Boij-Söderberg Theory

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Table of contents

Preface

In this presentation we try to give an outline on a new theory on free resolutions.
This theory is named after two Swedish mathematicians "Mats Boij" and "Jonas Söderberg".
Fløystad's paper, "Boij-Söderberg theory: Introduction and survey", provides a very nice introduction to this theory. (see [4].)

Betti Numbers

The coefficients of the Hilbert polynomial are the fundamental numerical invariants of a graded S-module.
The graded Betti numbers of a module are finer numerical invariants!
In this presentation we always assume that

- k is a field;
- $S = k[x_1, \ldots, x_n]$ is the polynomial ring with standard grading;
- $S(-i)$ denotes S with a grading shift, i.e., $S(-i)_j = S_{i-j}$;

Betti Numbers

Let $C^\bullet$ be a complex of graded free S-modules. Then $C^\bullet$ is of the form

$$\cdots \to \bigoplus_j S(-j)^{\beta_{i,j}} \to \bigoplus_j S(-j)^{\beta_{i+1,j}} \to \cdots$$

The number $\beta_{i,j}$ of the term $S(-j)$ in the i-th homological part of the complex, is called the i-th graded *Betti number* of the complex $C^\bullet$ in degree j.
In particular, when $C^\bullet$ is the minimal free resolution of a graded S-module M, these are called the Betti numbers of the S-module M.
The Betti numbers are usually displayed in an array called the *Betti diagram* of the module M.

Example of a Betti Table

Example

Let $S = k[x, y]$ and M be the quotient ring $S/(x^2, xy, y^3)$. Then its minimal resolution is

$$S \longleftarrow S(-2)^2 \oplus S(-3) \longleftarrow S(-3) \oplus S(-4).$$

The Betti diagram of M is displayed in an array as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Main Idea of Boij-Söderberg Theory

Attempting to prove the Multiplicity Conjectures of Herzog, Huneke and Srinivasan, Boij and Söderberg [1] made a big step forward towards an answer of the fundamental question:

What Betti tables are possible?

This problem is still out of reach! But the Boij-Söderberg theory describes Betti diagrams up to a multiple by a rational number.

That is, we do not determine if a diagram β is a Betti diagram of a module, but we can determine if $q\beta$ is a Betti diagram for some positive rational number q.

Degree Sequences

By a *degree sequence* we mean a strictly increasing sequence, $\mathbf{d} = (d_0, \dots, d_c)$ of integers. A resolution is called *pure* if it is of the form

$$S(-d_0)^{\beta_{0,d_0}} \leftarrow S(-d_1)^{\beta_{1,d_1}} \leftarrow \cdots \leftarrow S(-d_c)^{\beta_{c,d_c}}.$$

A *pure diagram* of type $\mathbf{d}$ is a diagram associated to a pure resolution of the form above.

Example of a Degree Sequence

Example

Let M be the quotient ring $S/(x^2, y^2)$, where S is the polynomial ring in two variables x, y. The minimal free resolution of M,

$$S \longleftarrow S(-2)^2 \longleftarrow S(-4),$$

is a pure resolution with degree sequence $(0, 2, 4)$.
The betti diagram

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has only one nonzero entry in each column.

Degree Sequences

We define a partial ordering on any set of degree sequences of length $c+1$ by the rule,

$$\mathbf{d} < \mathbf{d}' \Leftrightarrow d_i \leq d'_i, \forall i; 0 \leq i \leq c+1.$$

when $\mathbf{d} = (d_0, \ldots, d_c)$ and $\mathbf{d}' = (d'_0, \ldots, d'_c)$ are two degree sequences.

Degree Sequences

Let $\mathbf{d} = (d_0, \ldots, d_c)$ be a degree sequence. Herzog-Kühl equations show that $\mathbb{Q}$-vector space of Betti diagrams of Cohen-Macaulay modules with pure resolutions of type $\mathbf{d}$ is at most 1-dimensional. In this space we denote the vector with least integral coordinates by $\pi(\mathbf{d})$.

