

Cousin complexes and applications

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Ph.D. thesis

Defence: June 19, 2011

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2011

Acknowledgement

I wish to thank, first and foremost, professor **Mohammad T. Dibaei** who offered invaluable assistance, constant support and tactful guidance as my supervisor. His insights, passion and patience let me to overcome difficulties and win chills during my study and research life.

I am grateful to professors **H. Zakeri**, **M. Tousi**, **S. Yassemi** and **A. Taherizadeh**, for serving on my thesis committee, their helpful comments and suggestions, and for beneficial courses, seminars and discussions I had with them during my undergraduate years.

I wish to express my sincere thanks to professor **S. Zarzuela Armengou** for accepting me as an official visiting guest of the university of Barcelona and his hospitality. I was delighted to interact him by giving me the chance to involve in several academic activities.

As a great opportunity, I had a short but fruitful discussion with professor **R. Y. Sharp** about my thesis, during his visit to the university of Barcelona. I owe my gratitude to him for his comments and conversations.

Last but not the least, I owe my special gratitude and thanks to **my parents**, encouraging me towards higher educations by supporting spiritually and financially throughout my life and their dedication that provided the foundation for this work.

Raheleh Jafari

June 2011

Declaration

Chapters 2, 3 and 4 of this thesis are devoted to the original results.

Abstract

In this thesis, the class of modules whose Cousin complexes have finitely generated cohomologies are studied as a subclass of modules which have uniform local cohomological annihilators and it is shown that these two classes coincide over local rings with Cohen-Macaulay formal fibres. This point of view enables us to obtain some properties of modules with finite Cousin complexes and find some characterizations of them.

In this connection we discuss attached prime ideals of certain local cohomology modules in terms of cohomologies of Cousin complexes. In continuation, we study the top local cohomology modules with specified set of attached primes.

Our approach to study Cousin complexes leads us to characterization of generalized Cohen-Macaulay modules in terms of uniform annihilators of local cohomology. We use these results to study the Cohen-Macaulay loci of modules and find two classes of rings over which the Cohen-Macaulay locus of any finitely generated module is a Zariski-open subset of the spectrum of the ring.

Key words and phrases. Cousin complexes, uniform local cohomological annihilator, Cohen-Macaulay locus, local cohomology, attached primes.

2010 Mathematics Subject Classification. 13D02; 13D45; 13C14; 13D07.

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Introduction

Many concepts in commutative algebra are inspired by algebraic geometric objects. Of particular interest and effective tool in this thesis, is the Cousin complex of a module which is algebraic analogue of the Cousin complex introduced in 1963/64 by A. Grothendieck and R. Hartshorne [17, Chapter IV]. They used this notion to prove a duality theorem for cohomology of quasi-coherent sheaves, with respect to a proper morphism of locally noetherian preschemes.

In 1969, R. Y. Sharp presented the commutative algebraic analogue of the Cousin complex (see section 1.3) and approved it as a powerful tool by characterizing Cohen-Macaulay and Gorenstein rings in terms of Cousin complexes [29]. This concept is developed in [35] by Sharp and is discussed by S. Goto and K. Watanabe in the \mathbb{Z} -graded context [15]. In [29], Sharp shows that a commutative noetherian ring R is Cohen-Macaulay if and only if the Cousin complex $\mathcal{C}_R(R)$ of R is exact, which is improved to modules by himself in [30], while R is Gorenstein if and only if $\mathcal{C}_R(R)$ provides the minimal injective resolution of R . He also introduced Gorenstein modules and characterized them by using Cousin complexes in [30].

From the Cousin complex definition is apparent that its terms are very much look like non-finitely generated, and despite of it, R is Cohen-Macaulay if and only if $\mathcal{C}_R(R)$ is exact, i.e. its cohomologies are zero and so finitely generated. Now, one may ask what rings or modules admit finitely generated Cousin complex cohomologies and what properties these rings or modules have.

In 2001, M. T. Dibaei and M. Tousi, while studying the structure of dualizing complexes, found a class of modules whose Cousin complexes have finitely generated cohomologies. The theory of dualizing complexes comes also from algebraic geometry which was discussed firstly by Grothendieck and Hartshorne in 1963/64 and used to prove their duality theorem [17, Chapter V]. Afterwards Sharp and a number of authors studied its commutative algebraic analogue and found it as a useful tool.

For the rest of this section, R is a commutative noetherian ring and M is a finitely generated R -module.

A dualizing complex for a ring R is a bounded injective complex I^\bullet , where all cohomology modules $H^i(I^\bullet)$ are finitely generated R -modules and the natural map $M \rightarrow \text{Hom}_R(\text{Hom}_R(M, I^\bullet), I^\bullet)$ is quasi isomorphism for any finitely generated R -module M .

A dualizing complex I^\bullet is said to be fundamental whenever $\bigoplus_{i \in \mathbb{Z}} I^i \cong \bigoplus_{\mathfrak{p} \in \text{Spec } R} E(R/\mathfrak{p})$, where $E(R/\mathfrak{p})$ is the injective envelope of R/\mathfrak{p} as R -module, i.e. each prime ideal of R occurs in exactly one term of I^\bullet and exactly once [34, 1.1]. It is known that a ring R possesses a dualizing complex if and only if it possesses a fundamental dualizing complex (see [16, 3.6] and [34, 1.2]), which is unique up to isomorphism of complexes and shifting (see [32, 4.5] and [16, 4.2]).

Now, a natural and interesting treatment is to determine this unique complex. In 1998, Dibaei and Tousi described that if a local ring R which satisfies the condition (S_2) , possesses a dualizing complex, then the fundamental dualizing complex for R is isomorphic to the Cousin complex of the canonical module of R with respect to the height filtration (which is equal to the dimension filtration in this case)[8, 2.4]. As an application they proved that if a local ring R satisfies the condition (S_2) and has a canonical module K , then finiteness of cohomologies of the Cousin complex of K with respect to a certain filtration is necessary and sufficient condition for R to possess a dualizing complex [8, 3.4]. In 2001, they generalized their structural property of dualizing complex of [8] and showed that the Cousin cohomologies of M over a local ring R , are finitely generated if R has a dualizing complex and M is equidimensional which satisfies the condition (S_2) , in [9]. In continuation of [8] and [9], Dibaei studied some properties of Cousin complexes through the dualizing complexes in 2005, and proved the following result.

Theorem 1. [4, Theorem 2.1] Assume that all formal fibres of R are Cohen-Macaulay and M satisfies (S_2) . If \widehat{M} is equidimensional, then $\mathcal{C}_R(M)$ has finitely generated cohomology modules.

These ideas have been pursued in algebraic geometry by J. Lipman, S. Nayak and P. Sastry in [21]. Taking motivation from [8] and [9], Kawasaki studied Cousin complex of a module over a noetherian ring and improved results, independently from [4], in [20]. More precisely, he proved the following results.

Theorem 2. [20, Theorem 1.1] Assume that M is equidimensional and

- (i) R is universally catenary,
- (ii) all the formal fibers of all the localizations of R are Cohen-Macaulay,
- (iii) the Cohen-Macaulay locus of each finitely generated R -algebra is open,

Then all the cohomology modules of the Cousin complex of M are finitely generated and only finitely many of them are non-zero.

The assumptions of the above result are also necessary in a sense.

Theorem 3. [20, Theorem 1.4] Let R be a catenary ring. Then the following statements are equivalent.

- (i) R satisfies the conditions (i), (ii) and (iii) of Theorem 2.

- (ii) for any finitely generated equidimensional R -module M , all the cohomology modules of the Cousin complex of M are finitely generated and only finitely many of them are non-zero.

In special case when R is local, Kawasaki obtains a more simple but interesting version of his result, Theorem 2, as the following.

Theorem 4. [20, Theorem 5.5] Assume that R is a universally catenary local ring and M is an equidimensional R -module. If all formal fibres of R are Cohen-Macaulay, then all the cohomology modules of $\mathcal{C}_R(M)$ are finitely generated.

Note that if a local ring R is universally catenary, then \widehat{M} is equidimensional for each finitely generated equidimensional R -module M . In the proof of the above theorem, the assumption that R is universally catenary, is used to show that \widehat{M} is equidimensional. So one may consider this theorem as a generalization of Theorem 1.

After reviewing some well known results and basic concepts which we need throughout the thesis in Chapter 1, we start our study on Cousin complexes by discussing some useful techniques and essential properties of cohomology modules of Cousin complexes in the first section of Chapter 2. As a consequence we remove the condition (S_2) from Theorem 1 and recover Theorem 4, in Corollary 2.1.6 and Proposition 2.3.2.

In all above results about finiteness of the Cousin complex of an R -module M , there are some crucial common conditions on R and M :

- (a) M is equidimensional;
- (b) R is universally catenary;
- (c) all formal fibres of R are Cohen-Macaulay.

When R is a local ring, these conditions are sufficient for finiteness of cohomology modules of $\mathcal{C}_R(M)$ by Theorem 4, and conditions (b) and (c) are necessary for finiteness of cohomology modules of Cousin complexes of all equidimensional R -modules by Theorem 3. It is now natural to ask that which of these conditions are satisfied if $\mathcal{C}_R(M)$ has finitely generated cohomology modules for an R -module M .

In 2006, C. Zhou studied the properties of noetherian rings containing uniform local cohomological annihilators and showed that all such rings are universally catenary and locally equidimensional[37]. Recall that an element $x \in R$ is called a uniform local cohomological annihilator of M , if $x \in R \setminus \cup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$ and for each maximal ideal \mathfrak{m} of R , $xH_{\mathfrak{m}}^i(M) = 0$ for all $i < \dim M_{\mathfrak{m}}$.

We continue Chapter 2, improving some results of Zhou for modules which have uniform local cohomological annihilators and find some characterizations of these modules

in Section 2.2. We show that if a finitely generated R -module M has a uniform local cohomological annihilator, then M is locally equidimensional and $R/0 :_R M$ is universally catenary in Proposition 2.2.2 and Corollary 2.2.6. We also obtain that the property that M has a uniform local cohomological annihilator is independent of the module structure and depends only on the support of M in Corollary 2.2.5. Finally we investigate our main result in this section by proving that if $\mathcal{C}_R(M)$ has finitely generated cohomology modules, then M has a uniform local cohomological annihilator in Theorem 2.2.13 and so M is equidimensional and $R/0 :_R M$ is universally catenary.

Our approach in studying Cousin complexes is also useful for discussing about uniform local cohomological annihilators and helps us to recover some results in this context by a different and may be simple method, for instance see Corollary 2.2.9 and Proposition 2.2.11.

This point of view, also enables us to characterize modules with finite Cousin cohomologies over a local ring R with Cohen-Macaulay formal fibres. The last section of Chapter 2 is devoted to some applications of our approach. In Theorem 2.3.3, we show that over these rings, $\mathcal{C}_R(M)$ has finitely generated cohomologies if and only if M has a uniform local cohomological annihilator, if and only if \widehat{M} is equidimensional \widehat{R} -module. Our results about the annihilators of cohomologies of Cousin complexes in Section 2.1, lead us to present the height of an ideal of R in terms of Cousin complex in Theorem 2.3.5.

Another important subject which is strongly related to the uniform annihilators of local cohomology, is the notion $a(M)$ which is defined for a finitely generated R -module M over a local ring (R, \mathfrak{m}) as $a(M) = \bigcap_{i < \dim M} (0 :_R H_{\mathfrak{m}}^i(M))$. Note that by definition, an R -module M has a uniform local cohomological annihilator if and only if $a(M) \not\subseteq \cup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$. On the other hand if $\mathfrak{p} \in \text{Min } M$, then $a(M) \subseteq \mathfrak{p}$ if and only if $\mathfrak{p} \in \text{Att } H_{\mathfrak{m}}^i(M)$ for some $i < \dim M$ (see Lemma 2.2.7). These facts motivate us to study the relations between local cohomology modules and Cousin complexes.

It is well known that for a finitely generated R -module M with finite dimension $d = \dim M$, $\text{Att } H_{\mathfrak{m}}^d(M) = \text{Assh } M$ (see Theorem 1.2.4), we start Chapter 3 by discussing $\text{Att } H_{\mathfrak{m}}^t(M)$ for certain t , in particular $\text{Att } H_{\mathfrak{m}}^{d-1}(M)$, in terms of cohomologies of $\mathcal{C}_R(M)$. As a consequence we find a non-vanishing criterion of $H_{\mathfrak{m}}^{d-1}(M)$ when $\mathcal{C}_R(M)$ has finitely generated cohomologies in Corollary 3.1.6.

The main object of Section 3.2 is the following question which is raised by Dibaei and S. Yassemi in [10]. They investigate the set $\text{Att } H_{\mathfrak{a}}^d(M)$ for a finitely generated R -module M and an ideal \mathfrak{a} of R and show that $\text{Att } H_{\mathfrak{a}}^d(M) \subseteq \text{Assh } M$. Now it is natural to ask,

Question 5. [10, Question 2.9] For any subset T of $\text{Assh } M$, is there an ideal \mathfrak{a} of R such that $\text{Att } H_{\mathfrak{a}}^d(M) = T$?

Theorem 3.2.11 presents a positive answer to this question in the case where (R, \mathfrak{m})

is a complete local ring. In [11, Theorem 1.6], it is proved that if (R, \mathfrak{m}) is a complete local ring, then for any pair of ideals \mathfrak{a} and \mathfrak{b} of R , if $\text{Att } H_{\mathfrak{a}}^d(M) = \text{Att } H_{\mathfrak{b}}^d(M)$, then $H_{\mathfrak{a}}^d(M) \cong H_{\mathfrak{b}}^d(M)$. As a consequence we show that the number of non-isomorphic top local cohomology modules of M with respect to all ideals of R is equal to $2^{|\text{Assh } M|}$ in Corollary 3.2.12.

In last section 3.3, we use results of sections 3.1 and 3.2 and those of Chapter 2 for studying the class of generalized Cohen-Macaulay modules. In Corollary 3.3.4, we find a new characterization of generalized Cohen-Macaulay rings in terms of uniform annihilators of local cohomologies. Our results in this section are useful in the last chapter of thesis to study the Cohen-Macaulay loci of modules.

The Cohen-Macaulay locus of M is denoted by

$$\text{CM}(M) := \{\mathfrak{p} \in \text{Spec } R : M_{\mathfrak{p}} \text{ is Cohen-Macaulay as } R_{\mathfrak{p}}\text{-module}\}.$$

The topological property of Cohen-Macaulay loci of modules and determining when it is a zariski-open subset of $\text{Spec } R$ have been studied by many authors. Grothendieck in [14] states that $\text{CM}(M)$ is a Zariski-open subset of $\text{Spec } R$ whenever R is an excellent ring and in [17], Hartshorne shows that $\text{CM}(R)$ is open when R possesses a dualizing complex. In [26], C. Rotthaus and L. M. Şega study the Cohen-Macaulay loci of graded modules over a noetherian homogeneous graded ring $R = \bigoplus_{i \in \mathbb{N}} R_i$ considered as R_0 -modules.

Our aim in the first section of Chapter 4, is to determine when $\text{CM}(M)$ is a Zariski-open subset of $\text{Spec } R$. We find two classes of rings, over which, $\text{CM}(M)$ is open for all R -modules M . The first is the class of rings whose formal fibres are Cohen-Macaulay (see Remark 4.1.7) and the second is the class of catenary local rings R with finite non- $\text{CM}(R)$, where $\text{non-}\text{CM}(M) = \text{Spec } R \setminus \text{CM}(M)$ (see Corollary 4.1.9). Finally, we present examples to show that these two classes of rings are significant in 4.1.11 and 4.1.12.

Inspired by the above results, we study rings whose formal fibres are Cohen-Macaulay in Section 4.2. One of our main results in this section is Theorem 4.2.2 which gives a characterization of a finitely generated R -module which admits a uniform local cohomological annihilator in terms of certain set of formal fibres of R . In particular we show that, for a prime ideal \mathfrak{p} of R , R/\mathfrak{p} is universally catenary and the formal fibre of R over \mathfrak{p} is Cohen-Macaulay if and only if R/\mathfrak{p} has a uniform local cohomological annihilator (see Theorem 4.2.2 and Lemma 2.2.10).

