# Minimal free resolution of square free monomial ideals

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# Results via reduced simplicial homology

- Graded betti numbers and edge ideal

- Simplicial Trees

**Hochster's formula**:

$$\beta_{i,j}(I_\Delta) = \sum_{|W|=j, W\subset[n]} dim_{\mathbb{K}} \tilde{H}_{j-i-2}(\Delta_W; \mathbb{K})$$

# Simplicial complexes related to a simple graph

**Edge ideal of** $G = (V, E)$ $\qquad I(G) = \{x_i x_j \mid ij \in E\}$

**Independent complex of** $G = (V, E)$

$ind(G) := \{A \subset V | V \in \mathcal{I}(G)\}$

$$\star \mathrm{I}_{ind(G)} = I(G)$$

**Clique complex of** $G = (V, E)$ $\quad \Delta_c(G) = \{A \subset V | A \in \mathcal{C}(G)\}$

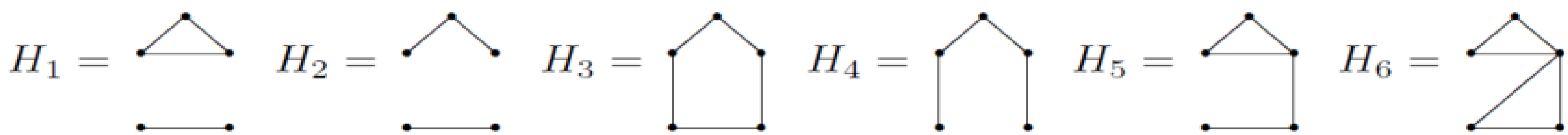
$$\star \mathrm{I}_{\Delta_c(G^c)} = I(G)$$

## Hochster's formula in terms of Garphs

$$\beta_{i,j}(I(G)) = \sum_{S \subset V, |S|=j} dim_{\mathbb{K}} \tilde{H}_{j-i-2}(\Delta_C(G_S^c), \mathbb{K})$$

## ⋆**Computing** $\beta_{2,5}(I(G))$

By Hochster's formula we must have $dim_{\mathbb{K}}\tilde{H}_1(\Delta_C(G_S^c),\mathbb{K}) \neq 0$ for subgraphs of $G$ with 5 vertices and the following subgaraphs are the only subgraphs of $G$ that satisfies these conditions.




Clique complexe of complement of each of these subgraphs is:

$$\beta_{2,5}(I(G)) = 2h_1(G) + h_2 + \cdots + h_6(G)$$

# Graded betti numbers and edge ideal

$$I(G) = (x_1x_2, x_1x_3, x_1x_7, x_1x_8, x_1x_{10}, x_2x_3, x_2x_8,$$
$$x_2x_9, x_2x_{12}, x_3x_7, x_3x_9, x_3x_{11}, x_4x_5, x_4x_6,$$
$$x_4x_8, x_4x_{11}, x_5x_6, x_5x_7, x_5x_{12}, x_6x_9, x_6x_{10},$$
$$x_7x_{10}, x_7x_{11}, x_7x_{12}, x_8x_{10}, x_8x_{11}, x_8x_{12},$$
$$x_9x_{10}, x_9x_{11}, x_9x_{12}, x_{10}x_{11}, x_{10}x_{12}, x_{11}x_{12})$$

$$Char\mathbb{K} = 2 \quad \Longrightarrow \quad \beta_{8,11}(I(G)) = 1$$

$$Char\mathbb{K} = 0 \quad \Longrightarrow \quad \beta_{8,11}(I(G)) = 0$$

# Graded betti numbers and edge ideal

Theorem: If $\beta_{i,j}(I(G)) \neq 0 \implies$ i+2 $\leq j \leq 2(i+1)$

Theorem: $\beta_{i,i+2}(I(G)) = \sum_{S\subset V,|S|=i+2} (\sharp\, comp(G_S^c) - 1)$

$$\beta_{i,2(i+1)}(\mathcal{I}(G)) = \left| \left\{ H \;\middle|\; \begin{array}{l} H \ \ is \ a \ induced \ subgraph \ of \ G \\ consisting \ of \ i+1 \ disjoint \ edges \end{array} \right\} \right|$$

