Path ideals of graphs

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Stanley-Reisner rings

Let $R = k[x_1, \ldots, x_n]$, where k is a field.

For a simplicial complex Δ with vertex set $\{x_1, \ldots, x_n\}$, the Stanley-Reisner ideal of Δ is defined as:

$$I_\Delta = (\prod_{x \in F} x : F \notin \Delta).$$

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Facet ideals

The facet ideal of Δ is

$$I(\Delta) = (\prod_{x \in F} x : F \textit{ is a facet of } \Delta).$$

Hypergraphs

Let V be a finite set and $E = \{e_1, \ldots, e_m\}$ a finite collection of distinct non-empty subsets of V. Then the pair $H = (V, E)$ is called a hypergraph.

Hypergraphs

A hypergraph H is said to be simple, if there is no containment between its edges.

A hypergraph H is said to be d-uniform if $|e_i| = d$, for every edge $e_i \in E$.

Edge Ideals

The edge ideal of a hypergraph H is defined as

$$I(H) = (\prod_{x \in e} x \ : \ e \in H).$$

Edge ideals

Let $G = (V, E)$ be a finite simple graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E.
Associated to G is a monomial ideal

$$I(G) = (x_i x_j : \{v_i, v_j\} \in E),$$

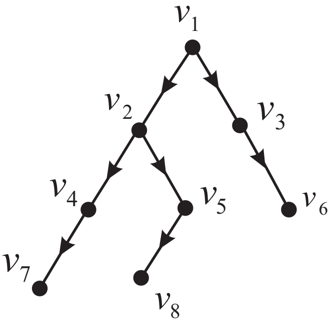
in $k[x_1, \ldots, x_n]$, called the edge ideal of G.

Let $G = (V, E)$ be a directed graph with vertex set $V = \{v_1, \dots, v_n\}$ and edge set E. Fix an integer $2 \leq t \leq n$. Associated to G is a monomial ideal

$$I_t(G) = (x_{i_1} \cdots x_{i_t} : v_{i_1}, \dots, v_{i_t} \text{ is a path of length t in } G),$$

called the path ideal of G.

Path ideals



$$I_3(G) = (x_1x_3x_6, x_1x_2x_4, x_2x_4x_7, x_1x_2x_5, x_2x_5x_8).$$

Path ideals

- (Conca - De Negri, 1999) The Rees algebra of the path ideal of directed trees
- (Brumatti - da Silva, 2001) The Rees algebra and symmetric algebra of the path ideal of cycles
- (Restuccia - Villarreal, 2001) The path ideal of complete bipartite graphs

Path ideals of cycles

It is known that

$$I(C_n) \textit{ is unmixed} \iff n = 3,\ 4,\ 5 \textit{ or } 7.$$

Path ideals of cycles

Kiani - me - Terai

Let $t \geq 3$. Then $I_t(C_n)$ is unmixed if and only if $n = 2t + 1$ or $t \leq n \leq \lfloor 3t/2 \rfloor + 1$.

A graded R-module M is called sequentially Cohen-Macaulay (over k) if there exists a finite filtration of graded R-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1})$$

Path ideals of cycles

Francisco - Van Tuyl

Let $n \geq 3$. Then the following are equivalent:

(i) $R/I(C_n)$ is Cohen-Macaulay.

(ii) $R/I(C_n)$ is sequentially Cohen-Macaulay.

(iii) $n = 3$ or 5.

Kiani - me - Terai

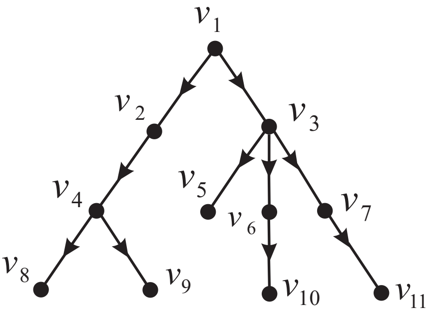
Let $n \geq t \geq 2$. Then the following are equivalent:

(i) $R/I_t(C_n)$ is Cohen-Macaulay.

(ii) $R/I_t(C_n)$ is sequentially Cohen-Macaulay.

(iii) $n = t$ or $t+1$ or $2t+1$.

Path ideals of trees



Path ideals of trees

Francisco -Van Tuyl

If G is a tree, then G is sequentially Cohen-Macaulay.

He -Van Tuyl

Let G be a directed tree and $t \geq 2$. Then $R/I_t(G)$ is sequentially Cohen-Macaulay.

Path ideals of trees

Theorem

Let I be a squarefree monomial ideal in R. Then R/I is Cohen-Macaulay if and only if R/I is sequentially Cohen-Macaulay and I is unmixed.

Haghighi - Terai - Yassemi - Zaare-Nahandi

Let I be a squarefree monomial ideal in R. Then R/I is S_r if and only if R/I is sequentially S_r and I is unmixed.

Some algebraic preliminaries

A graded R-module M is called sequentially S_r (over k) if there exists a finite filtration of graded R-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is S_r, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1})$$

Some algebraic preliminaries

A finitely generated graded module M over R is said to satisfy the Serre's condition S_r if depth $M_P \geq$min$(r,$ dim $M_P)$, for all $P \in \mathrm{Spec}(R)$.

Herzog - Hibi - Zheng

Let G be a chordal graph on the vertex set V. Let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ which admit a free vertex. Then the following conditions are equivalent:

(i) G is Cohen-Macaulay.

(ii) G is unmixed.

(iii) V is the disjoint union of $F_1, \ldots, F_m$.

level(v) :=the length of the unique path starting at the root and ending at v minus one.

