ON THE INDEX OF REDUCIBILITY OF NOETHERIAN MODULES

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1. Index of reducibility

- R: a commutative Noetherian local ring with the unique maximal ideal $\mathfrak{m}$, $\mathfrak{k} = R/\mathfrak{m}$ the residue field.
- I is a proper ideal of R, i.e $I \neq R$.
- Primary ideal I: $a, b \in R, ab \in I$ and $a \notin I$ $\Longrightarrow b \in \sqrt{I} = \{x \in R | \exists n > 0, x^n \in I\}$.
- Irreducible ideal I: If $I = I_1 \cap I_2 \Rightarrow I = I_1$ or $I = I_2$.

Facts:

- I is primary $\Rightarrow \sqrt{I} = \mathfrak{p}$ is a prime ideal. Then we call that I is $\mathfrak{p}$-primary.
- I is irreducible $\Rightarrow$ I is primary. The converse is not true.

Theorem I (Noether's Theorem on primary decomposition, 1921).

Let I be an ideal of R. Then there exist finite primary ideals $\mathfrak{q}_1, ..., \mathfrak{q}_n, \sqrt{\mathfrak{q}_i} = \mathfrak{p}_i \neq \mathfrak{p}_j = \sqrt{\mathfrak{q}_j}, \forall i \neq j$ such that

i) *$I = \mathfrak{q}_1 \cap ... \cap \mathfrak{q}_n$ is a reduced decomposition of I.*

ii) *The set $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ is independent of the choise of reduced primary decompositions of I, this set is called the set of asscociated prime ideals of I and denoted by $\mathrm{Ass}(I)$.*

Theorem II (Noether's Theorem on irreducible decomposition, 1921).

Let I be an ideal of R. Then there exist finite irreducible ideals $Q_1, ..., Q_m$ of R such that

i) $I = Q_1 \cap ..., Q_m$ *is a reduced decomposition of* I.

ii) *The number m is independent of the choise of reduced irreducible decompositions of* I.

This number of irreducible components in that a reduced irreducible decomposition of I is called the *index of reducibility* of I and denoted by $\mathrm{ir}(I)$.

Facts:

- M. Brodmann (1979): The set $\mathrm{Ass}(I^n)$ is asymtotic stability, i.e. $\mathrm{Ass}(I^n) = \mathrm{Ass}(I^{n+1})$ for all $n \gg 0$.
- Let $\mathrm{Ass}(I) = \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ as in Theorem I. Noether proved that there exist irreducible ideals $\mathfrak{q}_{i1}, ..., \mathfrak{q}_{ik_i}$ with $\sqrt{\mathfrak{q}_{ij}} = \mathfrak{p}_i, \forall j = 1, ..., k_i, i = 1, ..., n$ such that $I = \mathfrak{q}_{11} \cap ... \cap \mathfrak{q}_{1k_1} \cap ... \cap \mathfrak{q}_{n1} \cap ... \cap \mathfrak{q}_{nk_n}$ is a reduced irreducible decomposition of I. Moreover, the number k_i is dependent only on the prime ideal $\mathfrak{p}_i$ and therefore $\mathrm{ir}(I) = k_1 + ... + k_n$.

Let I be an $\mathfrak{m}$-primary ideal of R, i.e. $\ell_R(R/I) < \infty$. Then it is well-known that

$$\mathrm{ir}(I) = \dim_{\mathfrak{k}}(\mathrm{Soc}(R/I)) = \ell_R(I : \mathfrak{m} / I)$$

Question a. Find a generalization of above formula for $\mathrm{ir}(I)$ withouth the request that $\ell_R(R/I) < \infty$.

Question b. Let $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ be reduced primary decomposition of I as in Theorem I. When $\mathrm{ir}(I) = \mathrm{ir}(\mathfrak{q}_1) + \ldots + \mathrm{ir}(\mathfrak{q}_n)$?

Question c. Consider $\mathrm{ir}(I^n)$ as a function in n. Is this function a polynomial?

3. To Question a

The following result is an answer for Question a

Theorem 1.

Let I be an ideal of R. Then

$$\mathrm{ir}(I) = \sum_{\mathfrak{p} \in \mathrm{Ass}(I)} \dim_{k(\mathfrak{p})}(Soc((R/I)_{\mathfrak{p}}))$$

where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}\, R_{\mathfrak{p}}$ is the residue field of $R_{\mathfrak{p}}$.