- Let $\mathbb{Z}^{c+1}_{\text{deg}}$ be the set of strictly increasing integer sequences $(a_0, \ldots, a_c)$ in $\mathbb{Z}^{c+1}$.
- For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^{c+1}_{\text{deg}}$, let $\mathbb{D}(\mathbf{a}, \mathbf{b})$ be the set of diagrams $(\beta_{ij})_{i=0,\ldots,c, j\in\mathbb{Z}}$ such that β_{ij} may be nonzero only in the range $0 \leq i \leq c$ and $a_i \leq j \leq b_i$.
- Denote the set of all degree sequences $\mathbf{d}$ satisfying $a_i \leq d_i \leq b_i$ for all i, with $[\mathbf{a}, \mathbf{b}]_{\text{deg}}$.

Boij-S̈derberg Conjectures

Theorem (First Boij-Söderberg Conjecture)

Let $\mathbf{d}$ *be a degree sequence of length* $c+1$*. Does a Cohen-Macaulay module of codimension* c *with a resolution of type* $\mathbf{d}$ *exist?*

Theorem (Second Boij-Söderberg Conjecture)

Let M *be a Cohen-Macaulay module of codimension* c *with Betti diagram* $\beta(M)$ *in* $\mathbb{D}(\mathbf{a}, \mathbf{b})$*. There is a unique chain*

$$\mathbf{d}^1 < \mathbf{d}^2 < \cdots \mathbf{d}^r$$

in $[\mathbf{a}, \mathbf{b}]_{\text{deg}}$ *such that* $\beta(M)$ *is uniquely a linear combination*

$$c_1\pi(\mathbf{d}^1) + c_2\pi(\mathbf{d}^2) + \cdots + c_r\pi(\mathbf{d}^r)$$

where the c_i *are positive rational numbers.*

Boij-Söderberg Decomposition

Example

The diagram

$$\beta = \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

is a positive rational combination of pure diagrams

$$\pi(0,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \pi(0,2,4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\pi(0,3,4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 3 \end{bmatrix}.$$

Boij-Söderberg Decomposition

Example (Continue)

for degree sequences

$$(0,2,3) < (0,2,4) < (0,3,4).$$

The combination is

$$\beta = \frac{1}{2}\pi(0,2,3) + \frac{1}{4}\pi(0,2,4) + \frac{1}{4}\pi(0,3,4).$$

First Construction

- Here k is a field of characteristic 0.
- $\mathbf{d} = (d_0, \ldots, d_n)$ is a degree sequence.
- E is a n-dimensional vector space and $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition.
- $S_\lambda(E)$ denotes the Schur module of E with respect to λ.

To construct the desired complex, define the partitions $\alpha(\mathbf{d}, i)$ for $0 \leqslant i \leqslant n$ by

$$\begin{aligned}\alpha(\mathbf{d}, 0) =& \lambda,\\ \alpha(\mathbf{d}, i) =& (\lambda_1 + d_1 - d_0, \lambda_2 + d_2 - d_1, \ldots, \lambda_i + d_i - d_{i-1},\\ & \lambda_{i+1}, \ldots, \lambda_n),\end{aligned}$$

where $\lambda_i = \sum_{j>i}(d_j - d_{j-1} - 1)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$.

First Construction

Eisenbud, Floystad and Weyman, [2], show that there is a pure resolution of type **d** as

$$F(\mathbf{d}) : S \otimes_k S_{\alpha(\mathbf{d},0)} \longleftarrow S \otimes_k S_{\alpha(\mathbf{d},1)} \longleftarrow \cdots \longleftarrow S \otimes_k S_{\alpha(\mathbf{d},n)}.$$

Facts:

- This complex is uniquely defined up to multiplying the differentials by nonzero constants.
- This complex is $GL(n)$-equivariant.

Second Construction

This construction is characteristic free. To construct this resolution we need the following observation.

proposition

Let $\mathcal{F}$ be a sheaf on $X \times \mathbb{P}^m$, and let $p : X \times \mathbb{P}^m \to X$ be the projection. Suppose that $\mathcal{F}$ has a resolution of the form

$$\mathcal{G} : 0 \to \mathcal{G}_N \boxtimes \mathcal{O}(-e_N) \to \cdots \to \mathcal{G}_0 \boxtimes \mathcal{O}(-e_0) \to \mathcal{F} \to 0$$

with degrees $e_0 < \cdots < e_N$. If this sequence contains the subsequence $(e_{k+1}, \ldots, e_{k+m}) = (1, 2, \ldots, m)$ for some $k \geq -1$ then

$$R^\ell p_* \mathcal{F} = 0 \text{ for } \ell > 0$$

Second Construction

Proposition (Continue)

and $p_*\mathcal{F}$ has a resolution on X of the form

$$0 \to \mathcal{G}_N \otimes H^m\mathcal{O}(-e_N) \to \cdots \\ \to \mathcal{G}_{k+m+1} \otimes H^m\mathcal{O}(-e_{k+m+1}) \xrightarrow{\phi} \mathcal{G}_k \otimes H^0\mathcal{O}(-e_k) \to \\ \cdots \to \mathcal{G}_0 \otimes H^0\mathcal{O}(-e_0) \qquad (1)$$