Corollary 4.2.3 is a good summary of connections between uniform annihilators of local cohomologies and Cousin complexes which shows that a local ring R is universally catenary and all of its formal fibres are Cohen-Macaulay if and only if $\mathcal{C}_R(R/\mathfrak{p})$ has finitely generated cohomologies for all $\mathfrak{p} \in \text{Spec } R$, if and only if R/\mathfrak{p} has a uniform local cohomological annihilator for all $\mathfrak{p} \in \text{Spec } R$.

Note that for an R -module M , $\text{non-}\text{CM}(M) = V(\mathfrak{a}(M))$ whenever $\mathcal{C}_R(M)$ is finite (see Corollary 4.2.4). We close this section with Theorem which characterizes those modules

M satisfying $\text{non-CM}(M) = V(\mathfrak{a}(M))$ without assuming that the Cousin complex of M to be finite, which implies also that when $\text{CM}_R(M)$ is finite, then the formal fibres of R over some certain prime ideals are Cohen-Macaulay (see Corollary 4.2.9).

Observe that if $\mathcal{C}_R(M)$ has finitely generated cohomologies, then M is locally equidimensional and $R/0 :_R M$ is universally catenary by Corollary 2.3.1. On the other hand if all formal fibres of a universally catenary local ring R are Cohen-Macaulay, then $\mathcal{C}_R(M)$ has finitely generated cohomologies for all equidimensional R -module M (Theorem 4).

These results strengthen our guess that over a local ring R , if $\mathcal{C}_R(R)$ has finitely generated cohomologies, then all formal fibres of R are Cohen-Macaulay (Section 4.4).

Throughout this thesis, all known definitions and statements are quoted with a reference afterward and all others with no references are supposed to be new, most of them have been appeared in [5], [6] and [7].

Chapter 1

Preliminaries

Throughout this thesis R is a commutative, noetherian ring with non-zero identity and M is an R -module. In the case (R, \mathfrak{m}) is local, we use \widehat{M} as the \mathfrak{m} -adic completion of M . The set of all prime ideals of R is denoted by $\text{Spec } R$ and for an ideal I of R , $V(I)$ denotes the set of all prime ideals of R contain I . The support of M , denoted by $\text{Supp } M$, is the set $\{\mathfrak{p} \in \text{Spec } R : M_{\mathfrak{p}} \neq 0\}$ and the set of associated prime ideals of M , denoted by $\text{Ass } M$, is the set

$$\{\mathfrak{p} \in \text{Spec } R : \mathfrak{p} = (0 :_R x) \text{ for some non-zero element } x \in M\}.$$

The set of minimal primes of M , is the set of minimal elements of $\text{Supp } M$ with respect to inclusion and is denoted by $\text{Min } M$. The Krull dimension of M , denoted by $\dim M$, is defined to be the supremum of lengths of chains of prime ideals of $\text{Supp } M$ if this supremum exists, and ∞ otherwise. By convention, the zero module has Krull dimension -1 . For a prime ideal \mathfrak{p} of R , the M -height of \mathfrak{p} , denoted by $\text{ht}_M \mathfrak{p}$ is defined as $\dim M_{\mathfrak{p}}$. Note that $\dim M = \max\{\dim R/\mathfrak{p} : \mathfrak{p} \in \text{Supp } M\}$. The set of those prime ideals $\mathfrak{p} \in \text{Supp } M$ with $\dim R/\mathfrak{p} = \dim M$, is denoted by $\text{Assh } M$.

When $\dim M < \infty$, M is *equidimensional* if $\text{Min } M = \text{Assh } M$. We say that M is *locally equidimensional* if $M_{\mathfrak{m}}$ is equidimensional for every maximal ideal \mathfrak{m} of $\text{Supp } M$. An element $x \in R$ is said to be *M -regular* if $xm \neq 0$ for each non-zero element $m \in M$. A sequence x_1, \dots, x_n of elements of R is called an *M -regular sequence* or simply an *M -sequence* if x_1 is M -regular, x_i is $M/(x_1, \dots, x_{i-1})M$ -regular for all $2 \leq i \leq n$ and $M \neq (x_1, \dots, x_n)M$. If $IM \neq M$ for an ideal I of R , then the length of a maximal M -sequence in I is a well determined integer $\text{grade}(I, M)$ and is called the *grade of I on M* . We set $\text{grade}(I, M) = \infty$ if $IM = M$. If (R, \mathfrak{m}) is local ring, then the grade of \mathfrak{m} on M is called the *depth of M* and is denoted by $\text{depth } M$.

1.1 Secondary representation theory

The theory of secondary representation of a module is in a certain sense dual to the more familiar theory of primary decomposition and associated primes, and provides a very

satisfactory tool for studying artinian modules. In this section we recall the concepts and some main points of this theory which we need later on. One may find the developed theory by Macdonald in [22], it has been also mentioned by Matsumura in his book [24, Section 6, Appendix].

A non-zero R -module S is said to be *secondary* if its multiplication map by any element r of R is either surjective or nilpotent. If S is secondary, then $\mathfrak{p} = \text{rad}(0 :_R S)$ is a prime ideal, and S is said to be \mathfrak{p} -*secondary*. A *secondary representation* of an R -module S is an expression of S as a finite sum of secondary submodules

$$S = S_1 + S_2 + \cdots + S_n.$$

The representation is minimal if

- (a) the prime ideals $\mathfrak{p}_i = \text{rad}(0 :_R S_i)$ are all distinct, and
- (b) none of the S_i is redundant (i.e. $S_j \not\subseteq \sum_{\substack{i=1 \\ i \neq j}}^n S_i$).

An R -module S is called *representable* if it has a secondary representation. Note that the sum of the empty family of submodules is zero and so a zero R -module is representable. It is easy to see that if S has a secondary representation, then it has a minimal one.

Consider a minimal secondary representation $S = S_1 + \cdots + S_n$ of S , where S_i is \mathfrak{p}_i -secondary for all $1 \leq i \leq n$. Then the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is independent of the choice of minimal secondary representation of S , is called the set of *attached prime ideals of S* and is denoted by $\text{Att } S$ or $\text{Att}_R S$. By the *attached prime ideals* or the *attached primes* of S , we mean exactly the members of $\text{Att } S$.

Here we mention some essential properties and results about secondary representation and attached prime ideals which are interpreted by Macdonald in [22].

- [22, 5.2] Any artinian R -module has a secondary representation.
- [22, 2.2] Let S be a representable R -module and let \mathfrak{p} be a prime ideal of R . Then $\mathfrak{p} \in \text{Att } S$ if and only if there is a homomorphic image of S which has annihilator equal to \mathfrak{p} .
- [22, 4.1] Let $0 \rightarrow S' \rightarrow S \rightarrow S'' \rightarrow 0$ be an exact sequence of representable R -modules and R -homomorphisms. Then

$$\text{Att } S'' \subseteq \text{Att } S \subseteq \text{Att } S' \cup \text{Att } S''.$$

- [22, 2.6] Let S be a representable R -module and let r be an element of R . Then

$$(\alpha) \quad rS \neq S \text{ if and only if } r \in \bigcup_{\mathfrak{p} \in \text{Att } S} \mathfrak{p}.$$

$$(\beta) \operatorname{rad}(0 :_R S) = \bigcap_{\mathfrak{p} \in \operatorname{Att} S} \mathfrak{p}.$$

The following result follows easily from the above (β) .

Corollary 1.1.1. *Assume that S is a representable R -module.*

- (i) *If (R, \mathfrak{m}) is local and S is artinian, then S is finitely generated, and so of finite length, if and only if $\operatorname{Att} S \subseteq \{\mathfrak{m}\}$.*
- (ii) *If $\mathfrak{p} \in \operatorname{Min}(R/0 :_R S)$, then $\mathfrak{p} \in \operatorname{Att} S$.*
- (iii) *All elements of $\operatorname{Att} S$ contains $0 :_R S$.*

Finally, recall the following result which enables us to change the base ring in studying attached prime ideals.

Theorem 1.1.2. [2, 8.2.5] *Let $f : R \rightarrow R'$ be a homomorphism of noetherian rings. Assume that the R' -module S has a secondary representation. Then S has a secondary representation as an R -module by means of f , and*

$$\operatorname{Att}_R S = \{f^{-l}(\mathfrak{p}) : \mathfrak{p} \in \operatorname{Att}_{R'} S\}.$$

1.2 Local cohomology modules

In this section, we review some basic definitions and known results about the local cohomology modules. Our main references are the book of Brodmann and Sharp [2] and the lecture of Schenzel [28].

Given an ideal \mathfrak{a} of R , we consider the \mathfrak{a} -torsion functor over the category of R -modules, defined by

$$\Gamma_{\mathfrak{a}}(M) = \{x \in M : \mathfrak{a}^n x = 0 \text{ for some } n \in \mathbb{N}\},$$

for any R -module M . It is easy to see that $\Gamma_{\mathfrak{a}}(-)$ is an additive, covariant, R -linear, and left exact functor on the category of R -modules and R -homomorphisms. So it makes sense to consider the right derived functors of $\Gamma_{\mathfrak{a}}(-)$. For each $i \geq 0$, the i th right derived functor of $\Gamma_{\mathfrak{a}}(-)$ is denoted by $R^i \Gamma_{\mathfrak{a}}(-)$ and is called the i th local cohomology functor with respect to the ideal \mathfrak{a} , so that $\Gamma_{\mathfrak{a}}(-)$ and $H_{\mathfrak{a}}^0(-)$ are naturally equivalent. The module is called the i th local cohomology module of M with respect to \mathfrak{a} .

An R -module M is called \mathfrak{a} -torsion-free if $\Gamma_{\mathfrak{a}}(M) = 0$, while it is called \mathfrak{a} -torsion if $\Gamma_{\mathfrak{a}}(M) = M$.

The \mathfrak{a} -torsion functor can be expressed as

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \geq 0} (0 :_M \mathfrak{a}^n) \cong \varinjlim_n \text{Hom}_R(R/\mathfrak{a}^n, M);$$

so for all i , we have functorially

$$H_{\mathfrak{a}}^i(M) \cong \varinjlim_n \text{Ext}_R^i(R/\mathfrak{a}^n, M).$$

The following basic properties of local cohomology modules will be often used without further mention.

- [2, Remark 1.2.3] $H_{\mathfrak{a}}^r(M) = H_{\text{rad}(\mathfrak{a})}^r(M)$.
- [2, Corollary 2.1.7(i)] If M is \mathfrak{a} -torsion, then $H_{\mathfrak{a}}^i(M) = 0$ for all $i > 0$.
- [2, Corollary 2.1.7(iii)] $H_{\mathfrak{a}}^i(M) \cong H_{\mathfrak{a}}^i(M/\Gamma_{\mathfrak{a}}(M))$, for all $i > 0$.
- [2, Exercise 2.1.9] If M is \mathfrak{b} -torsion, then $H_{\mathfrak{a}+\mathfrak{b}}^i(M) \cong H_{\mathfrak{a}}^i(M)$ for all i .
- **Independence theorem** [2, 4.2.1]. Let $R \rightarrow S$ be a homomorphism of rings, \mathfrak{a} an ideal of R and let M be an S -module. Then $H_{\mathfrak{a}S}^i(M) \cong H_{\mathfrak{a}}^i(M)$ for all i .
- **Flat base change theorem** [2, 4.3.2]. Let $R \rightarrow S$ be a flat homomorphism of rings, \mathfrak{a} an ideal of R and let M be an R -module. Then $H_{\mathfrak{a}S}^i(M \otimes_R S) \cong H_{\mathfrak{a}}^i(M) \otimes_R S$ for all i .
- **Long exact sequence of local cohomology modules.** Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be an exact sequence of R -modules. Then we have the following long exact sequence.

$$\cdots \rightarrow H_{\mathfrak{a}}^i(M_1) \rightarrow H_{\mathfrak{a}}^i(M_2) \rightarrow H_{\mathfrak{a}}^i(M_3) \rightarrow H_{\mathfrak{a}}^{i+1}(M_1) \rightarrow \cdots$$

The vanishing of local cohomology modules is an important problem which there are many results concerning it. The following theorems are of most famous results.

Theorem 1.2.1. (Grothendieck's vanishing theorem)[2, 6.1.2] *Let M be an R -module and \mathfrak{a} be an ideal of R . Then $H_{\mathfrak{a}}^i(M) = 0$ for all $i > \dim M$.*

The following result identifies the grade of an ideal in terms of local cohomology modules.

Theorem 1.2.2. [2, Theorem 6.2.7] *Let M be a finitely generated R -module such that $\mathfrak{a}M \neq M$ for an ideal \mathfrak{a} of R . Then the least integer i for which $H_{\mathfrak{a}}^i(M) \neq 0$ is precisely $\text{grade}(\mathfrak{a}, M)$.*

Theorem 1.2.3. (The Lichtenbaum-Hartshorne vanishing theorem)[18, Theorem 3.1] *Assume that (R, \mathfrak{m}) is a local ring of dimension n and \mathfrak{a} is a proper ideal of R . Then the following statements are equivalent:*

- (i) $H_{\mathfrak{a}}^n(R) = 0$.
- (ii) $\mathfrak{a}\widehat{R} + \mathfrak{P}$ is not $\widehat{\mathfrak{m}}$ -primary for each prime ideal $\mathfrak{P} \in \text{Assh } \widehat{R}$.

The following results illustrate the utility of the secondary representation theory as an effective tool for studying local cohomology modules.

Theorem 1.2.4. (Grothendieck's non-vanishing theorem—textbf)[23, Theorem 2.2] *Assume that (R, \mathfrak{m}) is a local ring and let M be a non-zero finitely generated R -module. Then $\text{Att } H_{\mathfrak{m}}^{\dim M}(M) = \text{Assh } M$. In particular $H_{\mathfrak{m}}^{\dim M}(M) \neq 0$.*

Theorem 1.2.5. [12, Corollary 4] *Let (R, \mathfrak{m}) be local, \mathfrak{a} an ideal of R and let M be a finitely generated R -module. Then*

$$\text{Att } H_{\mathfrak{a}}^{\dim M}(M) = \{\mathfrak{q} \cap R : \mathfrak{q} \in \text{Assh } \widehat{R}, \dim \widehat{R}/(\mathfrak{a}\widehat{R} + \mathfrak{q}) = 0\}.$$

Localization is an important tool in commutative algebra. The following result provides a useful property of attached primes of local cohomology modules under localization with a condition on the base ring.

Theorem 1.2.6. (Shifted localization principle)[31, Theorem 3.7] *Let (R, \mathfrak{m}) be a local ring which is a homomorphic image of a Gorenstein local ring, M a finitely generated R -module, $\mathfrak{p} \in \text{Spec } R$ and $t = \dim R/\mathfrak{p}$. Assume that $\mathfrak{q} \in \text{Spec } R$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Then for all $i \in \mathbb{Z}$, $\mathfrak{q}R_{\mathfrak{p}} \in \text{Att } H_{\mathfrak{p}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$ if and only if $\mathfrak{q} \in \text{Att } H_{\mathfrak{m}}^{i+t}(M)$.*

If we remove the condition that R is a homomorphic image of a Gorenstein local ring, one have a weaker result.

Theorem 1.2.7. (Weak general shifted localization principle)[31, Theorem 4.8] *Let (R, \mathfrak{m}) be a local ring, M a finitely generated R -module, $\mathfrak{p}, \mathfrak{q} \in \text{Spec } R$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. If for $i \in \mathbb{Z}$, $\mathfrak{q}R_{\mathfrak{p}} \in \text{Att } H_{\mathfrak{p}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$, then $\mathfrak{q} \in \text{Att } H_{\mathfrak{m}}^{i+t}(M)$, where $t = \dim R/\mathfrak{p}$.*

The local cohomology modules of a finitely generated R -module M are rarely finitely generated. For instance, when (R, \mathfrak{m}) is local and M is finitely generated R -module, the fact that $H_{\mathfrak{m}}^i(M)$ is artinian for all $i \geq 0$ together with Corollary 1.1.1(i), implies that $H_{\mathfrak{m}}^i(M)$ is finitely generated if and only if $\text{Att } H_{\mathfrak{m}}^i(M) \subseteq \{\mathfrak{m}\}$. Now, by Grothendieck's non-vanishing theorem (1.2.4), we obtain that $H_{\mathfrak{m}}^{\dim M}(M)$ is not finitely generated if $\dim M > 0$. On the other hand for all ideals \mathfrak{a} of R , $H_{\mathfrak{a}}^0(M)$ is a submodule of M and so it is finitely generated. It is now interesting to find the least integer i for which $H_{\mathfrak{a}}^i(M)$ is not finitely generated.

