# Homological invariants of edge ideals

Somayeh Moradi

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# Stanley-Reisner correspondence

*Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over field $k$ and $I$ a squarefree monomial ideal. The Stanley-Reisner simplicial complex associated to $I$ on the vertex set $V = \{x_i : x_i \notin I\}$ is defined as:*

$$\Delta_I = \{\{x_{i_1}, \ldots, x_{i_k}\} : i_1 < \cdots < i_k,\ x_{i_1} \cdots x_{i_k} \notin I\}$$

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Let $\mathcal{X}$ be a finite set and $\mathcal{E} = \{E_1, \ldots, E_s\}$ a finite collection of non empty subsets of $\mathcal{X}$. The pair $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ is called a hypergraph. The elements of $\mathcal{X}$ and $\mathcal{E}$, respectively, are called the vertices and the edges of the hypergraph.

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For a hypergraph $\mathcal{H}$ with vertex set $\{x_1, \ldots, x_n\}$ the edge ideal of $\mathcal{H}$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ is defined as:

$$I(\mathcal{H}) = (x^E : \quad E \in \mathcal{E}(\mathcal{H})), \qquad \textit{where } x^E = \prod_{x_i \in E} x_i$$

For a hypergraph $\mathcal{H}$, the independence complex of $\mathcal{H}$ is defined as:

$$\Delta_{\mathcal{H}} = \{F \subseteq \mathcal{X}(\mathcal{H}) : \ E_i \not\subseteq F, \ \forall E_i \in \mathcal{E}(\mathcal{H})\}$$

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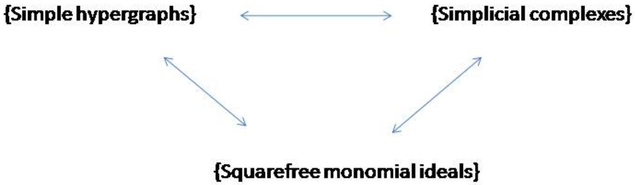
For a simplicial complex $\Delta$, let $\mathcal{H}$ be a hypergraph whose edge set is the minimal non-faces of $\Delta$. Then $\Delta = \Delta_{\mathcal{H}}$ and so $I_{\Delta} = I(\mathcal{H})$.



*A subset $C \subseteq X$ is called a vertex cover of $\mathcal{H}$ if it intersects all the edges of $\mathcal{H}$.*

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# Polarization

*Let* $I \subseteq S = k[x_1, \ldots, x_n]$*, with* $G(I) = \{u_1, \ldots, u_m\}$*, where* $u_i = \prod_{j=1}^{n} x_j^{a_{ij}}$*. For each* $j$*, let* $a_j = \max\{a_{ij} : 1 \leq i \leq m\}$ *and* $T$ *be the polynomial ring over* $k$ *in the variables*

$$x_{11}, \ldots, x_{1a_1}, \ldots, x_{21}, \ldots, x_{2a_2}, \ldots, x_{n1}, \ldots, x_{na_n}$$

*The ideal* $J \subset T$ *with generating set* $G(J) = \{v_1, \ldots, v_m\}$*, where*

$$v_i = \prod_{j=1}^{n} \prod_{k=1}^{a_{ij}} x_{jk}$$

*is called the polarization of* $I$*.*

# Polarization

*Let $I \subseteq S$ be a monomial ideal and $J \subseteq T$ its polarization. Then*

- $\beta_{i,j}(I) = \beta_{i,j}(J)$ *for all* $i$ *and* $j$.
- $H_{S/I}(t) = (1-t)^{\delta} H_{T/J}(t)$, *where* $\delta = \dim(T) - \dim(S)$.
- $ht(I) = ht(J)$.
- $\mathrm{pd}(S/I) = \mathrm{pd}(T/J)$ *and* $\mathrm{reg}(S/I) = \mathrm{reg}(T/J)$.
- $S/I$ *is Cohen-Macaulay if and only if* $T/J$ *is Cohen-Macaulay.*

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Let $\Delta$ be a simplicial complex and $F \in \Delta$. Then

$$\mathrm{del}_{\Delta}(F) = \{G \in \Delta : \ G \cap F = \emptyset\}$$

For a face $F \in \Delta$, link of $F$ is defined as:

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- The only facet of $\Delta$ is $\{x_1, \ldots, x_n\}$, or $\Delta = \emptyset$.
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The vertex $x$ is called a shedding vertex.