**Linear strand of** $M$: $l = max\{i \mid \beta_{i,i+d}(M) \neq 0\}$

Theorem: $G$ has no minimal 4-cycles

$$\beta_{i,i+2}(I(G)) = \sum_{v\in V} \binom{degv}{i+1} - \mathcal{K}_{i+2}(G)$$

# Graded betti numbers and edge ideal

Corollary: $G$ is a forest. Then

$$\beta_{0,2}(I(G)) = |E_G| \quad \text{and} \quad \beta_{i,i+2}(I(G)) = \sum_{v \in V} \binom{degv}{i+1}$$

Example:

$$\beta_{1,j}(I(G)) = \begin{cases} \sum_{v \in V} \binom{degv}{2} - \mathcal{K}_3(G) & \text{if } j = 3 \\ \mathcal{C}_4(G^c) & \text{if } j = 4 \\ 0 & \text{if } j \neq 3, 4 \end{cases}$$

Thm: $\forall G;$ $\quad \beta_{2,4}(I(G)) = \sum_{v \in V} \binom{degv}{3} - \mathcal{K}_4(G) + \mathcal{K}_{2,2}(G)$

$$\beta_{3,5}(I(G)) = \sum_{v \in V} \binom{degv}{4} - \mathcal{K}_5(G) + \mathcal{K}_{2,3}(G) + W_4(G) + d(G)$$

# Graded betti numbers and edge ideal

Example: $\beta_{2,j}(I(G))$

$$\begin{cases} \beta_{2,4}(I(G)) = \sum_{v\in V}\binom{degv}{3} - \mathcal{K}_4(G) + \mathcal{K}_{2,2}(G) \\ \beta_{2,5}(I(G)) = 2h_1 + h_2 + \cdots + h_6 \\ \beta_{2,6}(I(G)) = \left|\left\{ H \;\middle|\; \begin{array}{l} H \ is\ a\ induced\ subgraph\ of\ G \\ consisting\ of\ 3\ \ disjoint\ edges \end{array} \right\}\right| \end{cases}$$

Theorem:

$$\beta_{i,i+2}(I(G)) \geq \sum_{v\in V}\binom{degv}{i+1} - \mathcal{K}_{i+2}(G) + \mathcal{K}_{2,i}(G) + \cdots + \mathcal{K}_{\lfloor\frac{i+2}{2}\rfloor,\lceil\frac{i+2}{2}\rceil}(G)$$

# Graded betti numbers and edge ideal

Theorem:

$$\beta_{i,i+2}(I(G)) \geq \sum_{v \in V} \binom{deg v}{i+1} - \mathcal{K}_{i+2} + \sum_{\substack{i_1, i_2 \geq 2 \\ i_1+i_2=i+2}} \mathcal{K}_{i_1,i_2}$$

$$+ \; 2 \sum_{\substack{i_1,i_2,i_3 \geq 2 \\ i_1+i_2+i_3=i+2}} \mathcal{K}_{i_1,i_2,i_3} + \sum_{\substack{i_1,i_2 \geq 2 \\ i_1+i_2=i+1}} \mathcal{K}_{i_1,i_2,1}$$

$$\vdots$$

$$+ \; (t-1) \sum_{\substack{i_1,\ldots,i_t \geq 2 \\ i_1+\cdots+i_t=i+2}} \mathcal{K}_{i_1,\ldots,i_t} + \cdots + \sum_{\substack{i_1,i_2 \geq 2 \\ i_1+i_2=i-t}} \mathcal{K}_{i_1,i_2,1,\ldots,1}$$

$$\vdots$$

Graded betti numbers and edge ideal

Theorem: (Fröberg, 1990) I(G) has linear resolution $\iff$ $G^c$ is chordal.

$$\star\beta_{i,i+2}(I(K_{a,b})) = \binom{a+b}{i+2} - \binom{a}{i+2} - \binom{b}{i+2}$$

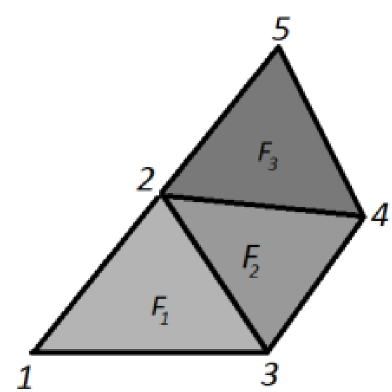
$$\star\beta_{i,i+2}(I(K_{d_1,\cdots,d_t})) = \sum_{l=2}^{t}(l-1)\sum_{\alpha_1+\cdots+\alpha_l=i+2 \;\; j_1<\cdots<j_l \;\; \alpha_1,\cdots,\alpha_l\geq 1}\binom{n_{j_1}}{\alpha_1}\cdots\binom{n_{j_l}}{\alpha_l}$$