Definition 1.2.8. [2, Definition 9.1.3] Let M be a finitely generated R -module. We define the *finiteness dimension of M relative to an ideal \mathfrak{a} of R* by

$$f_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N} : H_{\mathfrak{a}}^i(M) \text{ is not finitely generated}\}.$$

The following result provides further motivation for the concept of finiteness dimension.

Proposition 1.2.9. [2, Proposition 9.1.2] *The following statements are equivalent for a finitely generated R -module M , an ideal \mathfrak{a} of R and an integer $t \in \mathbb{N}$.*

- (i) $H_{\mathfrak{a}}^i(M)$ is finitely generated for all $i < t$.
- (ii) $\mathfrak{a} \subseteq \text{rad}(0 :_R H_{\mathfrak{a}}^i(M))$ for all $i < t$.

In light of the above result we have the following useful statement.

Theorem 1.2.10. *Let M be a finitely generated R -module and \mathfrak{a} an ideal of R . Then*

$$\begin{aligned} f_{\mathfrak{a}}(M) &= \inf\{i \in \mathbb{N} : H_{\mathfrak{a}}^i(M) \text{ is not finitely generated}\} \\ &= \inf\{i \in \mathbb{N} : \mathfrak{a} \not\subseteq \text{rad}(0 :_R H_{\mathfrak{a}}^i(M))\}. \end{aligned}$$

Some times it is more useful and fascinated to weaken the condition $\mathfrak{a} \not\subseteq \text{rad}(0 :_R H_{\mathfrak{a}}^i(M))$, using another ideal $\mathfrak{b} \subseteq \mathfrak{a}$.

Definition 1.2.11. [2, Definition 9.1.5] Let M be a finitely generated R -module and let \mathfrak{a} and \mathfrak{b} be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. We define *\mathfrak{b} -finiteness dimension, $f_{\mathfrak{a}}^{\mathfrak{b}}(M)$* , of M relative to \mathfrak{a} by

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M) := \inf\{i \in \mathbb{N} : \mathfrak{b} \not\subseteq \text{rad}(0 :_R H_{\mathfrak{a}}^i(M))\}.$$

Definition 1.2.12. [2, Definition 9.2.2] Let M be a finitely generated R -module. For a prime ideal $\mathfrak{p} \in \text{Spec } R \setminus V(\mathfrak{a})$, the *\mathfrak{a} -adjusted depth of M at \mathfrak{p}* , denoted by $\text{adj}_{\mathfrak{a}}\text{depth } M_{\mathfrak{p}}$, is defined by

$$\text{adj}_{\mathfrak{a}}\text{depth } M_{\mathfrak{p}} := \text{depth } M_{\mathfrak{p}} + \text{ht}(\mathfrak{a} + \mathfrak{p})/\mathfrak{p}.$$

Note that this is ∞ unless $\mathfrak{p} \in \text{Supp } M$ and $\mathfrak{a} + \mathfrak{p} \subset R$, and then it is positive integer.

Let \mathfrak{b} be another ideal of R such that $\mathfrak{b} \subseteq \mathfrak{a}$. We define *\mathfrak{b} -minimum \mathfrak{a} -adjusted depth of M* , denoted by $\lambda_{\mathfrak{a}}^{\mathfrak{b}}(M)$, by

$$\begin{aligned} \lambda_{\mathfrak{a}}^{\mathfrak{b}}(M) &= \inf\{\text{adj}_{\mathfrak{a}}\text{depth } M_{\mathfrak{p}} : \mathfrak{p} \in \text{Spec } R \setminus V(\mathfrak{b})\} \\ &= \inf\{\text{depth } M_{\mathfrak{p}} + \text{ht}(\mathfrak{a} + \mathfrak{p})/\mathfrak{p} : \mathfrak{p} \in \text{Spec } R \setminus V(\mathfrak{b})\}. \end{aligned}$$

Theorem 1.2.13. [2, Theorem 9.3.5] *Let \mathfrak{a} and \mathfrak{b} be ideals of R such that $\mathfrak{b} \subseteq \mathfrak{a}$, and let M be a finitely generated R -module. Then*

$$f_{\mathfrak{a}}^{\mathfrak{b}}(M) \leq \lambda_{\mathfrak{a}}^{\mathfrak{b}}(M).$$

Recall that a finitely generated R -module M , when (R, \mathfrak{m}) is local, is called *Cohen-Macaulay* if $\dim M = \text{depth } M$. In general, when R is not local, M is called Cohen-Macaulay if $M_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \text{Supp } M$. This class of modules are characterized in terms of local cohomology as well.

Remark 1.2.14. Assume that (R, \mathfrak{m}) is a local ring and M is a finitely generated R -module. Theorem 1.2.2 implies that M is Cohen-Macaulay if and only if for all $i < \dim M$.

The Cohen-Macaulay modules have very nice properties and are interesting to study. As a generalization of this important class of modules, one may consider those modules, their local cohomologies are finitely generated at indices less than dimension.

Definition 1.2.15. A finitely generated module M over a local ring (R, \mathfrak{m}) is called a *generalized Cohen-Macaulay* (g.CM) module whenever $\mathfrak{m}^n H_{\mathfrak{m}}^i(M) = 0$ for some $n \in \mathbb{N}$ and all $i < \dim M$. The module M is called *quasi-Buchsbaum* whenever $\mathfrak{m} H_{\mathfrak{m}}^i(M) = 0$ for all $i < \dim M$.

Theorem 1.2.16. *Assume that (R, \mathfrak{m}) is a local ring and M is a finitely generated R -module. Then the following statements hold true.*

(i) [2, 9.5.7(i)] *If M is g.CM, then $\dim R/\mathfrak{p} = \dim M$ for all $\mathfrak{p} \in \text{Ass } M \setminus \{\mathfrak{m}\}$ and $M_{\mathfrak{q}}$ is a Cohen-Macaulay $R_{\mathfrak{q}}$ -module for all $\mathfrak{q} \in \text{Supp } M \setminus \{\mathfrak{m}\}$.*

(ii) [2, 9.5.7(ii)] *As a converse of (i), if*

- *R is a homomorphic image of a regular ring,*
- *$\dim R/\mathfrak{p} = \dim M$ for all $\mathfrak{p} \in \text{Min } M$, and*
- *$M_{\mathfrak{q}}$ is a Cohen-Macaulay $R_{\mathfrak{q}}$ -module for all $\mathfrak{q} \in \text{Supp } M \setminus \{\mathfrak{m}\}$,*

Then M is g.CM.

(iii) [2, 9.5.8(i)] *Assume that M is g.CM and $r \in \mathfrak{m}$ is a parameter element of M (i.e. $\dim M/rM = \dim M - 1$). Then r is a non-zero-divisor on $M/\Gamma_{\mathfrak{m}}(M)$ and M/rM is g.CM.*

As another generalization of Cohen-Macaulay modules, an R -module M satisfies the *Serre's condition* (S_n) for some integer $n \geq 0$, if $\text{depth } M_{\mathfrak{p}} \geq \min\{n, \dim M_{\mathfrak{p}}\}$ for all $\mathfrak{p} \in \text{Supp } M$. A module satisfies (S_n) for all $n \geq 0$ just when it is a Cohen-Macaulay module.

We close this section by recalling the canonical module over a homomorphic image of a Gorenstein local ring and some properties which we will use later on.

Assume that (R, \mathfrak{m}) is a homomorphic image of a Gorenstein local ring (S, \mathfrak{n}) . We set

$$K_M = \text{Ext}_S^{\dim S - \dim M}(M, S),$$

where M is a finitely generated R -module, and call it the *canonical module of M* (see [28]).

There is a natural map $\tau_M : M \longrightarrow K_{K_M}$ by [28, Theorem 1.11], which has an important role in our later discussing.

Lemma 1.2.17. [28, Lemma 1.9] *In the above situation, we have the following statements.*

- (i) *If, for all $\mathfrak{p} \in \text{Supp } M$, $\dim M_{\mathfrak{p}} + \dim R/\mathfrak{p} = \dim M$, then $(K_M)_{\mathfrak{p}} \cong K_{M_{\mathfrak{p}}}$.*
- (ii) *$\text{Ass } K_M = \{\mathfrak{p} \in \text{Ass } M : \dim R/\mathfrak{p} = \dim M\}$, so $\dim M = \dim K_M$.*
- (iii) *K_M satisfies the condition (S_2) .*

Theorem 1.2.18. [28, Theorem 1.14] *Let M denote a finitely generated, equidimensional R -module with $d = \dim M$, where R is a homomorphic image of a Gorenstein ring. Then the following statements are equivalent for an integer $k \geq 1$.*

- (i) *M satisfies the condition S_k .*
- (ii) *The natural map $\tau_M : M \longrightarrow K_{K_M}$ is bijective (resp. injective for $k = 1$) and $H_{\mathfrak{m}}^n(K_M) = 0$ for all $d - k + 2 \leq n < d$.*

1.3 Cousin complexes

Definition 1.3.1. [35, Definition 1.1] A *filtration* of $\text{Spec } R$ is a descending sequence $\mathcal{F} = (F_i)_{i \geq 0}$ of subsets of $\text{Spec}(R)$, $F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_i \supseteq \cdots$, with the property that, for each $i \in \mathbb{N}_0$, every member of $\partial F_i = F_i \setminus F_{i+1}$ is a minimal member of F_i with respect to inclusion. We say the filtration \mathcal{F} *admits* M if $\text{Supp } M \subseteq F_0$.

Example and Notation 1.3.2. [35, Example 1.2] Let M be an R -module. For each $i \geq 0$, set

$$H_i = \{\mathfrak{p} \in \text{Supp } M \mid \text{ht}_M \mathfrak{p} \geq i\}.$$

The sequence $(H_i)_{i \geq 0}$ is a filtration of $\text{Spec } R$ which admits M and is called the *height filtration* of M and is denoted by $\mathcal{H}(M)$.

Notation and Remark 1.3.3. Let $\mathcal{F} = (F_i)_{i \geq 0}$ be a filtration of $\text{Spec } R$ which admits an R -module M . An obvious modification of the construction given in §2 of [29] produces a complex

$$0 \longrightarrow M \xrightarrow{d_M^{-1}} M^0 \xrightarrow{d_M^0} M^1 \longrightarrow \dots \longrightarrow M^{i-1} \xrightarrow{d_M^{i-1}} M^i \longrightarrow \dots,$$

denoted by $\mathcal{C}(\mathcal{F}, M)$ and called *Cousin complex for M with respect to \mathcal{F}* , such that $M^0 = \bigoplus_{\mathfrak{p} \in \partial F_0} M_{\mathfrak{p}}$ and $M^i = \bigoplus_{\mathfrak{p} \in \partial F_i} (\text{Coker } d_M^{i-2})_{\mathfrak{p}}$ for all $i > 0$. The component of $d_M^{-1}(m)$, for $m \in M$ and $\mathfrak{p} \in \partial F_0$, in $M_{\mathfrak{p}}$ is $m/1$. Note that for a prime ideal \mathfrak{p} of R , if $d_{\mathfrak{p}} : M \rightarrow M_{\mathfrak{p}}$ denotes the natural homomorphism given by $d_{\mathfrak{p}}(x) = x/1$ for $x \in M$, then for an element $x \in M$ and $i \geq 0$, $d_{\mathfrak{p}}(x) = 0$ for all but a finite number of prime ideals $\mathfrak{p} \in \partial F_i$ by [29, 2.2]. Consequently, for all $i \geq 0$, there is an R -homomorphism $d_M^{i-1} : M^{i-1} \rightarrow M^i$ for which, if $x \in M^{i-1}$ and $\mathfrak{q} \in \partial F_i$, the component of $d_M^{i-1}(x)$ in $(\text{Coker } d_M^{i-2})_{\mathfrak{q}}$ is $\pi(x)/1$, where $\pi : M^{i-1} \rightarrow \text{Coker } d_M^{i-2}$ is the canonical epimorphism.

We will denote the Cousin complex for M with respect to M -height filtration, $\mathcal{H}(M)$, by $\mathcal{C}_R(M)$. We also use the notation

$$\mathcal{C}_R(M)' : 0 \longrightarrow M^0 \xrightarrow{d_M^0} M^1 \xrightarrow{d_M^1} M^2 \xrightarrow{d_M^2} \dots \xrightarrow{d_M^{i-1}} M^i \xrightarrow{d_M^i} \dots$$

and for each $i \geq -1$,

$$K^i := \text{Ker } d_M^i, D^i := \text{Im } d_M^{i-1}, \mathcal{H}_M^i := K^i/D^i.$$

Then we have the following natural exact sequences,

$$0 \longrightarrow M^{l-1}/K^{l-1} \longrightarrow M^l \longrightarrow M^l/D^l \longrightarrow 0, \quad (1.3.1)$$

$$0 \longrightarrow \mathcal{H}_M^{l-1} \longrightarrow M^{l-1}/D^{l-1} \longrightarrow M^{l-1}/K^{l-1} \longrightarrow 0, \quad (1.3.2)$$

for all $l \geq -1$.

We call the Cousin complex $\mathcal{C}_R(M)$ *finite* whenever each \mathcal{H}_M^i is finitely generated as R -module.

The following Lemma has an important role in our approach for working with cohomologies of Cousin complexes. For the proof of the first part we adopt the argument in [33, Theorem].

Lemma 1.3.4. *Let M be an R -module. For any integer k with $0 \leq k < ht_M \mathfrak{a}$, the following statements are true.*

(a) $H_{\mathfrak{a}}^s(M^k) = 0$ for all integers $s \geq 0$.

(b) $\text{Ext}_R^s(R/\mathfrak{a}, M^k) = 0$ for all integers $s \geq 0$.

Proof. (a). Set $C_{k-1} := \text{Coker } d^{k-2} = M^{k-1}/D^{k-1}$ so that $M^k = \bigoplus_{\substack{\mathfrak{p} \in \text{Supp } M \\ \text{ht}_M \mathfrak{p} = k}} (C_{k-1})_{\mathfrak{p}}$. For

each $k < \text{ht}_M \mathfrak{a}$ and each $\mathfrak{p} \in \text{Supp } M$ with $\text{ht}_M \mathfrak{p} = k$, there exists an element $x \in \mathfrak{a} \setminus \mathfrak{p}$. Thus the multiplication map $(C_{k-1})_{\mathfrak{p}} \xrightarrow{x} (C_{k-1})_{\mathfrak{p}}$ is an automorphism and so the multiplication map $H_{\mathfrak{a}}^s((C_{k-1})_{\mathfrak{p}}) \xrightarrow{x} H_{\mathfrak{a}}^s((C_{k-1})_{\mathfrak{p}})$ is also an automorphism for all integers s . One may then conclude that $H_{\mathfrak{a}}^s((C_{k-1})_{\mathfrak{p}}) = 0$. Now, from additivity of local cohomology functors, it follows that $H_{\mathfrak{a}}^s(M^k) = 0$.