Second Construction

To simplify the notation we may harmlessly assume that $d_0 = 0$. Let $m_0 = m = n - 1$, and for $i = 1, \ldots, n$ set $m_i = d_i - d_{i-1} - 1$, and set $M = \sum_{j=0}^{k} m_j = d_n - 1$. Choose $M + 1$ homogenous forms without a common zero on

$$\mathbb{P} := \mathbb{P}^m \times \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n},$$

Let

$$\mathcal{K} : 0 \to \mathcal{K}_{M+1} \to \cdots \to \mathcal{K}_0 \to 0$$

be the tensor product of the Koszul complex of these forms on $\mathbb{P}$ and the line bundle $\mathcal{O}_{\mathbb{P}}(0, 0, d_1, \ldots, d_{n-1})$, so $\mathcal{K}_i = \mathcal{O}_{\mathbb{P}}(-i, -i, \ldots, d_{n-1} - i)^{\binom{d_n}{i}}$ for $i = 0, \ldots, d_n$.

Second Construction

Let

$$\pi : \mathbb{P}^m \times \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n} \to \mathbb{P}^m$$

be the projection onto the first factor. The complex $\mathcal{K}$ is exact because the forms have no common zero. If we think of $\mathcal{K}$ as a resolution of the zero sheaf $\mathcal{F} = 0$, and factor π into the successive projections along the factors of the product $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}$, then we may use the preceding Proposition repeatedly to get a resolution of $\pi_* \mathcal{F} = 0$ that has the form

$$0 \to \mathcal{O}^{\beta_n}(-d_n) \to \cdots \to \mathcal{O}^{\beta_1}(-d_1) \to \mathcal{O}^{\beta_0}.$$

Second Construction

Taking global sections in all twists, we get a complex

$$0 \to S^{\beta_n}(-d_n) \to \cdots \to S^{\beta_1}(-d_1) \to S^{\beta_0}$$

that has homology of finite length.
Now the Acyclicity Lemma of Peskine and Szpiro shows that the complex is actually acyclic.

Some definitions

Let $\mathbf{a}$ and $\mathbf{b}$ be in $\mathbb{Z}^{c+1}_{deg}$,

- $L(\mathbf{a}, \mathbf{b})$ is the $\mathbb{Q}$-vector subspace in $\mathbb{D}(\mathbf{a}, \mathbf{b})$ spanned by the Betti diagrams of CM-modules of codimension c, whose Betti diagrams are in this window.
- $B(\mathbf{a}, \mathbf{b})$ is the set of non-negative rays spanned by such Betti diagrams. This set is a convex cone.

Proposition

Given any maximal chain $\mathbf{a} = \mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^r = \mathbf{b}$, in $[\mathbf{a}, \mathbf{b}]_{deg}$. The associated pure diagrams

$$\pi(\mathbf{d}^1), \pi(\mathbf{d}^2), \ldots, \pi(\mathbf{d}^r)$$

form a basis for $L(\mathbf{a}, \mathbf{b})$. The length of such a chain, and hence the dimension of the latter vector space is $r = 1 + \sum(b_i - a_i)$.

Boij-Söderberg Fan

Definition

A fan Σ consists of a finite collection of cones such that

1. each face of a cone in the fan is also in the fan;
2. any pair of cones in the fan intersect in a common face.

We say a fan Σ is simplicial if the generators of each cone in Σ are linearly independent over $\mathbb{Q}$.

