ there exists a $y \in B_2 \setminus B_1$ such that $(B_2 \setminus y) \cup x \in \mathcal{B}$.

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**Exmp.** (Bipartite matroid) A graphic matroid is bipartite if and only if it comes from a bipartite graph.

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**Defn.** A simplicial complex whose facets are the bases of a matroid is called a matroid complex.

It is known that every matroid complex is pure shellable. To see this, consider the matroid complex $\mathcal{M}$ and let the facets $F_1, \ldots, F_r$ of $\mathcal{M}$ are lexicographically ordered with respect to an arbitrary ordering of vertices. Then it follows from definition that $F_1, \ldots, F_r$ is a shelling order for $\mathcal{M}$.
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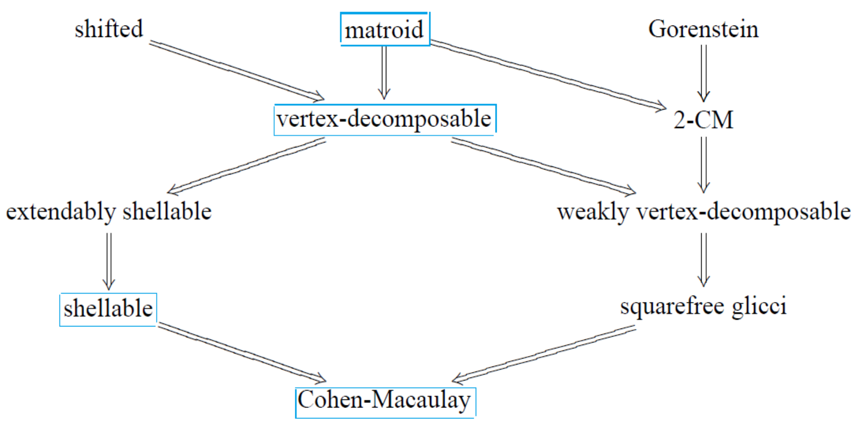
shifted
matroid
Gorenstein
vertex-decomposable
2-CM
extendably shellable
weakly vertex-decomposable
shellable
squarefree glicci
Cohen-Macaulay

# Greedoids

Greedoids were invented in 1981 by Korte and Lovasz. Originally, the main motivation for proposing this generalization of the matroid concept came from combinatorial optimization.
They had observed that the optimality of a greedy algorithm could in several instances be traced back to an underlying combinatorial structure that was not a matroid-but (as they named it) a greedoid.

# Greedoids

For a ground set $E$ we denote by $E^*$ the set of all sequences $\alpha := x_1 x_2 \ldots x_k$ of elements $x_i \in E$. Set $\tilde{\alpha} = \{x_1, x_2, \ldots, x_k\}$.
$E$ is called **alphabet**, its elements **letters** and the elements of $E^*$ **words**.
The collection of words $\mathcal{L} \subseteq E^*$ is called a **language** over the alphabet $E$.
The symbol $\emptyset$ will also be used to denote the empty word.
A language is called **simple** if no letter is repeated in any word.

## B. Kort, L. Lovász (1981)

A simple language $(E, \mathcal{L})$ is a greedoid if

(i) $\emptyset \in \mathcal{L}$;

(ii) $\alpha\beta \in \mathcal{L}$ implies $\alpha \in \mathcal{L}$;

(iii) if $\alpha, \beta \in \mathcal{L}$ with $|\alpha| > |\beta|$, then there exists an $x \in \tilde{\alpha}$ such that $\beta x \in \mathcal{L}$.

Set $\mathcal{G} := \{\tilde{\alpha} : \alpha \in \mathcal{L}\}$.
Since that $\mathcal{L}$ is simple, we can define:

A greedoid is a pair $(E, \mathcal{G})$ that satisfy the following conditions:

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Note that if $\mathcal{G}$ is closed under taking subsets, then (i) and (ii) are equivalent to the matroid axioms. So

every matroid is a greedoid.

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# Examples of greedoids

Exmp. Let $E = \{a, b, c, d\}$ and let

$$\mathcal{G} = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}, \{a, c, d\}, \{b, c, d\}\}.$$

Exmp. Let $G$ be a graph and $r \in V(G)$ a specified root. Let $\mathcal{G}$ be the family of edge-sets of subtrees of $G$ containing $r$.