(b). Assume in general that N is an R -module such that $H_{\mathfrak{a}}^s(N) = 0$ for all $s \geq 0$. We show, by induction on i , $i \geq 0$, that $\text{Ext}_R^i(R/\mathfrak{a}, N) = 0$. For $i = 0$, one has $\text{Hom}_R(R/\mathfrak{a}, N) = \text{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^0(N))$ which is zero. Assume that $i > 0$ and the claim is true for any such module N and all $j \leq i - 1$. Choose E to be an injective hull of N and consider the exact sequence $0 \rightarrow N \rightarrow E \rightarrow N' \rightarrow 0$, where $N' = E/N$. As $H_{\mathfrak{a}}^0(E) = 0$, it follows that $H_{\mathfrak{a}}^s(N') = 0$ for all $s \geq 0$. Thus $\text{Ext}_R^{i-1}(R/\mathfrak{a}, N') = 0$, by our induction hypothesis. As, by the above exact sequence $\text{Ext}_R^{i-1}(R/\mathfrak{a}, N') \cong \text{Ext}_R^i(R/\mathfrak{a}, N)$, the result follows. \square

We quote the following results as basic facts on Cousin complexes from [29], [30], [36] and [4].

Lemma 1.3.5. [4, Lemma 1.2] *Let $\overline{R} := R/0 :_R M$, then there exists an isomorphism of complexes $\mathcal{C}_R(M) \cong \mathcal{C}_{\overline{R}}(M)$.*

Theorem 1.3.6. [29, Theorem 3.5] *Suppose that S is a multiplicatively closed subset of R and M is an R -module. Then there is an isomorphism of complexes of $S^{-1}R$ -modules and $S^{-1}R$ -chain map,*

$$\Psi = \{\psi^n\}_{n \geq -1} : S^{-1}(\mathcal{C}_R(M)) \rightarrow \mathcal{C}_{S^{-1}R}(S^{-1}M)$$

which is such that $\psi^{-1} : S^{-1}M \rightarrow S^{-1}M$ is the identity.

The following result provide a characterization of Cohen-Macaulay modules in terms of Cousin complexes

Theorem 1.3.7. [30, Theorem 2.4] *Assume that M is a non-zero finitely generated R -module. Then M is Cohen-Macaulay if and only if the Cousin complex of M , $\mathcal{C}_R(M)$, is exact.*

The structure of Cousin complexes characterize the condition (S_n) as well.

Theorem 1.3.8. [36, Example 4.4] *Assume that M is a finitely generated R -module. Then M satisfies the condition (S_n) if and only if $\mathcal{C}_R(M)$ is exact at i th term for $i \leq n - 2$.*

Theorem 1.3.9. [29, 2.7] *Assume that M is an R -module. Then*

- (i) $\text{Supp Coker } d_M^{n-1} \subseteq H_n$, and
- (ii) $\text{Supp } \mathcal{H}_M^n \subseteq H_{n+2}$.

The following two lemmas may be considered as easy applications of the above theorem, which provide useful properties of Cousin cohomologies.

Lemma 1.3.10. *Assume that M is a finitely generated R -module of finite $\dim M = d$ and that $\mathcal{C}_R(M)$ is finite, then*

$$\bigcap_{i \geq -1} (0 :_R \mathcal{H}_M^i) \not\subseteq \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}.$$

Proof. By Theorem 1.3.9, for all $i \geq -1$, we have

$$V(0 :_R \mathcal{H}_M^i) = \text{Supp } \mathcal{H}_M^i \subseteq \{\mathfrak{p} \in \text{Supp } M : \dim M_{\mathfrak{p}} \geq i + 2\}.$$

So $0 :_R \mathcal{H}_M^i \not\subseteq \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$. As $\dim M = d$, we have $\mathcal{H}_M^i = 0$ for $i \geq d - 1$, hence Prime Avoidance Theorem implies that $\bigcap_{i \geq -1} (0 :_R \mathcal{H}_M^i) \not\subseteq \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$. \square

Lemma 1.3.11. *Assume that M is a finitely generated R -module with $d = \dim M$.*

- (i) $\text{Ass } M = \text{Min } M$, if and only if $\mathcal{H}_M^{-1} = 0$.
- (ii) $\mathcal{H}_M^{d-1} = \mathcal{H}_M^d = 0$.

Proof. (i) It is obvious by definition of (S_1) and Theorem 1.3.8.

(ii) It is clear by Theorem 1.3.9(ii). \square

Chapter 2

Finite Cousin complexes

The aim of this chapter is to study the class of modules whose Cousin complexes have finitely generated cohomology modules, as a subclass of modules which have uniform local cohomological annihilators. We describe some essential properties of the structure of Cousin complexes which are useful in the rest of the thesis, in the first section. In the second section, we study the theory of uniform annihilators of local cohomology and show that the class of finite Cousin complex modules is a subset of this class, and finally we use our approach to get some characterizations in the last section.

2.1 Cohomology modules of Cousin complexes

In this section we study the structure of cohomology modules of Cousin complexes and develop some techniques which will be used throughout the rest of the thesis.

Firstly, we discuss the relationship between Cousin complexes of modules satisfying a short exact sequence.

Lemma 2.1.1. *Assume that (R, \mathfrak{m}) is a local ring.*

- (a) *If $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N$ is an exact sequence of R -modules with the property that $\text{ht}_{M\mathfrak{p}} \geq 2$ for all $\mathfrak{p} \in \text{Supp } N$, then $\mathcal{C}_R(L)' \cong \mathcal{C}_R(M)'$; in particular, if L and M are finitely generated R -modules, then $\mathcal{C}_R(L)$ is finite if and only if $\mathcal{C}_R(M)$ is finite.*
- (b) *If $L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is an exact sequence of R -modules with the property that $\text{ht}_{M\mathfrak{p}} \geq 1$ for all $\mathfrak{p} \in \text{Supp } L$, then $\mathcal{C}_R(M)' \cong \mathcal{C}_R(N)'$; in particular, if M and N are finitely generated R -modules, $\mathcal{C}_R(M)$ is finite if and only if $\mathcal{C}_R(N)$ is finite.*

Proof. To prove (a), we argue on the following diagram.

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
& & & & \text{Ker } \psi & & \\
& & & & \downarrow & & \\
0 & & 0 & & & & \\
\downarrow & & \downarrow & & \downarrow & & \\
L & \xrightarrow{d_L^{-1}} & L^0 & \xrightarrow{\theta} & L^0/\text{Im } d_L^{-1} & \longrightarrow & 0 \\
\downarrow f & & \downarrow f^0 & & \downarrow \psi & & \\
M & \xrightarrow{d_M^{-1}} & M^0 & \xrightarrow{\lambda} & M^0/\text{Im } d_M^{-1} & \longrightarrow & 0 \\
\downarrow g & & \downarrow & & \downarrow & & \\
N & \longrightarrow & 0 & & 0 & & \\
\downarrow & & & & & & \\
0, & & & & & &
\end{array} \tag{2.1.1}$$

As $\text{Supp Im } g \subseteq \text{Supp } N$ we may replace N by $\text{Im } g$ and assume that

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is exact. Since $\text{Min } M \cap \text{Supp } N = \emptyset$, $\text{Min } M = \text{Min } L$ and it follows an isomorphism $f^0 : L^0 \longrightarrow M^0$ over f , where $L^0 = \bigoplus_{\mathfrak{p} \in \text{Min } L} L_{\mathfrak{p}}$ and $M^0 = \bigoplus_{\mathfrak{p} \in \text{Min } M} M_{\mathfrak{p}}$ and $f^0 d_L^{-1} = d_M^{-1} f$.

We next consider the natural epimorphisms

$$\theta : L^0 \longrightarrow L^0/\text{Im } d_L^{-1}, \quad \lambda : M^0 \longrightarrow d_M^{-1}, \quad \psi : L^0/\text{Im } d_L^{-1} \longrightarrow M^0/\text{Im } d_M^{-1},$$

which $\psi\theta = \lambda f^0$ and consider the map $\varphi := \theta(f^0)^{-1} d_M^{-1} : M \longrightarrow \text{Ker } \psi$, where $(f^0)^{-1}$ denotes the inverse map of f^0 , and show that φ is an epimorphism. In order to prove this, choose an element $x \in \text{Ker } \psi$, there is an element $m_0 \in M^0$ such that $x = \theta(f^0)^{-1}(m_0)$. It follows that $m_0 \in \text{Ker } \lambda \subseteq \text{Im } d_M^{-1}$. Thus $m_0 = d_M^{-1}(m)$ for some $m \in M$. Hence $x = \theta(f^0)^{-1} d_M^{-1}(m) = \varphi(m)$. As $\text{Im } f \subseteq \text{Ker } \varphi$, there is an epimorphism

$$(N \cong) M/\text{Im } f \longrightarrow M/\text{Ker } \varphi (\cong \text{Ker } \psi)$$

which implies that $\text{Supp } \text{Ker } \psi \subseteq \text{Supp } N$. As a result, for each $\mathfrak{p} \in \text{Supp } M$ with $\text{ht}_M \mathfrak{p} = 1$ we have $\mathfrak{p} \notin \text{Supp } \text{Ker } \psi$.

Now, summing up the localizations of the exact sequence

$$0 \longrightarrow \text{Ker } \psi \longrightarrow L^0/\text{Im } d_L^{-1} \longrightarrow M^0/\text{Im } d_M^{-1} \longrightarrow 0$$

at all prime ideals $\mathfrak{p} \in \text{Supp } M$ with $\text{ht}_M \mathfrak{p} = 1$, and taking the natural maps into consideration, we find the following commutative diagram

$$\begin{array}{ccccccc} L & \xrightarrow{d_L^{-1}} & L^0 & \xrightarrow{d_L^0} & L^1 & \xrightarrow{\theta^1} & L^1/\text{Im } d_L^0 & \longrightarrow & 0 \\ \downarrow f & & \downarrow f^0 & & \downarrow f^1 & & \downarrow \psi^1 & & \\ M & \xrightarrow{d_M^{-1}} & M^0 & \xrightarrow{d_M^0} & M^1 & \xrightarrow{\lambda^1} & M^1/\text{Im } d_M^0 & \longrightarrow & 0, \end{array}$$

with f^1 is an isomorphism, θ^1 and λ^1 are the natural epimorphisms and ψ^1 is the natural homomorphism. It is clear that ψ^1 is an isomorphism too. Now, by induction on i , we get a family of isomorphisms $(f^i)_{i \geq 0}$, $f^i : L^i \longrightarrow M^i$, $i > 0$, such that the diagram

$$\begin{array}{ccccccccc} 0 & & 0 & & 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & L^0 & \xrightarrow{d_L^0} & L^1 & \xrightarrow{d_L^1} & \dots & \longrightarrow & L^{i-1} & \xrightarrow{d_L^{i-1}} & L^i & \xrightarrow{d_L^i} & \dots \\ \downarrow f & & \downarrow f^0 & & \downarrow f^1 & & & & \downarrow f^{i-1} & & \downarrow f^i & & \\ M & \longrightarrow & M^0 & \xrightarrow{d_M^0} & M^1 & \xrightarrow{d_M^1} & \dots & \longrightarrow & M^{i-1} & \xrightarrow{d_M^{i-1}} & M^i & \xrightarrow{d_M^i} & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ N & \longrightarrow & 0 & & 0 & & 0 & & 0 & & 0 & & \\ \downarrow & & & & & & & & & & & & \\ 0, & & & & & & & & & & & & \end{array}$$

is commutative with exact rows. Now, it follows that there are an exact sequence $N \longrightarrow \mathcal{H}_L^0 \longrightarrow \mathcal{H}_M^0 \longrightarrow 0$ and isomorphisms $\mathcal{H}_L^i \cong \mathcal{H}_M^i$, $i > 0$. Thus the claim follows.

(b) As $\text{Supp } \text{Ker } f \subseteq \text{Supp } L$ we may replace L by $\text{Ker } f$ and assume that

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is exact. Since $\text{Min } M \cap \text{Supp } L = \emptyset$, we get $\text{Min } M = \text{Min } N$ and there is an isomorphism $g^0 : M^0 \longrightarrow N^0$ over g , where $M^0 = \bigoplus_{\mathfrak{p} \in \text{Min } M} M_{\mathfrak{p}}$ and $N^0 = \bigoplus_{\mathfrak{p} \in \text{Min } N} N_{\mathfrak{p}}$. So that we have

the commutative diagram

$$\begin{array}{ccccccc}
 0 & & & & & & \\
 \downarrow & & & & & & \\
 L & \longrightarrow & 0 & & & & \\
 \downarrow & & \downarrow & & & & \\
 M & \xrightarrow{d_M^{-1}} & M^0 & \xrightarrow{\gamma_M^0} & M^0/\text{Im } d_M^{-1} & \longrightarrow & 0 \\
 \downarrow g & & \downarrow g^0 & & \downarrow \lambda^0 & & \\
 N & \xrightarrow{d_N^{-1}} & N^0 & \xrightarrow{\gamma_N^0} & N^0/\text{Im } d_N^{-1} & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & & & \\
 0 & & 0, & & & &
 \end{array} \tag{2.1.2}$$

where γ_M^0 , γ_N^0 and λ^0 are natural homomorphisms. It is easy to see that λ^0 is an isomorphism. Therefore, there is an isomorphism $g^1 : M^1 \rightarrow N^1$ over g^0 , where $M^1 = \bigoplus_{\substack{\mathfrak{p} \in \text{Supp } M \\ \text{ht}_{M^{\mathfrak{p}}=1}} (M^0/\text{Im } d_M^{-1})_{\mathfrak{p}}$ and $N^1 = \bigoplus_{\substack{\mathfrak{p} \in \text{Supp } N \\ \text{ht}_{N^{\mathfrak{p}}=1}} (N^0/\text{Im } d_N^{-1})_{\mathfrak{p}}$. Hence we have the following commutative diagram where g^0 and g^1 are isomorphisms.

$$\begin{array}{ccccc}
 0 & \longrightarrow & M^0 & \xrightarrow{d_M^0} & M^1 \\
 & & \cong \downarrow g^0 & & \cong \downarrow g^1 \\
 0 & \longrightarrow & N^0 & \xrightarrow{d_N^0} & N^1
 \end{array} \tag{2.1.3}$$

Now, it follows by induction that $\mathcal{C}_R(M)' \cong \mathcal{C}_R(N)'$.

To prove the final claim, let us assume that $\mathcal{C}_R(M)$ is finite. It follows from the exact sequence

$$0 \longrightarrow \text{Im } d_M^{-1} \longrightarrow \text{Ker } d_M^0 \longrightarrow \mathcal{H}_M^0 \longrightarrow 0$$

that $\text{Ker } d_M^0$ is finitely generated. By (2.1.3), $\text{Ker } d_N^0$ is finitely generated and so \mathcal{H}_N^0 is finitely generated. The finiteness of \mathcal{H}_N^i , $i \geq 1$, follows by the isomorphism $\mathcal{C}_R(M)' \cong \mathcal{C}_R(N)'$. \square

As an application of the above lemma, we prove the following result which will be useful in our later methods.

Proposition 2.1.2. *Assume that (R, \mathfrak{m}) is local and M is a finitely generated R -module. Then there is a finitely generated R -module N which satisfies the condition (S_1) with $\text{Supp } N = \text{Supp } M$ and $\mathcal{C}_R(M)$ is finite if and only if $\mathcal{C}_R(N)$ is finite.*

Proof. There exists a submodule L of M such that $\text{Ass } L = \text{Ass } M \setminus \text{Min } M$, $\text{Ass } M/L = \text{Min } M$ (c.f. [1, Page 263, Proposition 4]). Set $N := M/L$. Now N satisfies (S_1) and considering the exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ and the fact that $\text{Supp } L \cap \text{Min } M = \emptyset$, the result follows by Lemma 2.1.1(b). \square

Corollary 2.1.3. *Assume that (R, \mathfrak{m}) is local and M is a finitely generated R -module such that $\mathcal{C}_R(M)$ is finite. Then there exists a finitely generated R -module N which satisfies (S_1) , $\text{Supp } N = \text{Supp } M$ and $\mathcal{C}_R(N)$ is finite.*

Consider the assumption and notation of Proposition 2.1.2. In the following result we find N satisfies the condition (S_2) whenever R is a homomorphic image of a Gorenstein ring. This result will be useful to find some sufficient conditions for finiteness of Cousin complexes.