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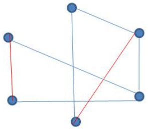
# Regularity of edge ideal of graphs

Let $G$ be a simple graph. Two edges $uv$ and $xy$ are called 3-disjoint if the induced subgraph of $G$ on $\{x, y, u, v\}$ has only two edges.

The maximum number of pairwise 3-disjoint edges in $G$ is denoted by $c(G)$.



# Regularity of edge ideal of graphs

[Zheng (2003)] For a tree graph $G$, $\mathrm{reg}(\mathrm{R}/\mathrm{I}(\mathrm{G})) = \mathrm{c}(\mathrm{G})$.

A graph $G$ is called chordal if any cycle of length $n \geq 4$ has a chord.

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# Regularity of edge ideal of graphs

[Van Tuyl (2009)] Let $G$ be a sequentially Cohen-Macaulay bipartite graph. Then $\mathrm{reg}(\mathrm{R}/\mathrm{I}(\mathrm{G})) = \mathrm{c}(\mathrm{G})$.

[Kummini], [Mohammadi, Moradi (2010)] Let $G$ be an unmixed bipartite graph. Then $\mathrm{reg}(\mathrm{R}/\mathrm{I}(\mathrm{G})) = \mathrm{c}(\mathrm{G})$.

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# Regularity of edge ideal of graphs

[Khosh-Ahang, Moradi (2012)] Let $G$ be a $C_5$-free vertex decomposable graph. Then $\mathrm{reg}(\mathrm{R}/\mathrm{I}(\mathrm{G})) = \mathrm{c}(\mathrm{G})$.

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# Regularity of edge ideal of graphs

Question 1: Does the equality $\mathrm{reg}(R/I(G)) = c(G)$ hold for any vertex decomposable graph?

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# Some bounds for the regularity of edge ideals

[Ha, Van Tuyl (2007)] For a graph $G$, $\mathrm{reg}(\mathrm{R}/\mathrm{I}(\mathrm{G})) \leq \mathrm{a}(\mathrm{G})$, where $a(G)$ is the matching number of $G$.

[Kiani, Moradi (2010)] Let $G$ be a shellable graph. Then $\mathrm{reg}(\mathrm{R}/\mathrm{I}(\mathrm{G})) \leq \mathrm{n}(\mathrm{G})$.

$$n(G) = \max\{|V(H)| : H \in \mathcal{S}(G), H \cup W(H) \in \mathcal{S}(G)\}$$

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# Some bounds for the regularity of edge ideals

[Kiani, Moradi (2010)] Let $G$ be a vertex decomposable graph. Then $\mathrm{reg}(R/I(G)) \leq \min\{a'(G), n(G)\}$.

$a'(G)$ : the maximum number of vertex disjoint paths of length at most two in $G$ such that paths of lengths one are pairwise 3-disjoint in $G$.

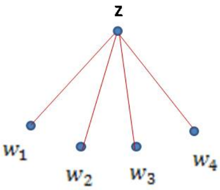
# Some bounds for the regularity of edge ideals

[Kiani, Moradi (2010)]

Let $G$ be a graph such that $G^c$ has no triangle, then $\mathrm{reg}(\mathrm{R}/\mathrm{I}(\mathrm{G})) \leq 2$. In addition if $G^c$ is not chordal, then $\mathrm{reg}(\mathrm{R}/\mathrm{I}(\mathrm{G})) = 2$.

# Projective dimension of edge ideal of graphs

The graph $B$ with vertex set $V(B) = \{z, w_1, \ldots, w_d\}$ and edge set $E(B) = \{\{z, w_i\} \mid 1 \leq i \leq d\}$ is called a bouquet. The vertex $z$ is called the root of $B$, the vertices $w_i$ flowers of $B$ and the edges $\{z, w_i\}$ the stems of $B$.



# Projective dimension of edge ideal of graphs

A subgraph of $G$ which is a bouquet is called a bouquet of $G$.

A set of bouquets $\mathcal{B} = \{B_1, \ldots, B_n\}$ is called strongly disjoint in $G$ if

(i) $V(B_i) \cap V(B_j) = \emptyset$ for all $i \neq j$,

(ii) we can choose a stem $e_i$ from each bouquet $B_i \in \mathcal{B}$ such that $\{e_1, \ldots, e_n\}$ is pairwise 3-disjoint in $G$.

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Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a set of bouquets of $G$.

$$F(\mathcal{B}) = \{w \in V(G) \mid w \text{ is a flower of some bouquet in } \mathcal{B}\}$$

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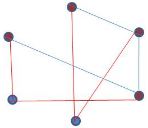
In the following graph $d(G) = 4$.