Boij-Söderberg Fan

Since for any chain $D : \mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^r$ in $[\mathbf{a}, \mathbf{b}]_{deg}$ the Betti diagrams $\pi(\mathbf{d}^1), \ldots, \pi(\mathbf{d}^r)$ are linearly independent diagrams in $\mathbb{D}(\mathbf{a}, \mathbf{b})$, their positive rational linear combinations give a simplicial cone $\sigma(D)$ in $\mathbb{D}(\mathbf{a}, \mathbf{b})$, Two such cones will intersect along another such cone. This means that,

The set of simplicial cones $\sigma(D)$ where D ranges over all chains $\mathbf{d}^1 < \cdots < \mathbf{d}^r$ in $[\mathbf{a}, \mathbf{b}]_{deg}$ form a simplicial fan, which we denote as $\Sigma(\mathbf{a}, \mathbf{b})$.
The second Boij-Söderberg conjecture is equivalent to the the following statement.

Geometric Interpretation of the Second Conjecture

The positive cone $B(\mathbf{a}, \mathbf{b})$ is contained in the realization of the fan $\Sigma(\mathbf{a}, \mathbf{b})$.

Second Boij-Söderberg Conjecture

The idea of the proof is now simple.

1. We describe all the external facets of the Boij-Söderberg fan (easy!);
2. We find the equation of the unique hyperplane containing each facet (not easy!);
3. Each of these supporting hyperplanes define a halfspace and the intersection of all these halfspaces is a positive cone contained in the Boij-Söderberg fan. (obvious!)
4. Show that every Betti diagram of a module is in all these positive halfspaces. (Hard!)
5. We are done!

Cohomology tables

For a coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^m$ our interest shall be the cohomological dimensions

$$\gamma_{i,d}(\mathcal{F}) = \dim_k H^i \mathcal{F}(d).$$

The indexed set $(\gamma_{i,d})_{i=0,\ldots,m,d\in\mathbb{Z}}$ is the *cohomology table* of $\mathcal{F}$, which lives in the vector space $\mathbb{T} = \mathbb{D}^* = \Pi_{d\in\mathbb{Z}}\mathbb{Q}^{m+1}$ with the $\gamma_{i,d}$ as coordinate functions. An element in this vector space will be called a *table*.

We shall normally display a table as follows.

$$\begin{array}{cccccc|c}
\cdots & \gamma_{n,-n-1} & \gamma_{n,-n} & \gamma_{n,-n+1} & \cdots & & n \\
 & \vdots & \vdots & \vdots & & & \\
\cdots & \gamma_{1,-2} & \gamma_{1,-1} & \gamma_{1,0} & \cdots & & 1 \\
\cdots & \gamma_{0,-1} & \gamma_{0,0} & \gamma_{0,1} & \cdots & & 0 \\
\hline
\cdots & -1 & 0 & 1 & \cdots & & d \backslash i
\end{array}.$$

Cohomology tables

Example

The cohomology table of the ideal sheaf of two points in $\mathbb{P}^2$ is

$\cdots$	6	3	1	$\lfloor 0$	0	0	0	$\cdots$
$\cdots$	2	2	2	2	1	$\lfloor 0$	0	$\cdots$
$\cdots$	0	0	0	0	1	3	8	$\cdots$
$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$

Forms of the External Facets

By studying many examples and a leap of insight, Eisenbud and Schreyer [3] defined for any integer e and $0 \leq \tau \leq n-1$ a pairing $\langle \beta, \gamma \rangle_{e,\tau}$ between diagrams and cohomology tables as the expression

$$\sum_{i<\tau, d\in\mathbb{Z}} (-1)^i \beta_{i,d} \gamma_{\leq i,-d}$$

$$+ \sum_{d\leq e} (-1)^\tau \beta_{\tau,d} \gamma_{\leq \tau,-d} + \sum_{d>e} (-1)^\tau \beta_{\tau,d} \gamma_{\leq \tau-1,-d}$$

$$+ \sum_{d\leq e+1} (-1)^{\tau+1} \beta_{\tau+1,d} \gamma_{\leq \tau,-d} + \sum_{d>e+1} (-1)^{\tau+1} \beta_{\tau+1,d} \gamma_{\leq \tau-1,-d}$$

$$+ \sum_{i>\tau+1, d\in\mathbb{Z}} (-1)^i \beta_{i,d} \gamma_{\leq i-2,-d}$$

Forms of the External Facets

Theorem

For any minimal free resolution $F_\bullet$ of length $\leq c$ and coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{c-1}$ the pairing

$$\langle F_\bullet, \mathcal{F} \rangle_{e,\tau} \geq 0.$$

The positivity of this form on Betti tables of graded modules shows that the positive cone $B(\mathbf{a}, \mathbf{b})$ is contained in the realization of the simplicial fan $\Sigma(\mathbf{a}, \mathbf{b})$.
It is worth mentioning at this point that the first Boij-Söderberg conjecture is equivalent to the inclusion of the fan $\Sigma(\mathbf{a}, \mathbf{b})$ in the cone $B(\mathbf{a}, \mathbf{b})$.
In conclusion, both of the Boij-Söderberg conjectures are equivalent to the following equality.