# Simplicial Trees

Definition: A facet $F \in \mathcal{F}(\Delta)$ is said to be a **leaf** of $\Delta$ if either $F$ is the only facet of $\Delta$, or there exists a facet $G \in \mathcal{F}(\Delta)$ with $G \neq F$, such that $H \cap F \subset G \cap F$ for all $H \in \mathcal{F}(\Delta)$ with $H \neq F$. $G$ called a **branch** of $F$.

Definition: $\Delta$ is called a **simplicial tree** if every subcomplex of $\Delta$ has a leaf.

Exapmle:



# Simplicial Trees

Definition: A **quasi tree** is a simplicial complex such that there exists a labelling $F_1, \ldots, F_q$ of the facets of $\Delta$, called a **leaf order**, such that for each $1 < i \leq q$ the facet $F_i$ is a leaf of the subcomplex $< F_1, \ldots, F_i >$.

Theorem: If $\Delta$ be a simplicial quasi-tree $\Longrightarrow I_\Delta$ has a linear resolution.

Theorem: If $\Delta$ be a simplicial quasi-tree on $[n]$. Then

$$\beta_{i,j}(I_\Delta) = \begin{cases} \sum_{A \subseteq [n], |A| = j} \sharp(comp\Delta_A - 1) & j = i + 2 \\ 0 & j \neq i + 2 \end{cases}$$

# Splittable monomial ideals

- Splittable monomial ideals
- Splitting edges
- Splitting vertices
- Splitting facets

# Splittable monomial ideals

Definition: I: monomial ideal is splittable when we have: $I = J + K$, such that

(1) $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$

(2) there is a splitting function $\mathcal{G}(J \cap k) \longrightarrow \mathcal{G}(J) \times \mathcal{G}(K)$

$$w \longmapsto (\varphi(w), \psi(w))$$

satisfying

(a) $\forall\ w \in \mathcal{G}(J \cap K)$, $w = lcm(\varphi(w), \psi(w))$.

(b) for every subset $S \subset \mathcal{G}(J \cap K)$,
both $lcm\varphi(S)$ and $lcm\psi(S)$ strictly divide $lcm(S)$

Then $I = J + K$ is a splitting of $I$.

# Splittable monomial ideals

Theorem(Eliahou, Kervaire, 1990 - Fatabbi, 2001): If $I = J + K$ be a splitting of $I$, then

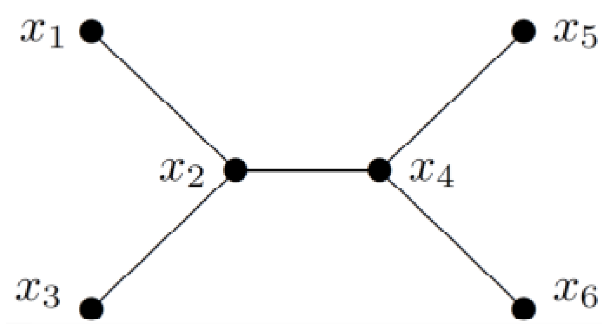
$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$

# Splitting edges

Theorem: An edge $e = uv$ is a splitting edge of $G \iff$

$$N(u) \subseteq (N(v) \cup \{v\}) \text{ or } N(v) \subseteq (N(u) \cup \{u\}).$$

Example:




$x_1 x_2$ is a splitting edge of $G$ but $x_2 x_4$ is not.

# Splitting edges

Theorem: $e = uv$: splitting edge of $G$, and set $H = G \setminus N(u) \cup N(v)$ and $n = \left|N(u) \cup N(v)\right| - 2$. Then $\forall$ $i \geq 1$ and $j \geq 0$

$$\beta_{i,j}(I(G)) = \beta_{i,j}(I(G \setminus e)) + \sum_{l=0}^{i} \binom{n}{l} \beta_{i-1-l,j-2-l}(I(H))$$

Corollary: $e = uv$ be a leaf of a forest $G$, if $degv = n$ and $N(v) = \{u, v_1, \cdots, v_{n-1}\}$. Then $\forall$ $i \geq 1$ and $j \geq 0$

$$\beta_{i,j}(I(G)) = \beta_{i,j}(I(T)) + \sum_{l=0}^{i} \binom{n-1}{l} \beta_{i-1-l,j-2-l}(I(H))$$

Where $T = G \setminus e = G \setminus \{u\}$ and $H = G \setminus \{u, v, v_1, \cdots, v_{n-1}\}$.