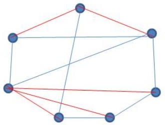
# Projective dimension of edge ideal of graphs

A set of bouquets $\mathcal{B} = \{B_1, \ldots, B_n\}$ is called semi strongly disjoint in $G$ if

(i) $V(B_i) \cap V(B_j) = \emptyset$ for all $i \neq j$, and

(ii) $R(\mathcal{B})$ is an independent set of $G$.

$d'_G := \max\{|F(\mathcal{B})| \ : \ \mathcal{B}$ is a semi-strongly disjoint set of bouquets of $G\}$.



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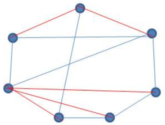

d’(G)=5

**Kimura (2011)**

(i) Suppose that $G$ is a chordal graph. Then $\beta_{i,i+j}(R/I(G)) \neq 0$ if and only if there exists a subset $W$ of $V$ such that the induced subgraph $G_W$ contains a strongly disjoint set of bouquets of type $(i,j)$.

(ii) When $G$ is a forest, the graded Betti number $\beta_{i,i+j}(R/I(G))$ coincides with the number of subsets $W$ of $V$ with the same condition as in (i).

# Projective dimension of edge ideal of graphs

For a hypergraph $\mathcal{H}$, big height of $\mathcal{H}$ is equal to:

$$\mathrm{bight}(I(\mathcal{H})) = \max\{|C_i| : \ C_i \text{ is a minimal vertex cover of } \mathcal{H}\}$$

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[Khosh-Ahang, Moradi (2012)] For a $C_5$-free vertex decomposable graph $G$, $\mathrm{pd}(R/I(G)) = \mathrm{d}'(G) = \mathrm{bight}(I(G))$.

A $d$-tree is a chordal graph defined inductively as follows:
($i$) $K_{d+1}$ is a $d$-tree.
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**Roth, Van Tuyl (2006)**

Let $G$ be a graph with no minimal cycle of length 4. Let $k_{i+2}(G)$ denote the number of $(i+2)$-cliques in $G$. Then, for any $i \geq 0$,

$$\beta_{i,i+2}(I(G)) = \sum_{u \in V(G)} \binom{\deg(u)}{i+1} - k_{i+2}(G).$$

**Roth, Van Tuyl (2006)**

Let $G$ be a forest. Then, for any $i \geq 1$,

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**Katzman (2004)**

For any $i \geq 1$, $\beta_{i,2(i+1)}(I(G))$ is equal to the number of induced subgraphs of $G$ consisting of exactly $i+1$, 3-disjoint edges.

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# Alexander dual ideal

*For any simplicial complex $\Delta$ with vertex set $X$, the Alexander dual simplicial complex $\Delta^\vee$ to $\Delta$ is defined as follows:*

$$\Delta^\vee = \{F \subseteq X; X \setminus F \notin \Delta\}$$

*For a squarefree monomial ideal $I = (x_{1,1}x_{1,2}\cdots x_{1,k_1}, \ldots, x_{n,1}x_{n,2}\cdots x_{n,k_n})$, Alexander dual ideal of $I$ is defined as:*

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*For a simplicial complex* $\Delta$, $\mathrm{pd}(\mathrm{I}_\Delta) = \mathrm{reg}(\mathrm{R}/\mathrm{I}_{\Delta^\vee})$.

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*Let* $\Delta$ *be a simplicial complex. Then*

$$k[\Delta] \text{ is Cohen} - \text{Macaulay} \;\Leftrightarrow\; I_{\Delta^\vee} \text{ has linear resolution}$$

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*A monomial ideal $I = (f_1, \ldots, f_m)$ has linear quotients, if there exists an order $f_1 < \cdots < f_m$ on the generators of $I$ such that the colon ideal $(f_1, \ldots, f_{i-1}) : f_i$ is generated by a subset of variables for all $2 \leq i \leq m$.*

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*For a monomial ideal $I \subset R$ and $d \geq 1$, let $I_{\langle d \rangle}$ be the ideal generated by all monomials of degree $d$ in $I$. The ideal $I$ is called componentwise linear if for each $d$, $I_{\langle d \rangle}$ has a linear resolution.*

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# Vertex splittable ideal

**Khosh-Ahang, Moradi (2013)**

*A monomial ideal $I$ of $R$ is called vertex splittable if it can be obtained by the following recursive procedure:*

