The $\mathfrak{p}$-standard system of parameters in a local ring and its applications

Nguyen Tu Cuong

Institute of Mathematics
Vietnam Academy of Science and Technology

Tehran, January 13, 2020

This talk is a short survey of the following works:

[1] N.T. Cuong, On the dimension of the non-Cohen-Macaulay locus of local rings admitting dualizing complexes, Math. Proc. Cambridge Philos. Soc., **109**(2) (1991), 479–488.

[2] N.T. Cuong, $\mathfrak{p}$-standard systems of parameters and $\mathfrak{p}$-standard ideals in local rings, Acta Math. Vietnamica, **20** (1995), 145–161.

[3] N.T. Cuong and D.T. Cuong, Local cohomology annihilators and Macaulayfication, Acta Math. Vietnamica, **42** (2017), 37–60.

[4] N.T. Cuong and P.H. Quy, On the index of reducibilily of parameter ideals: The stable and limit values, to appear in Acta Math. Vietnamica (2020).

[5] N.T. Cuong and P.H. Quy, On the structure of finitely generated modules over quotients of Cohen-Macaulay local rings, Prepint, arXiv: 1612.07638.

I. $\mathfrak{p}$-standard systems of parameters

- $(R, \mathfrak{m})$: a Noetherian local ring.
- M: a finitely generated R-module with $\dim M = d$.
- $\mathfrak{a}_i(M) = \mathrm{Ann}_R(H^i_{\mathfrak{m}}(M))$; $\mathfrak{a}(M) = \mathfrak{a}_0(M) \dots \mathfrak{a}_{d-1}(M)$.

Definition 1. ([1], [2])

A system of parameters $\underline{x} = (x_1, \dots, x_d)$ of M is called a $\mathfrak{p}$-*standard system of parameters*, if

$$\begin{cases} x_d \in \mathfrak{a}(M), \\ x_i \in \mathfrak{a}(M/(x_{i+1}, \dots, x_d)M), i = d-1, \dots, 1. \end{cases}$$

Facts. 1) Every system of parameters of a Cohen-Macaulay module is a $\mathfrak{p}$-standard system of parameters.
2) Let M be a generalized Cohen-Macauay module. Then $\underline{x}$ is a $\mathfrak{p}$-standard system of parameters of M iff $\underline{x}$ is a standard system of parameters introduced by M. Brodmann and N.V. Trung.
3) M always has a $\mathfrak{p}$-standard system of parameters if R admits a dualizing complex (because in that case $\dim(R/\mathfrak{a}(M)) < d$).
4) There are local rings never having a $\mathfrak{p}$-standard system of parameters (examplex: the two-dimensional domains R constructed by Ferrand-Reynaud or by M. Nagata never admit $\mathfrak{p}$-standard systems of parameters, because in that case $\dim(R/\mathfrak{a}(R) = 2)$).

Theorem 1 ([1], [2])

Let $\underline{x} = (x_1, \ldots, x_d)$ *be a* $\mathfrak{p}$*-standard system of parameters of* M*.*
(i) $\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_d^{n_d})$ *are again* $\mathfrak{p}$*-standard systems of parameters of* M *for all positive integer* $n_1, \ldots, n_d$*.*
(ii) $\underline{x}(\underline{n})$ *are* d*-sequences in sense of C. Huneke, i.e.,* $(x_1^{n_1}, \ldots, x_i^{n_i})M : x_j^{n_j} = (x_1^{n_1}, \ldots, x_i^{n_i})M : x_{i+1}^{n_{i+1}} x_j^{n_j}$ *for all* $0 \leq i < j \leq d$ *and positive integers* $n_1, \ldots, n_d$*.*
(iii) *There are non-negative integers* $a_0, \ldots, a_{d-1}$ *such that*

$$\ell_R(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = \sum_{i=0}^{d-1} n_1 \ldots n_i a_i.$$

Moreover, the degree of this polynomial is independent of the choice of the $\mathfrak{p}$*-standard system of parameters. This degree is called the polynomial type of* M *and denoted by* $\mathfrak{p}(M)$*. Then* $d - 1 \geq \mathfrak{p}(M) \geq \dim(\mathrm{nCM}(M))$ *and if* M *is equidimensional,* $\mathfrak{p}(M) = \dim(\mathrm{nCM}(M))$

Conjecture (1995)

The local ring R is a quotient of a Cohen-Macaulay ring if and only if any finitely generated R-module admits $\mathfrak{p}$-standard systems of parameters.

- When R admits a $\mathfrak{p}$-standard systems of parameters?

Theorem 2 ([3])

A Noetherian local ring R admits a $\mathfrak{p}$-standard system of parameters if and only if R is quotient of a Cohen-Macaulay ring.

- When M admits a $\mathfrak{p}$-standard systems of parameters?

Theorem 3 ([3])

The following statements are equivalent:
(i) M *admits a* $\mathfrak{p}$*-standard system of parameters.*
(ii) $R/\operatorname{Ann}_R(M)$ *admits a* $\mathfrak{p}$*-standard system of parameters.*
(iii) *Any finitely generated* R*-module* N *with* $\operatorname{Supp}(N) \subseteq \operatorname{Supp}(M)$ *admits a* $\mathfrak{p}$*-standard system of parameters.*

- **Theorem 2** and **Theorem 3** give a positive answer for the **Cojecture** and we also have the following useful consequence.