Note that by removing leaves at level strictly less than $(t-1)$ from a tree Γ and repeating this process, one obtains a tree denoted by $C(\Gamma)$. This process is called cleaning process and the tree $C(\Gamma)$ is called the clean form of Γ.

Path ideals of trees

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Note that by removing leaves at level strictly less than $(t-1)$ from a tree Γ and repeating this process, one obtains a tree denoted by $C(\Gamma)$. This process is called cleaning process and the tree $C(\Gamma)$ is called the clean form of Γ.

Let Γ be a tree over n vertices and $2 \leq t \leq n$. Suppose that $F_1, \ldots, F_m$ are all facets of $\Delta = \Delta_t(C(\Gamma))$ containing a leaf of $C(\Gamma)$ such that each leaf belongs to exactly one of them. If $V(\Delta)$ is the disjoint union of $F_1, \ldots, F_m$, then we say that Γ is t-partitioned (by $F_1, \ldots, F_m$).

Path ideals of trees

Let Γ be a t-partitioned tree (by $F_1, \ldots, F_m$). We define a t-branch of Γ, as a path of length $t+1$, say P, which starts at a vertex of some F_i, like x, and $P \cap F_i = \{x\}$. Then, for each $i = 1, \ldots, m$, we define degree of F_i, as

$\mathrm{Deg}_\Gamma(F_i)$:=the number of vertices of F_i which are the first vertices of a t-branch of Γ.

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We call a t-branch of Γ, initial if it intersects some F_i in the first vertex of F_i. Otherwise, we call it non-initial.

We define level of a t-branch P of Γ, denoted by level(P), as the level of the vertex x, where $P \cap F_i = \{x\}$ for some $i = 1, \ldots, m$.

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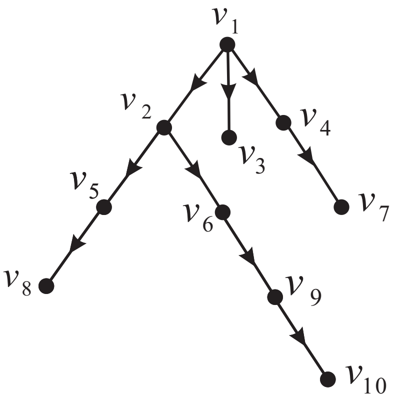
We define level of a t-branch P of Γ, denoted by $\text{level}(P)$, as the level of the vertex x, where $P \cap F_i = \{x\}$ for some $i = 1, \ldots, m$.

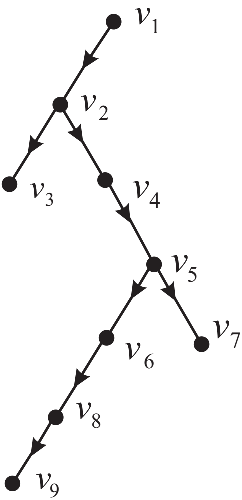
Let Γ be a t-partitioned tree over n vertices and $2 \leq t \leq n$. We say that Γ is fitting t-partitioned, if the following hold:

(1) $\mathrm{Deg}(\Gamma) \leq 1$; and
(2) $\mathrm{level}(P) \leq t-1$, for each non-initial t-branch P of Γ.



Γ_1 is fitting 3-partitioned by $F_1 = \{v_1, v_4, v_7\}$, $F_2 = \{v_2, v_5, v_8\}$ and $F_3 = \{v_6, v_9, v_{10}\}$. We have $\text{Deg}(\Gamma_1) = 1$.

Path ideals of trees



Γ_2 *is* $3-$ *partitioned but not fitting* $3-$ *partitioned.*

Kiani - me

Let Γ be a tree over n vertices and $2 \leq t \leq n$. Then the following conditions are equivalent:

(i) $I_t(\Gamma)$ is unmixed.

(ii) $R/I_t(\Gamma)$ is Cohen-Macaulay.

(iii) $R/I_t(\Gamma)$ is S_r.

(iv) Γ is fitting t-partitioned.

Path ideals of trees

Kiani - me

Let $2 \leq t \leq n$. Then $R/I_t(P_n)$ is Cohen-Macaulay if and only if $t = n$ or $n/2$.

Path ideals of trees

Kiani - me

Let Γ be a tree over n vertices and $2 \leq t \leq n$. If Γ is fitting t-partitioned (by $F_1, \ldots, F_m$), then $\mathrm{pd}(R/I_t(\Gamma)) = m$.

Herzog - Hibi - Zheng

Let G be a chordal graph. Then G is Gorenstein, if and only if G is a disjoint union of edges.

Kiani - me

Let Γ be a tree over n vertices, $2 \leq t \leq n$ and $\Delta_{n,t}$ be the Stanley-Reisner complex of $I_t(\Gamma)$. Then the following are equivalent:

(i) $R/I_t(\Gamma)$ is a complete intersection.

(ii) $R/I_t(\Gamma)$ is Gorenstein.

(iii) $\Delta_{n,t}$ is a matroid.

(iv) $C(\Gamma)$ is P_t.

Kiani - me

Let Γ be a tree over n vertices, $2 \leq t \leq n$ and $I := I_t(\Gamma)$. Then the following are equivalent:

(i) I^m (resp. $I^{(m)}$) is Cohen-Macaulay for every $m \geq 1$.

(ii) I^m (resp. $I^{(m)}$) is Cohen-Macaulay for some $m \geq 3$.

(iii) $C(\Gamma)$ is P_t.

Thanks for your attention.