Exmp. Let $G$ be a graph. Define a language $(E, \mathcal{L})$ with $E = V(G)$ consisting all words $x_1 \ldots x_k$ such that $x_i$ is a simplicial vertex in $G \backslash (N[x_1] \cup \ldots \cup N[x_{i-1}])$. $(E, \mathcal{L})$ is a greedoid.

$\leadsto$ B. Korte, L. Lovász, R. Schrader, *Greedoids*, Springer(1990), p. 51.

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# A result in combinatorics

**Thm.** The dual of a greedoid is vertex-decomposable and hence shellable.

# Open problems in Matroid Theory

Conj. The $h$-vector of a matroid complex is a pure $M$-vector.
⇝ R. Stanley, *Combinatorics and Commutative Algebra*, Birkhauser, Boston, MA, 1996.

Conj. For a matroid $\mathcal{M}$, the toric ideal $I_{\mathcal{M}}$, is generated by quadrics corresponding to double swaps, i.e.

$$I_{\mathcal{M}} = (y_u y_v - y_{(u/x_i)x_j} y_{(v/x_j)x_i} : x_i | u, x_j | v)$$

⇝ N. White, *A unique exchange property for bases*, Linear Algebra Appl. **31** (1980), 81-91.

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Some references for greedoids:


- V.E. Levit, E. Mandrescu, *Graph Operations that are Good for Greedoids*, arXiv:0809.1806v1.
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- E.W. McMahon , B. A. Shimkus, J. A. Wolfson, *Chordal graphs and the characteristic polynomial*, Discrete Mathematics **262** (2003) 211-219.
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# Discrete Polymatroids

Let $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ be two vectors belonging to $\mathbb{R}_+^n$. We write $\mathbf{u} \preceq \mathbf{v}$ if $u_i \leq v_i$ for all $i$, moreover, write $\mathbf{u} \prec \mathbf{v}$ if $\mathbf{u} \preceq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$. We say that $\mathbf{u}$ is a subvector of $\mathbf{v}$ if $\mathbf{u} \preceq \mathbf{v}$. In addition, we set

$$\mathbf{u} \vee \mathbf{v} = (\max\{u_1, v_1\}, \dots, \max\{u_n, v_n\}),$$

$$\mathbf{u} \wedge \mathbf{v} = (\min\{u_1, v_1\}, \dots, \min\{u_n, v_n\}).$$

Hence we have $\mathbf{u} \wedge \mathbf{v} \preceq \mathbf{u}, \mathbf{v} \preceq \mathbf{u} \vee \mathbf{v}$.
The **modulus** of a vector $\mathbf{u} = (u_1, \dots, u_n)$ is

$$|\mathbf{u}| = \sum_{i=1}^{n} u_i.$$

**Defn.** A polymatroid on the ground set $[n]$ is a nonempty compact subset $\mathcal{P}$ in $\mathbb{R}_+^n$, the set of **independent vectors**, such that
($P_1$) every subvector of an independent vector is independent;
($P_2$) if $\mathbf{u}, \mathbf{v} \in \mathcal{P}$ with $|\mathbf{v}| > |\mathbf{u}|$, then there is a vector $\mathbf{w} \in \mathcal{P}$ such that

$$\mathbf{u} \prec \mathbf{w} \preceq \mathbf{u} \vee \mathbf{v}.$$

A maximal independent vector of a polymatroid $\mathcal{P} \subset \mathbb{R}_+^n$ is an independent vector $\mathbf{u} \in \mathcal{P}$ with $\mathbf{u} \prec \mathbf{v}$ for no $\mathbf{v} \in \mathcal{P}$ and it is called a **base** of $\mathcal{P}$. Let $\mathcal{B}(\mathcal{P})$ denote the set of bases of $\mathcal{P}$.
If $\mathbf{u}$ and $\mathbf{v}$ are bases of $\mathcal{P}$ with $|\mathbf{u}| < |\mathbf{v}|$, then by $(P_2)$ there exists $\mathbf{w} \in \mathcal{P}$ with $\mathbf{u} \prec \mathbf{w} \preceq \mathbf{u} \vee \mathbf{v}$, which is a contradiction, since $\mathbf{u}$ is maximal. Thus every two elements of base of $\mathcal{P}$ have the same modulus $\mathrm{r}(\mathcal{P})$, the **rank** of $\mathcal{P}$.
Let $\epsilon_1, \ldots, \epsilon_n$ denote the canonical basis vectors of $\mathbb{R}^n$.