Corollary 1

Any finitely generated module over a quotient of a Cohen-Macaulay local ring admits a $\mathfrak{p}$*-standard system of parameters.*

Definition 2 (G. Faltings, 1978)

Let X be a Noetherian scheme. A *Macaulayfication* of X is a pair (Y, π) consiting of a Cohen-Macaulay scheme Y and a birational proper morphism $\pi : Y \to X$ such that $\pi : \pi^{-1}(\mathrm{nCM}(X)) \to \mathrm{nCM}(X)$ is an isomorphism.

Facts 1) Faltings (1978): X admits a Macaulayfication provided $\dim(\mathrm{nCM}(X)) \leq 1$ and X is over Noetherian ring admitting a dualizing complex.
2) T. Kawasaki (2000): Any scheme over Noetherian ring admitting dualizing complex has a Macaulayfication. Note that the key point in Kawasaki's contruction is to use the $\mathfrak{p}$-standard systems of parameter to determine the center for the blowing up the scheme X.

By applying Corollary 1, we can generalize Kawasaki's Theorem for local rings as follows:

Theorem 4 ([3])

Suppose that R is catenary. Then the following conditions are equivalent:
(i) R *is a quotient of a Cohen-Macaulay local ring.*
(ii) $\mathrm{Spec}(R)$ *has a Macaulayfication.*

Definition 3

Let I be a proper ideal of positive heigh of R. $\mathcal{R}(I) = \oplus_{n\geq 0} I^n$ is the Rees algebra of R with respect to I.
If $\mathcal{R}(I)$ is Cohen-Macaulay, then it is called an *arithmetic Macaulayfication* of the ring R.
Analogously, an arithmetic Macaulayfication of an R-module M is defined to be a Cohen-Macaulay Rees module
$\mathcal{R}(M, I) = \oplus_{n\geq 0} I^n M$ for some ideal I of R.

Theorem 5 ([3])

Suppose that M is unmixed (i.e. $\dim \hat{R}/P = d$ for all $P \in \mathrm{Ass}_{\hat{R}}(\hat{M})$). Then the following conditions are equivalent:
(i) *M has an arithmetic Macaulayfication.*
(ii) *M admits a $\mathfrak{p}$-standard system of parameters.*

Definition 4

Let N be a submodule of M. The number of irreducible components of an reduced irreducible decomposition of N, which is independent of the choise of the decomposition proved by E. Noether (1921), is called the *index of reducibility* of N in M and this number is denoted by $\mathrm{ir}_M(N)$. If I is an ideal of R we simply write $\mathrm{ir}_M(I)$ instead for $\mathrm{ir}_M(IM)$.

Facts. 1) (Northcott, 1957) The index of reducibility $\mathrm{ir}_M(\mathfrak{q})$ of a parameter ideal $\mathfrak{q}$ in a Cohen-Macaulay modunle M is independent of the choice of $\mathfrak{q}$, in fact $\mathrm{ir}_M(\mathfrak{q}) = \dim_{R/\mathfrak{m}} \mathrm{Soc}(H^d_{\mathfrak{m}}(M))$.
2) (H. L. Truong-C., 2008) Let M be a generalized Cohen-Macaulay module. Then there exists a positive integer n s.t. $\mathrm{ir}_M(\mathfrak{q})$ is independent of the choice of all parameter ideals of contained in $\mathfrak{m}^n$.

The following theorem is a generalization of 1) and 2).

Theorem 6 ([4])

Suppose that R is quotient of a Cohen-Macaulay ring. Then there exists an ideal $\mathfrak{b}(M) \subseteq \mathfrak{a}(M)$ with $\sqrt{\mathfrak{b}(M)} = \sqrt{\mathfrak{a}(M)}$ s.t. for any parameter ideal $\mathfrak{q} = (x_1, \ldots, x_d)$ of M satisfying

$$\begin{cases} x_d \in \mathfrak{b}(M), \\ x_i \in \mathfrak{b}(M/(x_{i+1}, \ldots, x_d)M), i = d-1, \ldots, 1, \end{cases}$$

the index of reducibility $\mathrm{ir}_M(\mathfrak{q})$ is independent of the choice of $\mathfrak{q}$. Moreover, denote this invariant by $\mathcal{N}_R(M)$ then M is a Cohen-Macaulay module if and only if $\mathcal{N}_R(M) = \dim_{R/\mathfrak{m}}(\mathrm{Soc}(H^d_{\mathfrak{m}}(M)))$.