Proof. Passing to R/I we may assume that $I = 0$. Let $\mathrm{Ass}(0) = \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$, $t_i = \dim_{k(\mathfrak{p}_i)}(\mathrm{Soc}(R_{\mathfrak{p}_i}))$. Let $\mathcal{F} = \{\mathfrak{p}_{11}, ..., \mathfrak{p}_{1t_1}, \mathfrak{p}_{21}, ..., \mathfrak{p}_{2t_2}, ..., \mathfrak{p}_{n1}, ..., \mathfrak{p}_{nt_n}\}$ be a family of prime ideals of R such that $\mathfrak{p}_{i1} = \cdots = \mathfrak{p}_{it_i} = \mathfrak{p}_i$ for all $i = 1, ..., n$. Denote $E(R)$ the injective envelop of R. Then we can write

$$E(R) = \bigoplus_{i=1}^{n} E(R/\mathfrak{p}_i)^{t_i} = \bigoplus_{\mathfrak{p}_{ij} \in \mathcal{F}} E(R/\mathfrak{p}_{ij}).$$

Let

$$\pi_i : \oplus_{i=1}^{n} E(R/\mathfrak{p}_i)^{t_i} \to E(R/\mathfrak{p}_i)^{t_i} \text{ and } \pi_{ij} : \oplus_{\mathfrak{p}_{ij}\in\mathcal{F}} E(R/\mathfrak{p}_{ij}) \to E(R/\mathfrak{p}_{ij})$$

be the canonical projections for all $i = 1, ..., n$ and $j = 1, ..., t_i$. Set $I_i = R \cap \operatorname{Ker} \pi_i \Rightarrow I_i$ is $\mathfrak{p}_i$-primary.

$\mathfrak{q}_{ij} = \cap \operatorname{Ker} \pi_{ij}, i = 1, ..., n, j = 1, ..., t_i. \Rightarrow \mathfrak{q}_{ij}$ is $\mathfrak{p}_i$-primary and irreducible. Then

$$I_i = \mathfrak{q}_{i1} \cap ... \cap \mathfrak{q}_{it_i}$$

and $0 = \bigcap_{i=1}^{n} I_i = \bigcap_{\substack{1\le i\le n,\\ 1\le j\le t_i}} \mathfrak{q}_{ij}$ is a reduced irreducible decomposition of I. Thus

$$\operatorname{ir}(0) = \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \dim_{k(\mathfrak{p})}(\operatorname{Soc}(R_{\mathfrak{p}_i})).$$

Let $\mathfrak{p} \in \mathrm{Ass}(I)$. We use $\wedge_{\mathfrak{p}}(I)$ to denote the set of all $\mathfrak{p}$-primary ideals of R which appear in a primary decomposition of I. Following Heinzer, Ratliff and Shah, an ideal $\mathfrak{q} \in \wedge_{\mathfrak{p}}(I)$ is called $\mathfrak{p}$-*maximal embedded component* of I if $\mathfrak{q}$ is a maximal element in $\wedge_{\mathfrak{p}}(I)$.

Theorem 2.

Let $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ *be a reduced primary decomposition of* I *with* $\mathrm{Ass}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$*. Then* $\mathrm{ir}(I) = \sum_{i=1}^{n} \mathrm{ir}(\mathfrak{q}_i)$ *if and only if* $\mathfrak{q}_i$ *is a* $\mathfrak{p}_i$*-maximal embedded component of* I *for all* $i = 1, \ldots, n$*.*

Combine Theorem 1 and Theorem 2 we get the following characterization of maximal embedded components in terms of the index reducibility.

Corollary 1.

Let $\mathfrak{p} \in \mathrm{Ass}(I)$ *and* $\mathfrak{q} \in \wedge_{\mathfrak{p}}(I)$. *Then* $\mathfrak{q}$ *is a* $\mathfrak{p}$*-maximal embedded component of* I *if and only if* $\mathrm{ir}(\mathfrak{q}) = \dim_{k(\mathfrak{p})}(Soc(R/I)_{\mathfrak{p}})$.

5. To Question c

By Brodmann's result: $\mathrm{Ass}(I^n) = \mathrm{Ass}(I^{n+1})$ for all $n \gg 0$. Denote this stable set by $A(I)$. Then by Theorem 1 we get for $n \gg 0$

$$\mathrm{ir}(I^n) = \sum_{\mathfrak{p} \in A(I)} \dim_{k(\mathfrak{p})}(\mathrm{Soc}(R/I^n)_{\mathfrak{p}}).$$

Thus $\mathrm{ir}(I^n)$ is a polynomial for $n \gg 0$ if the following result true:

Lemma.