**Defn.** A discrete polymatroid on the ground set $[n]$ is a nonempty finite set $\mathcal{P} \subset \mathbb{Z}_+^n$ satisfying
$(D_1)$ if $\mathbf{u} \in \mathcal{P}$ and $\mathbf{v} \in \mathbb{Z}_+^n$ with $\mathbf{v} \preceq \mathbf{u}$, then $\mathbf{v} \in \mathcal{P}$;
$(D_2)$ if $\mathbf{u} = (u_1, \ldots, u_n) \in \mathcal{P}$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathcal{P}$ with $|\mathbf{u}| < |\mathbf{v}|$, then there is $i \in [n]$ with $u_i < v_i$ such that $\mathbf{u} + \epsilon_i \in \mathcal{P}$.

Notice that a polymatroid $\mathcal{P} \subset \mathbb{R}_+^n \cap \mathbb{Z}^n$ is the same as a discrete polymatroid.
If $\mathcal{P} \subset \{0, 1\}^n$ then $\mathcal{P}$ may be regarded as a matroid.

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### Herzog, Hibi (2002)

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$$I_{\mathcal{P}} = (y_{\mathbf{u}} y_{\mathbf{v}} - y_{\mathbf{u}-\epsilon_i+\epsilon_j} y_{\mathbf{v}-\epsilon_j+\epsilon_i} : \mathbf{u}(i) > 0, \mathbf{v}(j) > 0).$$

# Monomial Ideals

## Herzog, Hibi (2002)

Defn. A monomial ideal $I$ of $S$ with $G(I) = \{\mathbf{x}^{\mathbf{u}_1}, \ldots, \mathbf{x}^{\mathbf{u}_t}\}$ is called polymatroidal if $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is the set of bases of a discrete polymatroid on $[n]$. In other words, all elements in $G(I)$ have the same degree, and if $\mathbf{x}^{\mathbf{u}_r} = x_1^{a_1} \ldots x_n^{a_n}$ and $\mathbf{x}^{\mathbf{u}_s} = x_1^{b_1} \ldots x_n^{b_n}$ belong to $G(I)$ with $a_i > b_i$, then there exists $j$ with $a_j < b_j$ such that $x_j(\mathbf{x}^{\mathbf{u}_r}/x_i) \in G(I)$.

A squarefree polymatroidal ideal is called matroidal.

# Monomial Ideals

## Herzog, Hibi (2002)

Defn. A monomial ideal $I$ of $S$ with $G(I) = \{\mathbf{x}^{\mathbf{u}_1}, \ldots, \mathbf{x}^{\mathbf{u}_t}\}$ is called polymatroidal if $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is the set of bases of a discrete polymatroid on $[n]$. In other words, all elements in $G(I)$ have the same degree, and if $\mathbf{x}^{\mathbf{u}_r} = x_1^{a_1} \ldots x_n^{a_n}$ and $\mathbf{x}^{\mathbf{u}_s} = x_1^{b_1} \ldots x_n^{b_n}$ belong to $G(I)$ with $a_i > b_i$, then there exists $j$ with $a_j < b_j$ such that $x_j(\mathbf{x}^{\mathbf{u}_r}/x_i) \in G(I)$.

A squarefree polymatroidal ideal is called matroidal.

# Some Properties

**Conca, Herzog (2003)**

A polymatroidal ideal $I$ has linear quotients with respect to the reverse lexicographical order of the generators.

**Herzog, Hibi (2002)**

Let $I$ and $J$ be polymatroidal monomial ideals. Then $IJ$ is polymatroidal.

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The base ring $K[\mathcal{P}]$ of a discrete polymatroid $P$ is normal.

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# Weakly polymatroidal ideals

**Hibi, KoKubo(2006)-Mohammadi, Moradi(2011)**

Defn. A monomial ideal $I$ is called weakly polymatroidal if for every two monomials $u = x_1^{a_1} \dots x_n^{a_n} >_{\text{lex}} v = x_1^{b_1} \dots x_n^{b_n}$ in $G(I)$ such that $a_1 = b_1, \dots, a_{t-1} = b_{t-1}$ and $a_t > b_t$, there exists $j > t$ such that $x_t(v/x_j) \in I$.

**Hibi, KoKubo(2006)-Mohammadi, Moradi(2011)**

Thm. A weakly polymatroidal ideal $I$ has linear quotients.