Proposition 2.1.4. *Assume that (R, \mathfrak{m}) is a homomorphic image of a Gorenstein ring and M is a finitely generated equidimensional R -module. Then there exists a finitely generated R -module N which satisfies the condition (S_2) , $\text{Supp } N = \text{Supp } M$ and $\mathcal{C}_R(M)$ is finite if and only if $\mathcal{C}_R(N)$ is finite.*

Proof. By Proposition 2.1.3, we may assume that M satisfies the condition (S_1) , i.e. $\text{Ass } M = \text{Min } M$, so that $\text{Ass } M = \text{Assh } M$. Since R is a homomorphic image of a Gorenstein ring, we may define the canonical module K_M . Set $N := K_{K_M}$. Then K_M and N satisfy the condition (S_2) by Lemma 1.2.17(iii). Now, by Theorem 1.2.18, we may consider the exact sequence

$$0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0.$$

Note that if $\mathfrak{p} \in \text{Supp } M$ such that $\text{ht}_M \mathfrak{p} \leq 1$, then $M_{\mathfrak{p}}$ is a Cohen-Macaulay module. Thus Theorem 1.2.18 implies that $L_{\mathfrak{p}} = 0$. By lemma 1.2.17, $\text{Ass } N = \text{Assh } K_M = \text{Assh } M = \text{Ass } M$ so that $\text{Supp } N = \text{Supp } M$ and we have $\text{ht}_N \mathfrak{p} \geq 2$ for all $\mathfrak{p} \in \text{Supp } L$. Now Lemma 2.1.1(a) implies the result. \square

Recall that, for a local ring (R, \mathfrak{m}) and the natural ring homomorphism $R \rightarrow \widehat{R}$ and any prime ideal $\mathfrak{p} \in \text{Spec } R$, the ring $\widehat{R} \otimes_R k(\mathfrak{p})$ is called the *formal fibre* of R over \mathfrak{p} , where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Note that all formal fibres of a homomorphic image of a Gorenstein ring are Cohen-Macaulay.

The following result of H. Petzl, helps us to use Proposition 2.1.4 more efficiently.

Lemma 2.1.5. [25, Theorem 3.5] *Assume that (R, \mathfrak{m}) is a local ring and all formal fibres of R satisfy the condition (S_1) . Let M be a finitely generated R -module. Then there is a*

monomorphism of complexes $u^\bullet : \mathcal{C}_R(M) \otimes_R \widehat{R} \rightarrow \mathcal{C}_{\widehat{R}}(\widehat{M})$. Moreover if all formal fibres of R are Cohen-Macaulay, the quotient complex Q^\bullet , in the exact sequence

$$0 \rightarrow \mathcal{C}_R(M) \otimes_R \widehat{R} \xrightarrow{u^\bullet} \mathcal{C}_{\widehat{R}}(\widehat{M}) \rightarrow Q^\bullet \rightarrow 0,$$

is an exact complex. In particular for each $i \geq 0$, there exists an \widehat{R} -isomorphism

$$\mathcal{H}_M^i \otimes_R \widehat{R} \cong \mathcal{H}_{\widehat{M}}^i.$$

In [4, Theorem 2.1], Dibaei proves that, over a local ring with Cohen-Macaulay formal fibres, the Cousin complex of a finitely generated module M is finite provided M satisfies (S_2) and \widehat{M} is equidimensional. We are now able to show that the condition (S_2) is superfluous. Kawasaki also obtains this result in the proof of [20, Theorem 5.5] by a different method.

Corollary 2.1.6. *Assume that (R, \mathfrak{m}) is a local ring such that all of its formal fibers are Cohen-Macaulay. Assume that M is a finitely generated R -module such that \widehat{M} is equidimensional \widehat{R} -module. Then the Cousin complex of M , $\mathcal{C}_R(M)$, is finite.*

Proof. We may assume that $\dim M \geq 2$, by Lemma 1.3.11(ii). Since for each i , by Lemma 2.1.5, there is an isomorphism $\mathcal{H}_M^i \otimes_R \widehat{R} \cong \mathcal{H}_{\widehat{M}}^i$, finiteness of $\mathcal{C}_R(M)$ is equivalent to finiteness of $\mathcal{C}_{\widehat{R}}(\widehat{M})$. Hence we may assume that R is complete, and so that R is a homomorphic image of a Gorenstein ring. Now, by Proposition 2.1.4, there exists a finitely generated R -module which satisfies the condition (S_2) and $\mathcal{C}_R(M)$ is finite if and only if $\mathcal{C}_R(N)$ is finite.

Since N satisfies the condition (S_2) and $\text{Supp } N = \text{Supp } M$, so that N is equidimensional, [4, Theorem 2.1] implies that $\mathcal{C}_R(N)$ is finite and so the result follows. \square

The following technical result will play a key role in the rest of the chapter.

Proposition 2.1.7. *Let M be an R -module and let \mathfrak{a} be an ideal of R such that $\mathfrak{a}M \neq M$. Then, for each non-negative integer r with $r < \text{ht}_M \mathfrak{a}$,*

$$\prod_{i=0}^r (0 :_R \text{Ext}_R^{r-i}(R/\mathfrak{a}, \mathcal{H}_M^{i-1})) \subseteq 0 :_R \text{Ext}_R^r(R/\mathfrak{a}, M).$$

Here \prod is used for product of ideals.

Proof. For each $j \geq -1$, recall the natural exact sequences 1.3.1 and 1.3.2.

$$0 \rightarrow M^{j-1}/K^{j-1} \rightarrow M^j \rightarrow M^j/D^j \rightarrow 0, \quad (2.1.4)$$

$$0 \rightarrow \mathcal{H}_M^{j-1} \rightarrow M^{j-1}/D^{j-1} \rightarrow M^{j-1}/K^{j-1} \rightarrow 0. \quad (2.1.5)$$

Let $0 \leq r < \text{ht}_M \mathfrak{a}$. We prove by induction on j , $0 \leq j \leq r$, that

$$\prod_{i=0}^j (0 :_R \text{Ext}_R^{r-i}(R/\mathfrak{a}, \mathcal{H}_M^{i-1})) \cdot (0 :_R \text{Ext}_R^{r-j}(R/\mathfrak{a}, M^{j-1}/K^{j-1})) \subseteq 0 :_R \text{Ext}_R^r(R/\mathfrak{a}, M). \quad (2.1.6)$$

In case $j = 0$, the exact sequence (2.1.5) implies the exact sequence

$$\text{Ext}_R^r(R/\mathfrak{a}, \mathcal{H}_M^{-1}) \longrightarrow \text{Ext}_R^r(R/\mathfrak{a}, M) \longrightarrow \text{Ext}_R^r(R/\mathfrak{a}, M^{-1}/K^{-1}),$$

so that

$$(0 :_R \text{Ext}_R^r(R/\mathfrak{a}, \mathcal{H}_M^{-1})) \cdot (0 :_R \text{Ext}_R^r(R/\mathfrak{a}, M^{-1}/K^{-1})) \subseteq 0 :_R \text{Ext}_R^r(R/\mathfrak{a}, M),$$

and thus the case $j = 0$ is justified.

Assume that $0 \leq j < r$ and formula (2.1.6) is settled for j . Therefore, by Lemma 1.3.4(b), the exact sequence 2.1.4 implies that

$$\text{Ext}_R^{r-j}(R/\mathfrak{a}, M^{j-1}/K^{j-1}) \cong \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, M^j/D^j). \quad (2.1.7)$$

On the other hand the exact sequence (2.1.5) implies the exact sequence

$$\text{Ext}_R^{r-j-1}(R/\mathfrak{a}, \mathcal{H}_M^j) \longrightarrow \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, M^j/D^j) \longrightarrow \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, M^j/K^j),$$

from which it follows that

$$(0 :_R \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, \mathcal{H}_M^j)) \cdot (0 :_R \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, M^j/K^j)) \subseteq 0 :_R \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, M^j/D^j). \quad (2.1.8)$$

Now (2.1.7) and (2.1.8) imply that

$$(0 :_R \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, \mathcal{H}_M^j)) \cdot (0 :_R \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, M^j/K^j)) \subseteq 0 :_R \text{Ext}_R^{r-j}(R/\mathfrak{a}, M^{j-1}/K^{j-1}). \quad (2.1.9)$$

From (2.1.9), it follows that

$$\begin{aligned} & \prod_{i=0}^{j+1} (0 :_R \text{Ext}_R^{r-i}(R/\mathfrak{a}, \mathcal{H}_M^{i-1})) \cdot (0 :_R \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, M^j/K^j)) = \\ & \prod_{i=0}^j (0 :_R \text{Ext}_R^{r-i}(R/\mathfrak{a}, \mathcal{H}_M^{i-1})) \cdot (0 :_R \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, \mathcal{H}_M^j)) \cdot (0 :_R \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, M^j/K^j)) \subseteq \\ & \prod_{i=0}^j (0 :_R \text{Ext}_R^{r-i}(R/\mathfrak{a}, \mathcal{H}_M^{i-1})) \cdot (0 :_R \text{Ext}_R^{r-j}(R/\mathfrak{a}, M^{j-1}/K^{j-1})), \end{aligned}$$

and, by the induction hypothesis (2.1.6), it follows that

$$\prod_{i=0}^{j+1} (0 :_R \text{Ext}_R^{r-i}(R/\mathfrak{a}, \mathcal{H}_M^{i-1})) \cdot (0 :_R \text{Ext}_R^{r-j-1}(R/\mathfrak{a}, M^j/K^j)) \subseteq 0 :_R \text{Ext}_R^r(R/\mathfrak{a}, M).$$

This is the end of the induction argument.

Note that there is an embedding $\text{Ext}_R^0(R/\mathfrak{a}, M^{r-1}/K^{r-1}) \hookrightarrow \text{Ext}_R^0(R/\mathfrak{a}, M^r)$, by the exact sequence (2.1.4). On the other hand $\text{Ext}_R^0(R/\mathfrak{a}, M^r) = 0$ by Lemma 1.3.4(b). So that $\text{Ext}_R^0(R/\mathfrak{a}, M^{r-1}/K^{r-1}) = 0$ and now putting $j = r$ in (2.1.6) gives the result. \square

An immediate corollary to the above result is the following.

Corollary 2.1.8. *Assume that \mathfrak{a} is an ideal of R such that $\mathfrak{a}M \neq M$. Then, for each integer r with $0 \leq r < \text{ht}_M \mathfrak{a}$,*

$$\prod_{i=-1}^{r-1} (0 :_R \mathcal{H}_M^i) \subseteq \bigcap_{i=0}^r (0 :_R \text{Ext}_R^i(R/\mathfrak{a}, M)) \subseteq \bigcap_{i=0}^r (0 :_R \mathbb{H}_{\mathfrak{a}}^i(M)).$$

Proof. It follows by Proposition 2.1.7 and the fact that the extension functors are linear and $\mathbb{H}_{\mathfrak{a}}^i(M) \cong \varinjlim_j (\text{Ext}_R^i(R/\mathfrak{a}^j, M))$. \square

2.2 Uniform annihilators of local cohomology

Recall that an element $x \in R \setminus \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$ is a *uniform local cohomological annihilator* of an R -module M if, for every maximal ideal \mathfrak{m} , $x\mathbb{H}_{\mathfrak{m}}^i(M) = 0$ for all $i < \text{ht}_M \mathfrak{m}$. Moreover, x is called a *strong uniform local cohomological annihilator* of M if x is a uniform local cohomological annihilator of $M_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} in $\text{Supp } M$. R is called *universally catenary* if every finitely generated R -algebra is catenary.

As a basic property of a ring R containing a uniform local cohomological annihilator, it is proved that R must be locally equidimensional and universally catenary (c.f. [37, Theorem 2.1]). One of essential results about uniform annihilators of local cohomology, is the following result due to Zhou which reduces the property that a ring R has a uniform local cohomological annihilator to the same property for R/\mathfrak{p} for each minimal prime \mathfrak{p} of R .

Theorem 2.2.1. [37, Theorem 3.2] *Let R be of finite dimension d . Then the following conditions are equivalent.*

- (i) *R has a uniform local cohomological annihilator.*
- (ii) *R is locally equidimensional, and R/\mathfrak{p} has a uniform local cohomological annihilator for each minimal prime ideal \mathfrak{p} of R .*

The module version of the above theorem is also true. One may use an almost similar method to prove it. We state the proof more precisely. First we show that a module which has a uniform local cohomological annihilator is locally equidimensional.

Proposition 2.2.2. *Let M be a finitely generated R -module such that it has a uniform local cohomological annihilator. Then M is locally equidimensional.*

Proof. Let $\mathfrak{m} \in \text{Max Supp } M$. We will show that $\dim M_{\mathfrak{m}} = \dim R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}}$ for all $\mathfrak{p} \in \text{Spec } R$ with $\mathfrak{p} \in \text{Min } M$ and $\mathfrak{p} \subseteq \mathfrak{m}$. By assumption, there exists an element $x \in R \setminus \cup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$ such that $xH_{\mathfrak{m}}^i(M) = 0$ for all $i < \dim M_{\mathfrak{m}}$. As $x \in R_{\mathfrak{m}} \setminus \cup_{\mathfrak{p}R_{\mathfrak{m}} \in \text{Min } M_{\mathfrak{m}}} \mathfrak{p}R_{\mathfrak{m}}$, and $H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{m}R_{\mathfrak{m}}}^i(M_{\mathfrak{m}})$ by using the definition of local cohomology, we may assume that (R, \mathfrak{m}) is local with the maximal ideal \mathfrak{m} and write $d := \dim M$.

Assume, to the contrary, that there exists $\mathfrak{p} \in \text{Min } M$ with $c := \dim R/\mathfrak{p} < d$. Set $S = \{\mathfrak{q} \in \text{Min } M : \dim R/\mathfrak{q} \leq c\}$ and $T = \text{Ass } M \setminus S$. There exists a submodule N of M such that $\text{Ass } N = T$ and $\text{Ass } M/N = S$. Note that $\dim M/N = c$ and that $\dim N = d$. As $\text{rad}(0 :_R N) = \cap_{\mathfrak{q} \in T} \mathfrak{q}$, it follows that there exists an element $y \in (0 :_R N) \setminus \cup_{\mathfrak{q} \in S} \mathfrak{q}$. Thus, trivially, $yH_{\mathfrak{m}}^i(N) = 0$ for all $i \geq 0$. The exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ implies the exact sequence

$$H_{\mathfrak{m}}^i(M) \rightarrow H_{\mathfrak{m}}^i(M/N) \rightarrow H_{\mathfrak{m}}^{i+1}(N).$$

As $xH_{\mathfrak{m}}^i(M) = 0$ for all $i < d$, it follows that $xyH_{\mathfrak{m}}^i(M/N) = 0$ for all $i < d$. In particular, $xyH_{\mathfrak{m}}^c(M/N) = 0$. Thus $xy \in \cap_{\mathfrak{q} \in \text{Assh } M/N} \mathfrak{q}$ (c.f. [2, Proposition 7.2.11 and Theorem 7.3.2]). Therefore $xy \in \mathfrak{p}$ by the choice of \mathfrak{p} . As $\mathfrak{p} \in S \cap \text{Min } M$, this is a contradiction. \square

We need the following technical lemma to extend Theorem 2.2.1 to module version. This result has been proved in [37, Lemma 3.1] for $M = R$, the same technique works also for an arbitrary R -module M .

Lemma 2.2.3. *Let (R, \mathfrak{m}) be a local ring, M a finitely generated R -module of dimension d , \mathfrak{p} be a minimal prime ideal of M and*

$$\begin{aligned} 0 &\longrightarrow R/\mathfrak{p} \longrightarrow M \longrightarrow N_1 \longrightarrow 0, \\ 0 &\longrightarrow R/\mathfrak{p} \longrightarrow N_1 \longrightarrow N_2 \longrightarrow 0, \\ &\vdots \\ 0 &\longrightarrow R/\mathfrak{p} \longrightarrow N_{t-2} \longrightarrow N_{t-1} \longrightarrow 0, \\ 0 &\longrightarrow R/\mathfrak{p} \longrightarrow N_{t-1} \longrightarrow N_t \longrightarrow 0. \end{aligned}$$

be a series of short exact sequences of finitely generated R -modules. Let y be an element of R such that $yN_t = 0$.