IV. Index of reducibility

Remark 1

Note that the assumption of Theorem 6 guarantees the existence of the ideal $\mathfrak{b}(M)$ and also for that a parameter ideal $\mathfrak{q}$ of M. The ideal $\mathfrak{b}(M)$ can be choosed as follows:
For a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of M we set
$\mathfrak{b}_{\underline{x}}(M) = \cap_{i=1}^{d} \operatorname{Ann}(0 : x_i)_{M/(x_1,\ldots,x_{i-1})}(M)$, and
$\mathfrak{b}(M) = (\cap_{\underline{x}} \mathfrak{b}_{\underline{x}}(M))^3$, where $\underline{x}$ runs over all systems of parameters of M. Then $\mathfrak{b}(M) \subseteq \mathfrak{a}(M)$ and $\sqrt{\mathfrak{b}(M)} = \sqrt{\mathfrak{a}(M)}$ by P. Schenzel (1982). Therefore the system of parameters defined in part a) of Theorem 6 is just a $\mathfrak{p}$-standard system of parameters and it was called by M. Morales and P.H. Quy (Def. 2.15, Proc. Edinb. Math. Soc. (2) 60 (2017), no. 3, 721–737) a C-system of parameters.

Open question

Is it true for any finitely generated R-module M that

$$\mathcal{N}_R(M) = \lim \alpha_n(M), \text{ where } \alpha_n(M) = \inf\{\mathrm{ir}_M(\mathfrak{q}) \mid \mathfrak{q} \subset \mathfrak{m}^n\}?$$

Definition 4 (W. V. Vasconcelos, 1998; E. Rossi, N.V. Trung, G. Valla, 2003)

Let I be an $\mathfrak{m}$-primary ideal. An *extended degree* on the catagory of finitely generated R-modules $\mathcal{M}(R)$ with respect to I is a numerical function

$$\mathrm{Deg}(I, -) : \mathcal{M}(R) \to \mathbb{R}$$

satisfying the following conditions
(1) $\mathrm{Deg}(I, M) = \mathrm{Deg}(I, \overline{M}) + \ell_R(H^0_{\mathfrak{m}}(M))$, where $\overline{M} = M/H^0_{\mathfrak{m}}(M)$.
(2) (Bertini's Rule) $\mathrm{Deg}(I, M) \geq \mathrm{Deg}(I, M/xM)$ for every generic element $x \in I \setminus \mathfrak{m}I$.
(3) If M is Cohen-Macaulay then $\mathrm{Deg}(I, M) = \deg(I, M)$, where $\deg(I, M) = e(I, M)$ is the Zariski-Samuel multiplicity of M with respect to I.

Facts. Up to nowaday, the only explicit extended degree is the homological degree introduced by Vasconcelos (1998): Suppose that R is a homomorphism of a Gorenstein $(S, \mathfrak{m})$ of dimension n. Then the homological degree $\mathrm{hdeg}(I, M)$ is defined by the following recursive formula

$$\begin{aligned} & \mathrm{hdeg}(I, M) \\ = \; & \deg(I, M) + \sum_{i=n-d+1}^{n} \binom{d-1}{i-n+d-1} \mathrm{hdeg}(I, \mathrm{Ext}_S^i(M, S)). \end{aligned}$$

(Note that $\dim \mathrm{Ext}_S^i(M, S) \leq n - i < d$ for all $i \geq n - d + 1$).

Cohen-Macaulay deviated sequence. Let $0 = \cap_{\mathfrak{p}\in \mathrm{Ass}(M)} N(\mathfrak{p})$ be a reduced primary decomposition of the zero submodule of M. We put

$$U(M) = \bigcap_{\mathfrak{p}\in \mathrm{Assh}(M)} N(\mathfrak{p}),$$

where $\mathrm{Assh}(M) = \{\mathfrak{p} \in \mathrm{Ass}(M) \mid \dim R/\mathfrak{p} = d\}$. This submodule $U(M)$ is called the unmixed component of M.

Theorem 7 ([5])

Let $\underline{x} = (x_1, \ldots, x_d)$ be a C-system of parameters as in Remark 1. Then the unmixed components $U(M/(x_{i+1}, \ldots, x_d)M)$ are independent (up to an isomorphism) of the choice of $\underline{x}$ for all $i = 1, \ldots, d$.

Denote by $U_i(M)$ the R-module satisfying

$$U_i(M) \cong U(M/(x_{i+2}, \ldots, x_d)M)$$

for a C-system of parameters $\underline{x} = (x_1, \ldots, x_d)$.

Definition 5

The sequence of modules $U_0(M), \ldots, U_{d-1}(M)$, where $U_i(M) \cong U(M/(x_{i+2}, \ldots, x_d)M$ for a C-system of parameters $\underline{x}$, is called the *Cohen-Macaulay deviated sequence* of M.

Remark 2

$\dim U_i(M) \leq i$ and $U_{d-1}(M) \cong U(M)$.

Theorem 8 ([5])

Let $\{U_0(M), \ldots, U_{d-1}(M)\}$ be a Cohen-Macaulay deviated sequence of M. Then the unmixed degree of M with respect to I is defined by

$$\operatorname{udeg}(I, M) = \sum_{i=0}^{d-1} \delta_{i,\dim U_i(M)} \deg(I, U_i(M))$$

is an extended degree, where $\delta_{i,j}$ is the Kronecker symbol.

Open question

Is it true that

$$\operatorname{udeg}(I, M) \leq \operatorname{hdeg}(I, M)?$$

Thank you for your attention.