$$\Sigma(\mathbf{a}, \mathbf{b}) = B(\mathbf{a}, \mathbf{b}).$$

Eisenbud-Schreyer Theory

In their paper [3], Eisenbud and Schreyer also achieved a complete classification of cohomology tables of vector bundles on projective spaces up to a rational multiple.
This runs fairly analogous to the classification of Betti diagrams of Cohen-Macaulay modules up to rational multiple.
In this theory, the role of a pure resolution is played by vector bundles with supernatural cohomology.

Vector Bundles with Supernatural Cohomology

A sheaf $\mathcal{F}$ on $\mathbb{P}^n$ has *supernatural cohomology* if its Hilbert polynomial $\chi(\mathcal{F}(d))$ is of the form

$$P_{\mathcal{F}}(d) = c.\prod_{i=1}^{n}(d - (z_i)),$$

for some constant c and also for each integer d, the cohomology group $\mathrm{H}^i\,\mathcal{F}(d)$ is nonzero only if $z_{i+1} < d < z_i$, when the numbers z_i are put in a decreasing order, $z_1 > z_2 > \cdots > z_n$.
The sequence $z_1 > z_2 > \cdots > z_n$ is called the *root sequence* of the sheaf $\mathcal{F}$.

Analogue of Boij-Söderberg Conjectures for Vector Bundles

In analogue to Boij-Söderberg conjectures, we have the following theorems for vector bundles. (see [3]).

Theorem

Any strictly decreasing sequence of n integers is the root sequence of a supernatural vector bundle on $\mathbb{P}^n$.

and also,

Theorem

The cohomology table of any vector bundle on $\mathbb{P}^n$ has a unique expression as a positive rational linear combination of the supernatural cohomology tables corresponding to a chain of root sequences.

Multiplicity Conjecture

Multiplicity Conjecture

Let h be the height of I. Suppose that S/I is Cohen-Macaulay (and thus h is also the projective dimension of S/I). Then the multiplicity e of S/I satisfies

$$\frac{1}{h!}\prod_{i=1}^{h} m_i \leq e \leq \frac{1}{h!}\prod_{i=1}^{h} M_i.$$

in which $m_i = \min_{j\geq 0} d_{ij}$ is the minimal shifts and $M_i = \max_{j\geq 0} d_{ij}$ is the maximal shifts in a minimal graded free resolution of S/I.

Non-Cohen-Macaulay Case

A few month after Eisenbud and Schreyer put their proofs on arXiv, Boij and Söderberg extended their result to Betti diagrams of all finite modules.
The modifications needed to extend the Boij-Söderberg conjectures (theorems actually) to graded modules in general are not great!
Let $\mathbb{Z}^{\leq n+1}_{deg}$ be the set of increasing sequences of integers $\mathbf{d} = (d_0, \dots, d_s)$ with $s \leq n$ and consider a partial order on this by letting

$$(d_0, \dots, d_s) \geq (e_0, \dots, e_t)$$

if $s \leq t$ and $d_i \geq e_i$ when i ranges from $0, \dots, s$. Note that if we identify the sequence $\mathbf{d}$ with the sequence $(d_0, \dots, d_n)$ where $d_{s+1}, \dots, d_n$ are all equal to $+\infty$, then this is completely natural.

Non-Cohen-Macaulay Case

Theorem

Let $\beta(M)$ be the Betti diagram of a graded S-module M. Then there exists positive rational numbers c_i and a chain of sequences $\mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^p$ in $\mathbb{Z}_{deg}^{\leq n+1}$ such that

$$\beta(M) = c_1\pi(\mathbf{d}^1) + \cdots + c_p\pi(\mathbf{d}^p).$$

End of Story

Thank you.

Mats Boij and Jonas Söderberg.
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