- (i) *If there is an element x of R such that $xH_{\mathfrak{m}}^i(M) = 0$ for $i < d$, then $(xy)^{t^{d-1}}H_{\mathfrak{m}}^i(R/\mathfrak{p}) = 0$ for $i < d$.*

- (ii) If there is an element x of R such that $xH_{\mathfrak{m}}^i(R/\mathfrak{p}) = 0$ for $i < d$, then $x^t y H_{\mathfrak{m}}^i(M) = 0$ for $i < d$.

Now, we are able to present the module version of Theorem 2.2.1.

Proposition 2.2.4. *Let M be a finitely generated R -module. Then the following conditions are equivalent.*

- (i) M has a uniform local cohomological annihilator.
(ii) M is locally equidimensional and R/\mathfrak{p} has a uniform local cohomological annihilator for all $\mathfrak{p} \in \text{Min } M$.

Proof. (i) \Rightarrow (ii). By Proposition 2.2.2, M is locally equidimensional. Assume that $\mathfrak{p} \in \text{Min } M$ and that \mathfrak{m} is a maximal ideal containing \mathfrak{p} . As $M_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$ -module of finite length, we set $t := l_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Then there exists a chain of submodules

$$N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_t \subseteq M,$$

such that $N_0 \cong N_i/N_{i-1} \cong R/\mathfrak{p}$, for $1 \leq i \leq t$, and

$$\begin{aligned} 0 &\longrightarrow R/\mathfrak{p} \longrightarrow M \longrightarrow M/N_0 \longrightarrow 0, \\ 0 &\longrightarrow R/\mathfrak{p} \longrightarrow M/N_0 \longrightarrow M/N_1 \longrightarrow 0, \\ &\vdots \\ 0 &\longrightarrow R/\mathfrak{p} \longrightarrow M/N_{t-2} \longrightarrow M/N_{t-1} \longrightarrow 0, \\ 0 &\longrightarrow R/\mathfrak{p} \longrightarrow M/N_{t-1} \longrightarrow M/N_t \longrightarrow 0, \end{aligned}$$

are exact sequences. Since $M_{\mathfrak{m}}$ is equidimensional, $\text{ht}_M \mathfrak{m}/\mathfrak{p} = \text{ht}_M \mathfrak{m}$. As, by definition of t , $l_{R_{\mathfrak{p}}}((M/N_t)_{\mathfrak{p}}) = 0$, it follows that $0 :_R (M/N_t) \not\subseteq \mathfrak{p}$. Choose an element $y \in 0 :_R (M/N_t) \setminus \mathfrak{p}$. Localizing the above exact sequences at \mathfrak{m} implies the following exact sequences.

$$\begin{aligned} 0 &\longrightarrow (R/\mathfrak{p})_{\mathfrak{m}} \longrightarrow M_{\mathfrak{m}} \longrightarrow (M/N_0)_{\mathfrak{m}} \longrightarrow 0, \\ 0 &\longrightarrow (R/\mathfrak{p})_{\mathfrak{m}} \longrightarrow (M/N_0)_{\mathfrak{m}} \longrightarrow (M/N_1)_{\mathfrak{m}} \longrightarrow 0, \\ &\vdots \\ 0 &\longrightarrow (R/\mathfrak{p})_{\mathfrak{m}} \longrightarrow (M/N_{t-2})_{\mathfrak{m}} \longrightarrow (M/N_{t-1})_{\mathfrak{m}} \longrightarrow 0, \\ 0 &\longrightarrow (R/\mathfrak{p})_{\mathfrak{m}} \longrightarrow (M/N_{t-1})_{\mathfrak{m}} \longrightarrow (M/N_t)_{\mathfrak{m}} \longrightarrow 0. \end{aligned}$$

By assumption, there is an element $x \in R \setminus \bigcup_{\mathfrak{q} \in \text{Min } M} \mathfrak{q}$ such that $xH_{\mathfrak{m}R_{\mathfrak{m}}}^i(M_{\mathfrak{m}}) = 0$ for all $i < \text{ht}_M \mathfrak{m}$. Now, by Lemma 2.2.3, we have $(xy)^l H_{\mathfrak{m}}^i(A/\mathfrak{p})_{\mathfrak{m}} = 0$ for all $i < \text{ht}_M \mathfrak{m}$ and for some integer $l > 0$.

- (ii) \Rightarrow (i). One may use a similar method as above to get the result. \square

The following result is an easy application of the above proposition which shows that the property that a module M has a uniform local cohomological annihilator is independent of the module structure and depends only on the support of M .

Corollary 2.2.5. *Let M and N be finitely generated R -modules of finite dimensions such that $\text{Supp } M = \text{Supp } N$. Then M has a uniform local cohomological annihilator if and only if N has a uniform local cohomological annihilator.*

Corollary 2.2.6. *Let M be a finitely generated R -module that has a uniform local cohomological annihilator. Then $R/0 :_R M$ is universally catenary.*

Proof. It follows by Corollary 2.2.5 and [37, Theorem 2.1]. \square

Recall that for a finitely generated R -module M over a local ring (R, \mathfrak{m}) ,

$$a(M) = \bigcap_{i < \dim M} (0 :_R H_{\mathfrak{m}}^i(M)).$$

Note that by definition, an R -module M has a uniform local cohomological annihilator if and only if $a(M) \not\subseteq \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$. On the other hand if $\mathfrak{p} \in \text{Min } M$, then $a(M) \subseteq \mathfrak{p}$ if and only if $\mathfrak{p} \in \text{Att } H_{\mathfrak{m}}^i(M)$ for some $i < \dim M$. More precisely we have the following result.

Lemma 2.2.7. *Assume that (R, \mathfrak{m}) is local and that M is a finitely generated R -module of dimension d . Then M has a uniform local cohomological annihilator if and only if $\text{Att } H_{\mathfrak{m}}^i(M) \cap \text{Min } M = \emptyset$ for all $i = 0, \dots, d - 1$.*

Proof. Assume that M has a uniform local cohomological annihilator. Therefore there is an element $x \in R \setminus \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$ satisfying $xH_{\mathfrak{m}}^i(M) = 0$ for all $i = 0, \dots, d - 1$. Thus, by Corollary 1.1.1(iii), $x \in \bigcap_{\mathfrak{q} \in \text{Att } H_{\mathfrak{m}}^i(M)} \mathfrak{q}$ for all $0 \leq i \leq d - 1$. Now the claim is clear.

Conversely, note that if $a(M) \subseteq \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$, then $a(M) \subseteq \mathfrak{p}$ for some prime ideal $\mathfrak{p} \in \text{Min } M$, by prime avoidance theorem. Therefore, $0 :_R H_{\mathfrak{m}}^i(M) \subseteq \mathfrak{p}$ for some $0 \leq i \leq d - 1$. On the other hand one has $0 :_R M \subseteq 0 :_R H_{\mathfrak{m}}^i(M) \subseteq \mathfrak{p}$ which gives $\mathfrak{p} \in \text{Supp } M$. Thus $\mathfrak{p} \in \text{Min } (R/0 :_R H_{\mathfrak{m}}^i(M))$ and so $\mathfrak{p} \in \text{Att } H_{\mathfrak{m}}^i(M)$ by Corollary 1.1.1(ii), which contradicts our assumption. Hence $a(M) \not\subseteq \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$ and the result follows. \square

The following lemma provides a relation between prime ideals containing $a(M)$ and those \mathfrak{p} which $M_{\mathfrak{p}}$ is not Cohen-Macaulay.

Lemma 2.2.8. *Assume that (R, \mathfrak{m}) is a local ring, M is a finitely generated R -module and $\mathfrak{p} \in \text{Supp } M$ such that $M_{\mathfrak{p}}$ is not Cohen-Macaulay. Then $a(M) \subseteq \mathfrak{p}$.*

Proof. Let $d := \dim M$ and assume contrarily that $\mathfrak{a}(M) \not\subseteq \mathfrak{p}$. Then by Theorem 1.2.13, we have

$$d = f_{\mathfrak{m}}^{\mathfrak{a}(M)}(M) \leq \lambda_{\mathfrak{m}}^{\mathfrak{a}(M)}(M) \leq \text{depth } M_{\mathfrak{p}} + \dim R/\mathfrak{p} \leq \dim M_{\mathfrak{p}} + \dim R/\mathfrak{p} \leq d.$$

Hence $\text{depth } M_{\mathfrak{p}} = \dim M_{\mathfrak{p}}$ which is a contradiction. \square

As an immediate corollary of the above lemma, we have the following result which Zhou has also proved it, in [37, Corollary 2.3], for $M = R$ and by a different method.

Corollary 2.2.9. *Assume that x is a uniform local cohomological annihilator of a finitely generated R -module M . Then M_x is a Cohen-Macaulay R_x -module.*

Proof. Let \mathfrak{m} be a maximal ideal of R with $\dim M_{\mathfrak{m}} > 0$. Since $x \in \mathfrak{a}(M_{\mathfrak{m}})$, for any prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ with $x \notin \mathfrak{p}$, we have $M_{\mathfrak{p}}$ is Cohen-Macaulay by Lemma 2.2.8. \square

Another property of rings which contain uniform local cohomological annihilators is a result of Zhou [37, Theorem 2.2] which proves that if x is a uniform local cohomological annihilator of R , then a power of x is a strong uniform local cohomological annihilator of R . Using the above result and our approach to uniform annihilators of local cohomology, we are able to recover this result in special case when R is local, by a different method. Before that, we mention the following well known fact.

Lemma 2.2.10. [24, Theorem 31.7] *Assume that (R, \mathfrak{m}) is universally catenary local ring and M is finitely generated and equidimensional. Then \widehat{M} is equidimensional \widehat{R} -module.*

Proposition 2.2.11. *Assume that (R, \mathfrak{m}) is a local ring, M is a finitely generated R -module and x is a uniform local cohomological annihilator of M , then a power of x is a strong uniform local cohomological annihilator of M .*

Proof. Let $d = \dim M_{\mathfrak{m}}$. Note that M is equidimensional by Proposition 2.2.2 and $R/0 :_{\widehat{R}} \widehat{M}$ is universally catenary by Corollary 2.2.6. Thus Lemma 2.2.10, implies that \widehat{M} is equidimensional.

Let $\mathfrak{p} \in \text{Supp } M$ with $r = \text{ht}_M \mathfrak{p}$. We may choose elements x_1, \dots, x_r in \mathfrak{p} such that $\text{ht}_M(x_1, \dots, x_r) = r$ and $\dim R/(x_1, \dots, x_r) = d - r$. Set $I = (x_1, \dots, x_r)$, then $\dim \widehat{R}/I\widehat{R} = d - r$ and so $\text{ht}_{\widehat{M}}(I\widehat{R}) = r$.

Note that $\mathcal{C}_{\widehat{R}}(\widehat{M})$ is finite, by Corollary 2.1.6 and \widehat{M}_x is Cohen-Macaulay by Corollary 2.2.9, which means that $\mathcal{H}_{\widehat{M}_x}^i = (\mathcal{H}_{\widehat{M}}^i)_x = 0$ for all $i \geq -1$. Since $\mathcal{H}_{\widehat{M}}^i$ is finitely generated \widehat{R} -module, there exists a positive integer n such that $x^n \in \cap_{i \geq -1} (0 :_{\widehat{R}} \mathcal{H}_{\widehat{M}}^i)$ (note that $\mathcal{H}_{\widehat{M}}^i = 0$ for $i \geq d - 1$).

Now, Corollary 2.1.8 implies that $x^{nd} \in 0 :_{\widehat{R}} \mathcal{H}_{I\widehat{R}}^i(\widehat{M})$ for $i < \text{ht}_{\widehat{M}}(I\widehat{R}) = r$. Hence $x^{nd}\mathcal{H}_I^i(M) = 0$ for $i < r = \text{ht}_M I$ and the result follows by the fact that $H_{IR_{\mathfrak{p}}}^i(M_{\mathfrak{p}}) \cong H_{\mathfrak{p}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$. \square

The following theorem gives a characterization for a finitely generated module M over a local ring to have a uniform local cohomological annihilator in terms of the existence of a specific parameter element of M . In proving (ii) \implies (i) of this theorem, A. Talemi–Eshmanani had a fruitful cooperation.

Theorem 2.2.12. *Let (R, \mathfrak{m}) be local and M be a finitely generated R -module with dimension $d = \dim M > 1$. Then the following statements are equivalent.*

- (i) M has a uniform local cohomological annihilator.
- (ii) $R/0 :_R M$ is catenary and equidimensional, there exists a parameter element x of M such that $\text{Min}(M/xM) \cap \text{Ass } M = \emptyset$ and all modules $M/x^t M$, $t \in \mathbb{N}$, have a common uniform local cohomological annihilator.

Proof. In the following argument we fix N to be a submodule of M such that $\text{Ass } N = \text{Min } M$ and $\text{Ass } M/N = \text{Ass } M \setminus \text{Min } M$ (see [1, Page 263, Proposition 4] for existence of N).

(i) \implies (ii) By Proposition 2.2.2, M is equidimensional and $R/0 :_R M$ is catenary by Corollary 2.2.6. Set $X = \{\mathfrak{p} \in \text{Ass } M : \text{ht}_M \mathfrak{p} = 1\}$. It can be easily checked that $X = \{\mathfrak{p} \in \text{Supp } M/N : \text{ht}_M \mathfrak{p} = 1\}$ and it is a subset of $\text{Min } M/N$ so that X is a finite set. Assume that r is an element of $R \setminus \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$ which is a uniform local cohomological annihilator of M . If r is unit element then M is Cohen–Macaulay and the claim follows by choosing x to be a non–zero–divisor on M .

Therefore we assume that $r \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$, so that $\dim M/rM = d - 1$. Note that, as M is equidimensional and $\dim M > 1$, $\mathfrak{m} \notin \text{Min } M$, $\mathfrak{m} \notin X$ and $\mathfrak{m} \notin \text{Min } M/rM$. Hence there exist

$$x \in \mathfrak{m} \setminus \left(\bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p} \right) \cup \left(\bigcup_{\mathfrak{p} \in X} \mathfrak{p} \right) \cup \left(\bigcup_{\mathfrak{p} \in \text{Min } M/rM} \mathfrak{p} \right). \quad (2.2.1)$$

It follows that $\text{Min } M/xM \cap \text{Ass } M = \emptyset$.

We claim that $r \notin \bigcup_{\mathfrak{p} \in \text{Min } M/xM} \mathfrak{p}$. Otherwise $r \in \mathfrak{p}$ for some $\mathfrak{p} \in \text{Min } M/xM$ and so $\text{ht}_M \mathfrak{p} = 1$ which implies that $\mathfrak{p} \in \text{Min } M/rM$. This contradicts with the chosen x in (2.2.1).

Next we claim that $0 :_R M/N \not\subseteq \cup_{\mathfrak{p} \in \text{Min } M/xM} \mathfrak{p}$. Otherwise $0 :_R M/N \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \text{Min } M/xM$. Thus $\mathfrak{p} \in \text{Supp } M/N$ and $\text{ht}_M \mathfrak{p} = 1$ which shows that $\mathfrak{p} \in X$. This also contradicts with (2.2.1). As a result there exists an element $s \in 0 :_R M/N \setminus \cup_{\mathfrak{p} \in \text{Min } M/xM} \mathfrak{p}$. Now consider the induced exact sequences

$$\mathbf{H}_m^{i-1}(M/N) \longrightarrow \mathbf{H}_m^i(N) \longrightarrow \mathbf{H}_m^i(M).$$

As $s\mathbf{H}_m^{i-1}(M/N) = 0$ for all i , and $r\mathbf{H}_m^i(M) = 0$ for all $i < d$, we get $rs\mathbf{H}_m^i(N) = 0$ for all $i < d$.