Let I be an ideal in the local ring $(R, \mathfrak{m})$. Then $\dim_{\mathfrak{k}}(Soc(R/I^n)) = \ell_R(I^n : \mathfrak{m} / I^n)$ is a polynomial for all $n \gg 0$.

5. To Question c

We can also estimate the degree of the polynomial $\mathrm{ir}(I^n)$.

- $\mathrm{bight}(I) = \max\{\dim R_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Ass}(I), \mathfrak{p} \text{ is minimal in } \mathrm{Ass}(I)\}$. This invariant is called the *big hight* of I.
- The *analytic spread* of I is defined by $\ell(I) = \dim F(I)$ where $F(I) = \bigoplus_{i=0}^{\infty} I^i / \mathfrak{m} I^i$ is the ring of fiber cone w.r.t I.
- It should note that $\mathrm{bight}(I) \leq \ell(I)$.

5. To Question c

The following theorem is an answer for Question 3.

Theorem 3.

Let I be an ideal of R. Then there exists a polynomial $Ir_I(n)$ with rational coefficients such that

$$\mathrm{ir}(I^n) = Ir_I(n)$$

for all $n \gg 0$. Moreover, we have

$$bight(I) - 1 \leq \deg Ir_I(n) \leq \ell(I) - 1.$$

Corollary 2.

If I is and $\mathfrak{m}$-primary ideal then $\deg Ir_I(n) = \dim R - 1$.

5. To Question c

There are examples to show that although the bounds in Theorem 3 are sharp, neither $\operatorname{bight}(I) - 1$ nor $\ell(I) - 1$ equal to $\deg \operatorname{Ir}_I(n)$ in general:

- Let $I = (X^2, XY)$ be an ideal of the polynomial ring $R = K[X, Y]$. Then

$$\operatorname{bight}(I) - 1 = 0 < 1 = \deg \operatorname{Ir}_I(n) = \ell(I) - 1.$$

- Let $I = (X_1X_2, X_2X_3, X_3X_4, X_4X_5, X_5X_6, X_6X_1)$ be an ideal of the polynomial ring $(K[X_1, \ldots, X_6])_{(X_1,\ldots,X_6)}$. Then

$$\operatorname{bight}(I) - 1 = 3 = \deg \operatorname{Ir}_I(n) < 4 = \ell(I) - 1.$$

Open problem: Find a formula for $\deg \operatorname{Ir}_I(n)$ in terms of known invariants associated to I and R.

- Let $\dim R = d$. A system of d elements $\underline{a} = \{a_1, ..., a_d\}$ in $\mathfrak{m}$ is called a *system of parameters* of R if $\ell_R(R/(a_1, ..., a_d)) < \infty$, and ideal I is called a *parameter ideal* if I generated by a system of parameters of R.
- A system of parameters $\underline{a} = \{a_1, ..., a_t\}$ of element in $\mathfrak{m}$ is called a *regular sequence* of R, if a_{i+1} is a nonzero divisor of $R/(a_1, ..., a_i)$ for all $i = 1, ..., t-1$.
- R is called a *Cohen-Macaulay* ring if any system of parameters is a regular sequence of R.

6. Application

- D.G. Northcott (1957): Any parameter ideal in a Cohen-Macaulay has the same index of reducibility. But, S. Endo and M. Narita, (1964) showed that the converse is not true.

Theorem 3.

Let R be a local ring of dimension d. Then the following conditions are equivalent:

(i) *R is a Cohen-Macaulay ring.*

(ii) $\mathrm{ir}(\mathfrak{q}^{n+1}) = \dim_k(Soc(H^d_{\mathfrak{m}}(R)))\binom{n+d-1}{d-1}$ *for all parameter ideals $\mathfrak{q}$ of R and $n \geq 0$.*

(iii) $\mathrm{ir}(\mathfrak{q}) = \dim_k(Soc(H^d_{\mathfrak{m}}(R)))$ *for all parameter ideal $\mathfrak{q}$ of R.*

($H^d_{\mathfrak{m}}(R) = \varinjlim_n R/(a_1^n, ..., a_d^n)$, where $\{a_1, ..., a_d\}$ is a system of parameters of R)

Thank you for your attention.