# Splitting edges

Corollary: $e = uv$ be asplitting edge of $G$, and

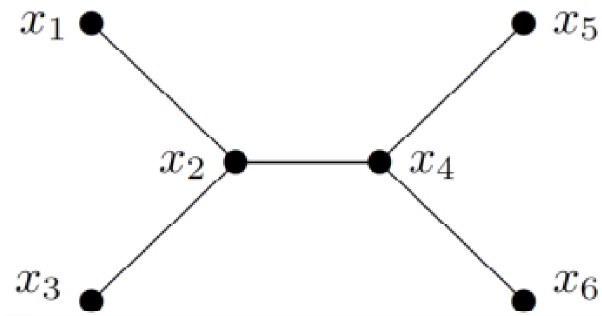
$H = G \setminus (N(u) \cup N(v))$. Let $n = |N(u) \cup N(v)| - 2$. Then

1 $Reg(I(G)) = max\{2, Reg(I(G \setminus e)), Reg(I(H)) + 1\}$

2 $Pd(I(G)) = max\{Pd(I(G \setminus e)), Pd(I(H)) + n + 1\}$

# Splitting edges

Example:




Consider the minimal free resolution of $I(G)$:

$$0 \longrightarrow S^2(-4) \longrightarrow S^6(-3) \longrightarrow S^5(-2) \longrightarrow I(G) \longrightarrow 0$$

And the minimal free resolution of $I(G \setminus e))$ is:

$$0 \to S(-6) \to S^4(-5) \to S^2(-3) \oplus S^4(-4) \to S^4(-4) \to I(G \setminus e) \to 0$$

Then we see that: $Pd(I(G \setminus e)) = 3 > 2 = Pd(I(G))$ and $Reg(I(G \setminus e) = 3 > 2 = Reg(I(G))$.

# Splitting vertices

Definition: A vertex $v \in V_G$ is a **splitting vertex** if $degv = d > 0$ and $G \smallsetminus \{v\}$ is not the graph of isolated vertices.

Theorem: $v$ is a splitting vertex of $G$ and $N(v) = \{v_1, \cdots, v_d\}$. If $J = (vv_1, \cdots, vv_d)$ and $K = I(G \smallsetminus \{v\})$. Then $I(G) = J + K$ is a splitting of $I(G)$.

Corollary: Let $G_i := G \smallsetminus (N(v) \cup N(v_i))$ $i = 1, \cdots, d$ and $G_{(v)} := G_{\{v_1,\ldots,v_d\}} \cup \{e \in E \mid e \text{ incident to } v_1, \ldots, v_d \text{ but not } v\}$. Then

$$J \cap K = vI(G_{(v)}) + vv_1 I(G_1) + \cdots + vv_d I(G_d)$$

## Splitting vertices

Corollary: If $v \in V$ be a splitting vertex of degree $d$ in $G$. Then

$$(i)\ \ Reg(I(G)) \geq max\{2, Reg(I(G \setminus \{v\}))\}$$

$$(ii)\ \ Pd(I(G)) \geq max\{d-1, Pd(I(G \setminus \{v\}))\}$$

Theorem: Suppose that $v$ is a splitting vertex of $G$ with $N(v) = \{v_1, \cdots, v_d\}$. Then

$$\beta_{i,j}(I(G)) = \beta_{i,j}(I(K_{1,d})) + \beta_{i,j}(I(G \setminus \{v\})) + \beta_{i-1,j}(L)$$

Where $L = vI(G_{(v)}) + vv_1 I(G_1) + \cdots + vv_d I(G_d)$

# Splitting vertices

Theorem: $v$ is a splitting vertex of $G$. Then $\forall\ i \geq 0$,

$$\beta_{i,i+2}(I(G)) = \beta_{i,i+2}(I(K_{1,d})) + \beta_{i,i+2}(I(G \setminus \{v\})) + \beta_{i-1,i+1}(I(G_{(v)})).$$

# Splitting facets

Definition: $\Delta$ is a simplicial complex and $F$ is a facet of $\Delta$. The **connected component** of $F$ in $\Delta$, $conn_{\Delta}(F)$ is the connected component of $\Delta$ containing $F$.