Now, for the module N , we have $\text{Ass } N = \text{Min } M$ and so, by (2.2.1), each of the elements x^t is a non-zero-divisor on N . Choose an arbitrary positive integer t and consider the exact sequence

$$0 \longrightarrow N \xrightarrow{x^t} N \longrightarrow N/x^t N \longrightarrow 0$$

which induces the exact sequence

$$\mathbf{H}_m^i(N) \longrightarrow \mathbf{H}_m^i(N/x^t N) \longrightarrow \mathbf{H}_m^{i+1}(N).$$

As $rs\mathbf{H}_m^i(N) = 0$ for $i < d$, we get $(rs)^2\mathbf{H}_m^i(N/x^t N) = 0$ for all $i < d - 1$.

On the other hand the exact sequence

$$0 \longrightarrow N/x^t N \longrightarrow M/x^t N \longrightarrow M/N \longrightarrow 0$$

implies the exact sequences

$$\mathbf{H}_m^i(N/x^t N) \longrightarrow \mathbf{H}_m^i(M/x^t N) \longrightarrow \mathbf{H}_m^i(M/N)$$

from which it follows that $(rs)^2s\mathbf{H}_m^i(M/x^t N) = 0$ for all $i < d - 1$.

Finally, the exact sequence

$$0 \longrightarrow x^t M/x^t N \longrightarrow M/x^t N \longrightarrow M/x^t M \longrightarrow 0$$

implies the exact sequences

$$\mathbf{H}_m^i(M/x^t N) \longrightarrow \mathbf{H}_m^i(M/x^t M) \longrightarrow \mathbf{H}_m^{i+1}(x^t M/x^t N).$$

Note that $s(M/N) = 0$ implies that $s\mathbf{H}_m^i(x^t M/x^t N) = 0$ for all i . Therefore

$$(rs)^2s^2\mathbf{H}_m^i(M/x^t M) = 0,$$

for all $i < d - 1$.

As $r, s \in R \setminus \bigcup_{\mathfrak{p} \in \text{Min } M/xM} \mathfrak{p}$ and $\dim M/x^t M = d - 1$, the element $r^2 s^4$ is a uniform local cohomological annihilator for $M/x^t M$.

(ii) \Rightarrow (i). Assume that all modules $M/x^t M$, $t \in \mathbb{N}$, have a common uniform local cohomological annihilator r , say. We first observe that

$$0 :_R M/N \not\subseteq \bigcup_{\mathfrak{p} \in \text{Min } M/xM} \mathfrak{p}.$$

Otherwise there is a prime ideal $\mathfrak{p} \in \text{Min } M/xM$ which $0 :_R M/N \subseteq \mathfrak{p}$. As $\dim M/N \leq d - 1$ and $\text{ht}_M \mathfrak{p} = 1$, we find that $\mathfrak{p} \in \text{Min } M/N$ and so $\mathfrak{p} \in \text{Ass } M$ which contradicts our assumption $\text{Min } M/xM \cap \text{Ass } M = \emptyset$. As a result there is an element

$$s \in 0 :_R M/N \setminus \bigcup_{\mathfrak{p} \in \text{Min } M/xM} \mathfrak{p}.$$

Consider an arbitrary positive integer t . From the exact sequence

$$0 \longrightarrow x^t M/x^t N \longrightarrow M/x^t N \longrightarrow M/x^t M \longrightarrow 0$$

it follows the induced exact sequences

$$H_{\mathfrak{m}}^i(x^t M/x^t N) \longrightarrow H_{\mathfrak{m}}^i(M/x^t N) \longrightarrow H_{\mathfrak{m}}^i(M/x^t M).$$

Since $s(M/N) = 0$ and r is a uniform local cohomological annihilator of $H_{\mathfrak{m}}^i(M/x^t M)$, it follows that $rsH_{\mathfrak{m}}^i(M/x^t N) = 0$ for all $i < d - 1$.

On the other hand, the exact sequence $0 \longrightarrow N/x^t N \longrightarrow M/x^t N \longrightarrow M/N \longrightarrow 0$ implies the exact sequences

$$H_{\mathfrak{m}}^{i-1}(M/N) \longrightarrow H_{\mathfrak{m}}^i(N/x^t N) \longrightarrow H_{\mathfrak{m}}^i(M/x^t N)$$

from which it follows that $rs^2 H_{\mathfrak{m}}^i(N/x^t N) = 0$ for all $i < d - 1$.

From the fact that M is catenary and equidimensional and that x is a parameter element of M it follows that $\bigcup_{\mathfrak{p} \in \text{Min } N} \mathfrak{p} \subseteq \bigcup_{\mathfrak{q} \in \text{Min } M/xM} \mathfrak{q}$ and so $rs^2 \notin \bigcup_{\mathfrak{p} \in \text{Min } N} \mathfrak{p}$.

Our next step is to show that $rs^2 H_{\mathfrak{m}}^i(N) = 0$ for all $i < d$. Let $i < d$ and choose an arbitrary element $\alpha \in H_{\mathfrak{m}}^i(N)$. By torsionness of local cohomology modules, there is a positive integer t such that $\alpha \in (0 :_{H_{\mathfrak{m}}^i(N)} x^t)$. As $\text{Ass } N = \text{Min } M$ and $x^t \notin \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$, x^t

is a non-zero-divisor on N . Thus the exact sequence $0 \longrightarrow N \xrightarrow{x^t} N \longrightarrow N/x^t N \longrightarrow 0$ implies the exact sequence $H_{\mathfrak{m}}^{i-1}(N/x^t N) \longrightarrow H_{\mathfrak{m}}^i(N) \xrightarrow{x^t} H_{\mathfrak{m}}^i(N)$ and also the exact sequence

$$H_{\mathfrak{m}}^{i-1}(N/x^t N) \longrightarrow (0 :_{H_{\mathfrak{m}}^i(N)} x^t) \longrightarrow 0.$$

As $rs^2H_m^{i-1}(N/x^tN) = 0$, we obtain that $rs^2(0 :_{H_m^i(N)} x^t) = 0$. In particular, $rs^2\alpha = 0$. Therefore, as $r, s \notin \bigcup_{\mathfrak{p} \in \text{Min } N} \mathfrak{p}$, rs^2 is a uniform local cohomological annihilator of N . Since $\text{Supp } M = \text{Supp } N$, M has a uniform local cohomological annihilator, by Corollary 2.2.5. \square

Now we introduce an important class of modules which have uniform local cohomological annihilators. This is the class of modules with finite Cousin complexes. In next chapter, we will show that these two classes coincide, if all formal fibres of R are Cohen-Macaulay.

Theorem 2.2.13. *Assume that M is a finitely generated R -module of finite $\dim M = d$ and that $\mathcal{C}_R(M)$ is finite, then M has a uniform local cohomological annihilator.*

Proof. By Lemma 1.3.10, there exists an element $x \in \bigcap_{i \geq -1} \mathcal{H}_M^i \setminus \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$. Now x^d is a uniform local cohomological annihilator of M , by Corollary 2.1.8. \square

In Theorem 2.2.12, we have shown that if M has a uniform local cohomological annihilator, then there exists a parameter element x of M which M/xM has also a uniform local cohomological annihilator. Now, when $\mathcal{C}_R(M)$ is finite, we may show this property for all parameter elements of M . Firstly we see the result for each non-zero-divisor x of M .

Proposition 2.2.14. *Let (R, \mathfrak{m}) be local ring and let M be a finitely generated R -module with $\dim M = d > 1$ such that $\mathcal{C}_R(M)$ is finite. Then, for each non-zero-divisor x of M , the quotient module M/xM has a uniform local cohomological annihilator. Moreover, $\text{Min}(M/xM) \cap \text{Ass } M = \emptyset$ and all modules M/x^tM , $t \in \mathbb{N}$, have a common uniform local cohomological annihilator.*

Proof. Let $\mathfrak{p} \in \text{Min } M/xM$. Then $\text{ht}_M \mathfrak{p} = 1$ and so $\mathfrak{p} \notin \text{Supp } \mathcal{H}_M^i$ for all $i \geq 0$ by Theorem 1.3.9(ii). If $\mathfrak{p} \in \text{Supp } \mathcal{H}_M^{-1}$ then $\mathfrak{p} \in \text{Min } \mathcal{H}_M^{-1}$ and so $\mathfrak{p} \in \text{Ass } M$ which contradicts with the fact that x is a non-zero-divisor on M . Hence

$$\bigcap_{i \geq -1} (0 :_R \mathcal{H}_M^i) \not\subseteq \bigcup_{\mathfrak{p} \in \text{Min } M/xM} \mathfrak{p}.$$

Therefore, there is an element $r \in \bigcap_{i \geq -1} (0 :_R \mathcal{H}_M^i) \setminus \bigcup_{\mathfrak{p} \in \text{Min } M/xM} \mathfrak{p}$. Now, from the exact sequence $0 \rightarrow M \rightarrow M \rightarrow M/xM \rightarrow 0$ we get the exact sequence

$$\cdots \rightarrow H_m^i(M) \rightarrow H_m^i(M/xM) \rightarrow H_m^{i+1}(M) \rightarrow \cdots$$

By Corollary 2.1.8, $rH_m^i(M) = 0$ for all $i < d$. Then the above exact sequence implies that $r^2H_m^i(M/xM) = 0$ for all $i < d - 1$. As $r \in \bigcup_{\mathfrak{p} \in \text{Min } M/xM} \mathfrak{p}$, r^2 is a uniform local

cohomological annihilator of $M/x^n M$ for all positive integers n and $\text{Min } M/xM \cap \text{Ass } M = \emptyset$. \square

Theorem 2.2.15. *Let (R, \mathfrak{m}) be a local ring and let M be a finitely generated R -module with finite Cousin complex. Then M/xM has a uniform local cohomological annihilator for any parameter element x of M .*

Proof. There is a submodule N of M such that $\text{Ass } M/N = \text{Min } M$ and $\text{Ass } N = \text{Ass } M \setminus \text{Min } M$ (c.f. [1, Page 263, Proposition 4]). As in the exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ we have $\text{ht}_M \mathfrak{p} \geq 1$ for all $\mathfrak{p} \in \text{Supp } N$, Lemma 2.1.1(b) implies that $\mathcal{C}_R(M/N)$ is finite. Assume that x is parameter element of M . As x is a non-zero-divisor on M/N , Proposition 2.2.14 implies that $(M/N)/x(M/N)$ has a uniform local cohomological annihilator. Note that $\text{Supp } M/xM = \text{Supp } (M/N)/x(M/N)$ so, by Corollary 2.2.5, M/xM has a uniform local cohomological annihilator. \square

2.3 Applications

Our study of properties of Cousin cohomologies in Section 2.1, provides a new approach to the property of uniform annihilators of local cohomologies. This point of view enabled us to study both classes of modules, more deeply in section 2.2. In this section we present some more applications to characterize modules with finite cousin complexes (over rings with some extra conditions) and also to investigate a new formula for height of an ideal in terms of cohomologies of Cousin complexes.

2.3.1 Partial characterizations

We start with the following result which has an essential role in our approach.

Corollary 2.3.1. *Assume that M is a finitely generated R -module with finite dimension and that $\mathcal{C}_R(M)$ is finite. Then M is locally equidimensional and $R/0 :_R M$ is universally catenary.*

Proof. It is clear from Theorem 2.2.13, Proposition 2.2.2 and Corollary 2.2.6. \square

Now it is easy to provide an example of a module whose Cousin complex has at least one non-finitely generated cohomology.

Example. Consider a noetherian local ring R of dimension $d > 2$. Choose any pair of prime ideals \mathfrak{p} and \mathfrak{q} of R with conditions $\dim R/\mathfrak{p} = 2$, $\dim R/\mathfrak{q} = 1$, and $\mathfrak{p} \not\subseteq \mathfrak{q}$. Then $\text{Min } R/\mathfrak{p}\mathfrak{q} = \{\mathfrak{p}, \mathfrak{q}\}$ and so $R/\mathfrak{p}\mathfrak{q}$ is not an equidimensional R -module and thus its Cousin

complex is not finite.

In [37, Corollary 3.3], Zhou proves that any locally equidimensional noetherian ring has a uniform local cohomological annihilator provided it is a homomorphic image of a Cohen-Macaulay ring of finite dimension. Note that any homomorphic image of a Cohen-Macaulay ring is universally catenary and all of its formal fibres are Cohen-Macaulay. Here we extend this result to any universally catenary local ring with Cohen-Macaulay formal fibres, by showing that over these rings, every equidimensional module has finite Cousin complex, which also recovers the result of Kawasaki, [20, Theorem 5.5], by a different method.

Proposition 2.3.2. *Assume that R is universally catenary and all formal fibres of R are Cohen-Macaulay. If M is a finitely generated and equidimensional R -module, then $\mathcal{C}_R(M)$ is finite; in particular M has a uniform local cohomological annihilator.*

Proof. By Lemma 2.2.10, \widehat{M} is equidimensional, so Corollary 2.1.6 implies the result. \square

We are now ready to present the following result which, for a finitely generated R -module M , shows connections of finiteness of its Cousin complex, existence of a uniform local cohomological annihilator for M , and equidimensionality of \widehat{M} .

Theorem 2.3.3. *Assume that (R, \mathfrak{m}) is local and all formal fibers of R are Cohen-Macaulay. Then the following statements are equivalent for a finitely generated R -module M .*

- (i) \widehat{M} is an equidimensional \widehat{R} -module.
- (ii) The Cousin complex of M is finite.
- (iii) M has a uniform local cohomological annihilator.

Proof. (i) \Rightarrow (ii). This is Corollary 2.1.6.

(ii) \Rightarrow (iii). This is Theorem 2.2.13.

(iii) \Rightarrow (i). There exists an element $x \in R \setminus \cup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$ such that $xH_{\mathfrak{m}}^i(M) = 0$ for all $i < \dim M$, so that $xH_{\mathfrak{m}}^i(\widehat{M}) = 0$ for all $i < \dim \widehat{M}$. Now \widehat{M} is equidimensional by Proposition 2.2.2. \square

2.3.2 Height of an ideal

We use some results about the annihilators of cohomologies of Cousin complexes, to present the height of an ideal in terms of Cousin complexes.

As mentioned in Corollary 2.1.8, we may write the following result.

Corollary 2.3.4. *For any finitely generated R -module M and any ideal \mathfrak{a} of R with $\mathfrak{a}M \neq M$,*

$$\prod_{-1 \leq i} (0 :_R \mathcal{H}_M^i) \subseteq 0 :_R H_{\mathfrak{a}}^{\text{ht}_M \mathfrak{a} - 1}(M).$$

We now raise the question that whether it is possible to improve the upper bound restriction.

Question. Does the inequality

$$\prod_{-1 \leq i} (0 :_R \mathcal{H}_M^i) \subseteq 0 :_R H_{\mathfrak{a}}^{\text{ht}_M \mathfrak{a}}(M)$$

hold?

It will be proved that the answer is negative for the class of finitely generated R -modules with finite Cousin cohomologies. More precisely,

Theorem 2.3.5. *Assume that M is a finitely generated R -module of finite dimension and that its Cousin complex $\mathcal{C}_R(M)$ is finite. Then*

$$\text{ht}_M \mathfrak{a} = \inf \left\{ r : \prod_{-1 \leq i} (0 :_R \mathcal{H}_M^i) \not\subseteq 0 :_R H_{\mathfrak{a}}^r(M) \right\},$$

for all ideals \mathfrak{a} with $\mathfrak{a}M \neq M$.

Proof. $\prod_{i \geq -1} (0 :_R \mathcal{H}_M^i) \subseteq 0 :_R H_{\mathfrak{a}}^r(M)$ for all $r < \text{ht}_M \mathfrak{a}$. Hence we have

$$\text{ht}_M \mathfrak{a} \leq \inf \left\{ r : \prod_{-1 \leq i} (0 :_R \mathcal{H}_M^i) \not\subseteq 0 :_R H_{\mathfrak{a}}^r(M) \right\}.$$

Thus it is sufficient to show that $\prod_{-1 \leq i} (0 :_R \mathcal{H}_M^i) \not\subseteq 0 :_R H_{\mathfrak{a}}^{\text{ht}_M \mathfrak{a}}(M)$. By Independence Theorem of local cohomology, $H_{\mathfrak{a}}^{\text{ht}_M \mathfrak{a}}(M) \cong H_{\mathfrak{b}}^{\text{ht}_M \mathfrak{b}}(M)$ as $\overline{R} = R/0 :_R M$ -module, where $\mathfrak{b} = (\mathfrak{a} + 0 :_R M)/0 :_R M$. Note that $\text{ht}_M \mathfrak{a} = \text{ht}_M \mathfrak{b}$ and that $\mathcal{C}_R(M) \cong \mathcal{C}_{\overline{R}}(M)$ by Lemma 1.3.5.