(i) *If $u$ is a monomial and $I = (u)$, then $I$ is a vertex splittable ideal.*

(ii) *If there is a variable $x \in X$ and vertex splittable ideals $I_1$ and $I_2$ of $k[X \setminus \{x\}]$ so that $I = xI_1 + I_2$ and $I_2 \subseteq I_1$, then $I$ is a vertex splittable ideal.*

*With the above notations if $I = xI_1 + I_2$ is a vertex splittable ideal, then $xI_1 + I_2$ is called a vertex splitting for $I$.*

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*A simplicial complex $\Delta$ is vertex decomposable if and only if $I_{\Delta^\vee}$ is a vertex splittable ideal.*

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# Betti splitting

## Definition

*Let $I$, $J$ and $K$ be monomial ideals such that $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$. Then $I = J + K$ is a Betti splitting if*

$$\beta_{i,j}(I) = \beta_{i.j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K)$$

*for all $i \in \mathbb{N}$ and (multi)degrees $j$.*

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For a vertex splittable ideal $I$ with vertex splitting $I = xI_1 + I_2$, the graded Betti numbers of $I$ can be computed by the following recursive formula

$$\beta_{i,j}(I) = \beta_{i,j-1}(I_1) + \beta_{i,j}(I_2) + \beta_{i-1,j-1}(I_2).$$

Let $\Delta$ be a vertex decomposable simplicial complex, $x$ a shedding vertex of $\Delta$, $\Delta_1 = \mathrm{del}_\Delta(x)$ and $\Delta_2 = \mathrm{lk}_\Delta(x)$. Then

$$\beta_{i,j}(I_{\Delta^\vee}) = \beta_{i,j-1}(I_{\Delta_1^\vee}) + \beta_{i,j}(I_{\Delta_2^\vee}) + \beta_{i-1,j-1}(I_{\Delta_2^\vee}).$$

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## Corollary

Let $\Delta$ be a vertex decomposable simplicial complex, $x$ a shedding vertex of $\Delta$ and $\Delta_1 = \mathrm{del}_\Delta(x)$ and $\Delta_2 = \mathrm{lk}_\Delta(x)$. Then

$$\mathrm{pd}(R/I_\Delta) = \max\{\mathrm{pd}(R/I_{\Delta_1}) + 1, \mathrm{pd}(R/I_{\Delta_2})\},$$

$$\mathrm{reg}(R/I_\Delta) = \max\{\mathrm{reg}(R/I_{\Delta_1}), \mathrm{reg}(R/I_{\Delta_2}) + 1\}.$$

# Vertex cover ideal of a vertex decomposable graph

*Let $G$ be a vertex decomposable simplicial complex, $v \in V(G)$ be a shedding vertex of $G$, $G' = G \setminus \{v\}$, $G'' = G \setminus N_G[v]$ and* $\deg_G(v) = t$. *Then*

$$\beta_{i,j}(I(G)^\vee) = \beta_{i,j-1}(I(G')^\vee) + \beta_{i,j-t}(I(G'')^\vee) + \beta_{i-1,j-t-1}(I(G'')^\vee).$$

**Francisco, Ha, Van Tuyl (2009)**

*Let $G$ be a Cohen-Macaulay bipartite graph, $x, y \in V(G)$ be adjacent vertices with* $\deg_G(x) = 1$ *such that $G' = G \setminus N_G[x]$ and $G'' = G \setminus N_G[y]$ are Cohen-Macaulay and* $\deg_G(y) = t$. *Then*

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# splitting of edge ideals

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*Let $e = uv$ be an edge of the graph $G$. $I(G) = (uv) + I(G \setminus e)$ is a splitting if and only if $N(u) \subseteq N[v]$ or $N(v) \subseteq N[u]$.*

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*For a graph $G$, the edge ideal $I(G)$ has linear resolution if and only if $G^c$ is a chordal graph.*

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A graph $G$ is called perfect if for every induced subgraph $G_S$, with $S \subseteq V(G)$, we have $\chi(G_S) = \omega(G_S)$.

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For an integer $t \geq 1$, the partial t-cover ideal of G is the monomial ideal

$$J_t(G) = \bigcap_{x \in V(G)} \Big( \bigcap_{\{x_{i_1},\ldots,x_{i_t}\} \subseteq N(x)} (x, x_{i_1}, \ldots, x_{i_t}) \Big).$$

**Bhat, Biermann, Van Tuyl (2013)**

Let G be a tree. Then for any integer $t \geq 1$, the partial t-cover ideal $J_t(G)$ satisfies the persistence property.

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Open question: Do all square-free monomial ideals have the persistence property?

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Thanks!