If $conn_{\Delta}(F) \smallsetminus F =< G_1, \cdots, G_p >$, then **reduced connected component** of $F$ in $\Delta$, $\overline{conn}_{\Delta}(F)$, is the simplicial complex whose facets are given by $G_1 \smallsetminus F, \cdots, G_p \smallsetminus F$.

where if there exist $G_i$ and $G_j$ such that; $0 \neq G_i \smallsetminus F \subseteq G_j \smallsetminus F$, then we shall disregard the bigger facet $G_j \smallsetminus F$ in $\overline{conn}_{\Delta}(F)$.

# Splitting facets

Example: Consider the simplicial complex $\Delta$ with

$\mathcal{F}(\Delta) = \Big\{\{1,2,3\},\{1,3,4\},\{1,4,5\},\{1,5,6\}\Big\}$ and let $F = \{1,5,6\}$. Then

$$conn_\Delta(F) = \Delta \text{ and } \overline{conn}_\Delta(F) = \langle\{2,3\},\{4\}\rangle.$$

Consider $\{1,3,4\} \smallsetminus F = \{3,4\}$ contains $\{4\} = \{1,4,5\} \smallsetminus F$ so we disregard the bigger set in obtaining $\overline{conn}_\Delta(F)$.

# Splitting facets

- $F$ is a facet of a simplicial complex $\Delta$, $\Delta' = \Delta \setminus F$: the simplicial complex obtained by removing $F$ from the facet set of $\Delta$. Let $J = (x_F)$ , $K = I(\Delta')$.

Note that $\mathcal{G}(I(\Delta))$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$.

Theorem: $F$ splitting facet of $\Delta$, then $\forall\ i, j \geq 0$

$$\beta_{i,j}(I(\Delta)) = \beta_{i,j}(I(\Delta')) + \sum_{l_1=0}^{i} \sum_{l_2=0}^{j-|F|} \beta_{l_1-1,l_2}(I(\overline{conn}_\Delta(F)))\beta_{i-l_1-1,j-|F|-l_2}(I(\Omega))$$

Where $\Omega = \Delta \setminus conn_\Delta(F)$

# Splitting facets

Theorem: If $F$ is a leaf of $\Delta$, then $F$ is a splitting facet of $\Delta$.

Theorem: Let $F$ be a facet of a forest $\Delta$. Then $\overline{conn}_{\Delta}(F)$ is a forest.

Theorem: Let F be a leaf of a simplicial forest $\Delta$, and let $\Delta' = \Delta \setminus F$ and $\Omega = \Delta \setminus conn_{\Delta}(F)$. Then $\Delta'$, $\Omega$ , and $\overline{conn}_{\Delta}(F)$ are also simplicial forests and for all $i \geq 1$ and $j \geq 0$

$$\beta_{i,j}(I(\Delta)) = \beta_{i,j}(I(\Delta'))+\sum_{l_1=0}^{i} \sum_{l_2=0}^{j-|F|} \beta_{l_1-1,l_2}(I(\overline{conn}\Delta(F)))\beta_{i-l_1-1,j-|F|-l_2}(I(\Omega))$$

# Splitting facets

For a face $G$ of dimension $d-2$ of a pure $(d-1)$–dimensional simplicial complex $\Delta$ we define:

$$deg_{\Delta}(G) := \left|\left\{F \in \mathcal{F}(\Delta) \mid G \subseteq F\right\}\right|$$

$\mathcal{A}(\Delta)$: the set of $(d-2)$–dimensional faces of $\Delta$.

Theorem: Let $\Delta$ be a pure $(d-1)$–dimensional forest. Then

$$\beta_{i,i+d}(I(\Delta)) = \begin{cases} |\mathcal{F}(\Delta)| & \text{if } i = 0 \\ \sum_{G \in \mathcal{A}(\Delta)} \binom{deg_{\Delta}(G)}{i+1} & \text{if } i \geq 1 \end{cases}$$

Thank you