# Generated in one degree

$$\textit{polymatroidal} \Downarrow \textit{weakly polymatroidal} \Downarrow \textit{linear quotients} \Downarrow \textit{linear resolution}$$

# Generated in different degrees

$$\textit{weakly polymatroidal}$$
$$\Downarrow$$
$$\textit{linearquotients}$$
$$\Downarrow$$
$$\textit{linearresolution}$$

# Questions

## $G$ is a graph

$$G \text{ is ? } \Longleftrightarrow \begin{cases} I(G) \text{ has linear resolution or} \\ I(G) \text{ has linear quotients or} \\ I(G) \text{ is weakly polymatroidal or} \\ I(G) \text{ is polymatroidal} \end{cases}$$

$$G \text{ is ? } \Longleftrightarrow \begin{cases} I(G)^\vee \text{ has linear resolution or} \\ I(G)^\vee \text{ has linear quotients or} \\ I(G)^\vee \text{ is weakly polymatroidal or} \\ I(G)^\vee \text{ is polymatroidal} \end{cases}$$

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# Notations

Let $\mathcal{V} = \{x_1, \ldots, x_n\}$ be a finite set and let $\mathcal{E} = \{e_1, \ldots, e_m\}$ be a finite collection of distinct subsets of $\mathcal{V}$. The pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is called a hypergraph if $e_i \neq \emptyset$ for each $i$.

The elements of $\mathcal{V}$ and $\mathcal{E}$ are called the **vertices** and **edges**, respectively, of $\mathcal{H}$. We may write $\mathcal{V}(\mathcal{H})$ and $\mathcal{E}(\mathcal{H})$ for the vertices and edges of $\mathcal{H}$, respectively. The hypergraph $\mathcal{H}$ is a **simple hypergraph** if:

(1) $|e| \geq 2$ for all $e \in \mathcal{E}$ and

(2) whenever $e_i, e_j \in \mathcal{E}$ and $e_i \subseteq e_j$, then $i = j$.

In the literature, a simple hypergraph is also called a *clutter*.

The vertex $x \in \mathcal{V}(\mathcal{H})$ is called **isolated** if it belongs to no edge of $\mathcal{H}$. The hypergraph $\mathcal{H}$ is called **$d$-uniform** if $|e| = d$ for each $e \in \mathcal{E}(\mathcal{H})$. A hypergraph with vertex set $[n] := \{x_1, \ldots, x_n\}$ is **complete** if its edge set is the set of all subsets of $[n]$ and it is denoted by $\mathcal{K}_n$. We will also denote by $\mathcal{K}_n^d$ the complete $d$-uniform hypergraph. If $n < d$, we interpret $\mathcal{K}_n^d$ as $n$ isolated points.

For a $d$-uniform hypergraph $\mathcal{H}$, a subset $C$ of $\mathcal{V}(\mathcal{H})$ is called a **$d$-clique** of $\mathcal{H}$ if $\mathcal{H}_C$ is a complete $d$-uniform subhypergraph. The **$d$-clique complex** of $\mathcal{H}$, denoted by $\Delta(\mathcal{H})$, is a simplicial complex whose facets are all $d$-cliques of $\mathcal{H}$. Notice that $\Delta(\mathcal{H})$ is a simplicial complex with the vertex set $\mathcal{V}(\mathcal{H})$.

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A subset $F$ of $\mathcal{V}(\mathcal{H})$ is called an **independent set** if $e \nsubseteq F$ for any $e \in \mathcal{E}(\mathcal{H})$. The **independence complex** of $\mathcal{H}$, $\Delta_{\mathcal{H}}$, is a simplicial complexs whose every face is a independent set of $\mathcal{H}$.

The **edge ideal** of a hypergraph $\mathcal{H}$ is the ideal $I(\mathcal{H}) \subset S$ which is generated by the monomials $\mathbf{x}^e = \prod_{x_i \in e} x_i$, where $e \in \mathcal{E}(\mathcal{H})$. It is known that

$$I(\mathcal{H}) = I_{\Delta_{\mathcal{H}}}.$$

In other words, every edge of $\mathcal{H}$ is a minimal nonface of $\Delta_{\mathcal{H}}$.

The **complementary hypergraph** $\mathcal{H}^c$, of a $d$-uniform hypergraph $\mathcal{H}$, is defined as the hypergraph on the same set of vertices as $\mathcal{H}$, and edge set

$$\mathcal{E}(\mathcal{H}^c) = \{e \subseteq \mathcal{V}(\mathcal{H}) : |e| = d, e \notin \mathcal{E}(\mathcal{H})\}.$$

For a $d$-uniform hypergraph $\mathcal{H}$, one can easily obtain that

$$(\mathcal{H}^c)^c = \mathcal{H}.$$

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For every $d$-uniform hypergraph $\mathcal{H}$ we have

$$\Delta_{\mathcal{H}} = \Delta(\mathcal{H}^c).$$

A hypergraph is as **graph** if every edge has cardinality 2. For the graph $G$, $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. Also, we will denote a complete graph on $n$ vertices by $K_n$. We call the simplicial complex $\Delta(G)$, **clique complex** of $G$. A **vertex cover** of a graph $G$ on $V(G)$ is a subset $C \subset V(G)$ such that $e \cap C \neq \emptyset$ for all $e \in E(G)$. A vertex cover $C$ is called minimal if $C$ is a vertex of $G$, and no proper subset of $C$ is a vertex cover of $G$. Obviously, $U$ is an independent set of $G$ if and only if $V(G) \backslash U$ is a vertex cover of $G$. Thus the maximal independent sets of $G$ correspond to the minimal vertex covers of $G$.

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A hypergraph $\mathcal{H}$ on the vertex set $V^{(1)}\dot{\sqcup}\ldots\dot{\sqcup}V^{(r)}$ (a disjoint union of vertex sets) is a **multipartite** hypergraph or ***r*-partite** hypergraph if for every edge $e=\{i_1,\ldots,i_s\}\in\mathcal{E}(\mathcal{H})$, $e\nsubseteq V^{(j)}$ for all $j$.

**Fröberg (1990)**

Thm. A graph $G$ is chordal if and only if $I(G^c)$ has a linear resolution.

Recall that a graph is chordal if each of its cycle of length four or more has an edge joining two nonadjacent vertices in the cycle.

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## Improved

Theorem. A graph $G$ is chordal if and only if $I(G^c)$ is weakly polymatroidal.

## Emtander (2008)

Definition. A chordal hypergraph is a $d$-uniform hypergraph, obtained inductively as follows:

- $\mathcal{K}_n^d$ is a chordal hypergraph, $n, d \in \mathbb{N}$.
- If $\mathcal{G}$ is chordal, then so is $\mathcal{C} = \mathcal{G} \cup_{\mathcal{K}_j^d} \mathcal{K}_i^d$ for $0 \leq j < i$. ($\mathcal{K}_i^d$ is attached to $\mathcal{G}$ in a common (under identification) $\mathcal{K}_j^d$.)

## Emtander, Mohammadi, Moradi (2008)

Theorem. If a $d$-uniform hypergraph is chordal, then the edge ideal of its complementary hypergraph, $I(\mathcal{C}^c)$, has linear quotients.

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## Yassemi and me (2012)

Theorem. Let $\mathcal{C}$ be a $d$-uniform hypergraph which is chordal. Then the edge ideal of $\mathcal{C}^c$, $I(\mathcal{C}^c)$, is weakly polymatroidal.

The converse does not hold.

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## Open problem

For a $d$-uniform hypergraph $\mathcal{H}$ which $I(\mathcal{H})$ has linear resolution (l.q. or is w.p.), what do you say about $\mathcal{H}$?

## Nagel, Reiner (2009)

Definition. A $d$-partite $d$-uniform hypergraph $\mathcal{F}$ on the vertex set $V^{(1)}\dot{\sqcup}\dots\dot{\sqcup}V^{(d)}$ and with the edge set $\mathcal{E}(\mathcal{F}) = \{\{x_{i_1},\dots,x_{i_d}\} : x_{i_j} \in V^{(j)} \text{ for all } j\}$ is a Ferrers hypergraph if for $\{x_{i_1},\dots,x_{i_d}\} \in \mathcal{E}(\mathcal{F})$ and $\{x_{i'_1},\dots,x_{i'_d}\}$ with $i'_j \leq i_j$ for all $j$, one also has $\{x_{i'_1},\dots,x_{i'_d}\} \in \mathcal{E}(\mathcal{F})$.

A hypergraph $\mathcal{H}$ on the vertex set $V^{(1)}\dot{\sqcup}\dots\dot{\sqcup}V^{(r)}$ (a disjoint union of vertex sets) is an $r$-partite hypergraph if for every edge $e = \{i_1,\dots,i_s\} \in \mathcal{E}(\mathcal{H})$, $e \not\subseteq V^{(j)}$ for all $j$.

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Theorem. If $\mathcal{F}$ is a $d$-partite Ferrers $d$-uniform hypergraph without isolated vertices then $I(\mathcal{F})$ has a linear resolution.

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The converse does not hold.

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# References


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Thanks for your attention