Hence we may assume that $0 :_R M = 0$. Set $h := \text{ht}_M \mathfrak{a}$. Let $x \in 0 :_R H_{\mathfrak{a}}^h(M)$. As $\mathfrak{a}M \neq M$, there exists a minimal prime \mathfrak{q} over \mathfrak{a} in $\text{Supp } M$ such that $\dim R_{\mathfrak{q}} = \text{ht}_M \mathfrak{a}$. Hence $x/1 \in 0 :_{R_{\mathfrak{q}}} H_{\mathfrak{q}R_{\mathfrak{q}}}^h(M_{\mathfrak{q}})$. Thus, by any choice of $\mathfrak{p}R_{\mathfrak{q}} \in \text{Assh } M_{\mathfrak{q}}$ we have $x/1 \in \mathfrak{p}R_{\mathfrak{q}}$ (see Theorem 1.2.4 and Corollary 1.1.1(iii)) and so $x \in \mathfrak{p}$. Therefore, one has

$$0 :_R H_{\mathfrak{a}}^h(M) \subseteq \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}.$$

On the other hand, by Lemma 1.3.10, $\prod_{i \geq -1} (0 :_R \mathcal{H}_M^i) \not\subseteq \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$, from which it follows that

$$\prod_{i \geq -1} (0 :_R \mathcal{H}_M^i) \not\subseteq 0 :_R H_{\mathfrak{a}}^h(M).$$

□

Chapter 3

Attached primes of local cohomology modules

Throughout this chapter (R, \mathfrak{m}) is a local ring and M is a finitely generated R -module of dimension d . Recall that M has a uniform local cohomological annihilators if and only if $\mathfrak{a}(M) \not\subseteq \cup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}$. On the other hand we have seen in Lemma 2.2.7, that if $\mathfrak{p} \in \text{Min } M$, then $\mathfrak{a}(M) \subseteq \mathfrak{p}$ if and only if $\mathfrak{p} \in \text{Att } H_{\mathfrak{m}}^i(M)$ for some $i < \dim M$. Inspired by these facts, we study $\text{Att } H_{\mathfrak{m}}^t(M)$ for certain t , in particular $\text{Att } H_{\mathfrak{m}}^{d-1}(M)$, in terms of cohomologies of $\mathcal{C}_R(M)$ and obtain a non-vanishing criterion of $H_{\mathfrak{m}}^{d-1}(M)$ when $\mathcal{C}_R(M)$ is finite, in section 3.1. We continue by study the attached prime ideals of the top local cohomology module $H_{\mathfrak{m}}^d(M)$ in the second section and present a positive answer to a question of [10], in the case when R is complete. The last section of this chapter is devoted to some applications of our results to find a new characterization of generalized Cohen-Macaulay modules.

3.1 Attached primes related to cohomologies of Cousin complexes

In this section we study some relations between the set of attached primes of local cohomology modules $H_{\mathfrak{m}}^i(M)$ and those of cohomologies of the Cousin complex of M .

The following result describes the situation when all cohomology modules \mathcal{H}_M^i of the Cousin complex of M are local cohomology modules of M .

Lemma 3.1.1. *Assume that i is an integer with $0 \leq i < d$. The following statements are equivalent.*

- (i) $\dim \mathcal{H}_M^j \leq 0$ for all j with $-1 \leq j < i$.
- (ii) $H_{\mathfrak{m}}^{j+1}(M) \cong \mathcal{H}_M^j$ for all j with $-1 \leq j < i$.

Proof. Assume that s is an integer such that $0 \leq s < d$ and $\dim \mathcal{H}_M^{s-1} \leq 0$. Considering the exact sequence (1.3.1) with $i = s$, gives the exact sequence

$$\mathrm{H}_\mathfrak{m}^{t-1}(M^s) \longrightarrow \mathrm{H}_\mathfrak{m}^{t-1}(M^s/D^s) \longrightarrow \mathrm{H}_\mathfrak{m}^t(M^{s-1}/K^{s-1}) \longrightarrow \mathrm{H}_\mathfrak{m}^t(M^s)$$

for all integers t . As $s < d$, by Lemma 1.3.4 (a), we get

$$\mathrm{H}_\mathfrak{m}^{t-1}(M^s/D^s) \cong \mathrm{H}_\mathfrak{m}^t(M^{s-1}/K^{s-1}). \quad (3.1.1)$$

Next consider the exact sequence (1.3.2) with $i = s$ which gives the exact sequence

$$\mathrm{H}_\mathfrak{m}^t(\mathcal{H}_M^{s-1}) \longrightarrow \mathrm{H}_\mathfrak{m}^t(M^{s-1}/D^{s-1}) \longrightarrow \mathrm{H}_\mathfrak{m}^t(M^{s-1}/K^{s-1}) \longrightarrow \mathrm{H}_\mathfrak{m}^{t+1}(\mathcal{H}_M^{s-1}).$$

Choosing $t > 0$ in the above exact sequence we obtain

$$\mathrm{H}_\mathfrak{m}^t(M^{s-1}/D^{s-1}) \cong \mathrm{H}_\mathfrak{m}^t(M^{s-1}/K^{s-1}). \quad (3.1.2)$$

As a consequence, from (3.1.1) and (3.1.2), we get

$$\mathrm{H}_\mathfrak{m}^t(M^{s-1}/D^{s-1}) \cong \mathrm{H}_\mathfrak{m}^{t-1}(M^s/D^s) \quad (3.1.3)$$

for all $t > 0$.

(i) \Rightarrow (ii). Let $-1 \leq j < i$. By repeated use of (3.1.3), we get

$$\mathrm{H}_\mathfrak{m}^{j+1}(M^{-1}/D^{-1}) \cong \mathrm{H}_\mathfrak{m}^0(M^j/D^j).$$

From the exact sequence (1.3.1) with $i = j + 1$ we have $\mathrm{H}_\mathfrak{m}^0(M^j/K^j) = 0$ (because $j + 1 \leq i < d$ and Lemma 1.3.4). Hence the exact sequence (1.3.2) with $i = j + 1$ implies that $\mathrm{H}_\mathfrak{m}^0(M^j/D^j) \cong \mathrm{H}_\mathfrak{m}^0(\mathcal{H}_M^j) \cong \mathcal{H}_M^j$. Therefore $\mathrm{H}_\mathfrak{m}^{j+1}(M) \cong \mathcal{H}_M^j$.

(ii) \Rightarrow (i) is clear. \square

Theorem 1.3.9, implies that $\dim \mathcal{H}_M^i \leq d - i - 2$ for all $i \geq -1$. The following lemma states some properties for t th local cohomology modules of M , whenever $\dim \mathcal{H}_M^i \leq t - i - 1$ for all $i \geq -1$, in particular for $\mathrm{H}_\mathfrak{m}^{d-1}(M)$.

Lemma 3.1.2. *Assume that $0 \leq t < d$ is an integer such that $\dim \mathcal{H}_M^i \leq t - i - 1$, for all $i \geq -1$. Then the following statements hold true.*

(i) $\mathrm{Att} \mathrm{H}_\mathfrak{m}^t(M) \subseteq \bigcup_{i=-1, \dots, t-1} \mathrm{Att} \mathrm{H}_\mathfrak{m}^{t-i-1}(\mathcal{H}_M^i).$

(ii) *There is an epimorphism $\mathrm{H}_\mathfrak{m}^t(M) \twoheadrightarrow \mathrm{H}_\mathfrak{m}^0(\mathcal{H}_M^{t-1})$.*

(iii) *Assume that $\mathcal{C}_A(M)$ is finite. Then \mathcal{H}_M^{t-1} is non-zero if and only if $\mathfrak{m} \in \mathrm{Att} \mathrm{H}_\mathfrak{m}^t(M)$.*

Proof. (i). We prove by induction on j , $-1 \leq j \leq t-1$, that

$$\text{Att } H_{\mathfrak{m}}^t(M) \subseteq \bigcup_{i \geq -1}^j \text{Att } H_{\mathfrak{m}}^{t-i-1}(\mathcal{H}_M^i) \bigcup \text{Att } H_{\mathfrak{m}}^{t-j-1}(M^j/K^j). \quad (3.1.4)$$

Due to $\dim \mathcal{H}_M^i \leq t-i-1$, the Grothendieck's vanishing theorem implies that $H_{\mathfrak{m}}^{t-i}(\mathcal{H}_M^i) = 0$. The exact sequence (1.3.2) with $l = 0$, implies the exact sequence $H_{\mathfrak{m}}^t(\mathcal{H}_M^{-1}) \rightarrow H_{\mathfrak{m}}^t(M) \rightarrow H_{\mathfrak{m}}^t(M^{-1}/K^{-1}) \rightarrow 0$. Thus we get

$$\text{Att } H_{\mathfrak{m}}^t(M) \subseteq \text{Att } H_{\mathfrak{m}}^t(\mathcal{H}_M^{-1}) \bigcup \text{Att } H_{\mathfrak{m}}^t(M^{-1}/K^{-1}).$$

Assume that $-1 \leq j < t-1$ and (3.1.4) holds. First note that (1.3.1) with $l = j+1$ implies the exact sequence

$$H_{\mathfrak{m}}^{t-j-2}(M^{j+1}) \rightarrow H_{\mathfrak{m}}^{t-j-2}(M^{j+1}/D^{j+1}) \rightarrow H_{\mathfrak{m}}^{t-j-1}(M^j/K^j) \rightarrow H_{\mathfrak{m}}^{t-j-1}(M^{j+1}).$$

As $-1 \leq j < t-1$, $H_{\mathfrak{m}}^{t-j-2}(M^{j+1}) = 0$ and $H_{\mathfrak{m}}^{t-j-1}(M^{j+1}) = 0$, by Lemma 1.3.4, therefore

$$H_{\mathfrak{m}}^{t-j-2}(M^{j+1}/D^{j+1}) \cong H_{\mathfrak{m}}^{t-j-1}(M^j/K^j). \quad (3.1.5)$$

On the other hand from the exact sequence (1.3.2) with $l = j+2$ we have the exact sequence

$$H_{\mathfrak{m}}^{t-j-2}(\mathcal{H}_M^{j+1}) \rightarrow H_{\mathfrak{m}}^{t-j-2}(M^{j+1}/D^{j+1}) \rightarrow H_{\mathfrak{m}}^{t-j-2}(M^{j+1}/K^{j+1}) \rightarrow H_{\mathfrak{m}}^{t-j-1}(\mathcal{H}_M^{j+1}). \quad (3.1.6)$$

As $H_{\mathfrak{m}}^{t-j-1}(\mathcal{H}_M^{j+1}) = 0$, (3.1.5) with the exact sequence (3.1.6) imply that

$$\begin{aligned} \text{Att } H_{\mathfrak{m}}^{t-j-1}(M^j/K^j) &= \text{Att } H_{\mathfrak{m}}^{t-j-2}(M^{j+1}/D^{j+1}) \\ &\subseteq \text{Att } H_{\mathfrak{m}}^{t-j-2}(\mathcal{H}_M^{j+1}) \bigcup \text{Att } H_{\mathfrak{m}}^{t-j-2}(M^{j+1}/K^{j+1}). \end{aligned} \quad (3.1.7)$$

Now, (3.1.7) and (3.1.4) complete the induction argument. Thus we have

$$\text{Att } H_{\mathfrak{m}}^t(M) \subseteq \bigcup_{i=-1,0,\dots,t-1} \text{Att } H_{\mathfrak{m}}^{t-i-1}(\mathcal{H}_M^i) \bigcup \text{Att } H_{\mathfrak{m}}^0(M^{t-1}/K^{t-1}).$$

On the other hand, considering the fact that $H_{\mathfrak{m}}^0(M^t) = 0$, it follows from the exact sequence (1.3.1) with $l = t$, that $H_{\mathfrak{m}}^0(M^{t-1}/K^{t-1}) = 0$.

(ii). Consider the exact sequence (1.3.1) with $l = t-i$ which imply the exact sequence

$$H_{\mathfrak{m}}^{i-1}(M^{t-i}) \rightarrow H_{\mathfrak{m}}^{i-1}(M^{t-i}/D^{t-i}) \rightarrow H_{\mathfrak{m}}^i(M^{t-i-1}/K^{t-i-1}) \rightarrow H_{\mathfrak{m}}^i(M^{t-i}).$$

Taking $0 \leq i \leq t$ we get $0 \leq t-i \leq t < d$ and so $H_{\mathfrak{m}}^i(M^{t-i}) = 0 = H_{\mathfrak{m}}^{i-1}(M^{t-i})$, by Lemma 1.3.4. Therefore we have isomorphisms

$$H_{\mathfrak{m}}^{i-1}(M^{t-i}/D^{t-i}) \cong H_{\mathfrak{m}}^i(M^{t-i-1}/K^{t-i-1}) \quad (3.1.8)$$

for all i , $0 \leq i \leq t$.

Consider the exact sequence (1.3.2) with $l = t - i$ which induces the exact sequence

$$H_{\mathfrak{m}}^i(\mathcal{H}_M^{t-i-1}) \longrightarrow H_{\mathfrak{m}}^i(M^{t-i-1}/D^{t-i-1}) \longrightarrow H_{\mathfrak{m}}^i(M^{t-i-1}/K^{t-i-1}) \longrightarrow H_{\mathfrak{m}}^{i+1}(\mathcal{H}_M^{t-i-1}).$$

As, by assumption $\dim \mathcal{H}_M^{t-i-1} \leq i$, we have $H_{\mathfrak{m}}^{i+1}(\mathcal{H}_M^{t-i-1}) = 0$ and so one obtains an epimorphism

$$H_{\mathfrak{m}}^i(M^{t-i-1}/D^{t-i-1}) \twoheadrightarrow H_{\mathfrak{m}}^i(M^{t-i-1}/K^{t-i-1}). \quad (3.1.9)$$

By successive use of (3.1.9) and (3.1.8) one obtains an epimorphism

$$H_{\mathfrak{m}}^t(M^{-1}/D^{-1}) \twoheadrightarrow H_{\mathfrak{m}}^0(M^{t-1}/D^{t-1}).$$

On the other hand, we have seen at the end of part (i) that, we have $H_{\mathfrak{m}}^0(M^{t-1}/K^{t-1}) = 0$. Therefore, from the exact sequence (1.3.2) with $l = t$, we get $H_{\mathfrak{m}}^0(\mathcal{H}_M^{t-1}) \cong H_{\mathfrak{m}}^0(M^{t-1}/D^{t-1})$ which results an epimorphism

$$H_{\mathfrak{m}}^t(M) \twoheadrightarrow H_{\mathfrak{m}}^0(\mathcal{H}_M^{t-1}). \quad (3.1.10)$$

(iii). Assume that $\mathcal{H}_M^{t-1} \neq 0$. As, by assumption $\dim \mathcal{H}_M^{t-1} \leq 0$, we have $H_{\mathfrak{m}}^0(\mathcal{H}_M^{t-1}) = \mathcal{H}_M^{t-1}$ and so $\text{Att } H_{\mathfrak{m}}^0(\mathcal{H}_M^{t-1}) = \{\mathfrak{m}\}$. Now (3.1.10) implies that $\mathfrak{m} \in \text{Att } H_{\mathfrak{m}}^t(M)$.

Conversely, assume that $\mathfrak{m} \in \text{Att } H_{\mathfrak{m}}^t(M)$. By part (i), $\mathfrak{m} \in \text{Att } H_{\mathfrak{m}}^{t-i-1}(\mathcal{H}_M^i)$ for some i , $-1 \leq i \leq t-1$, and thus $\dim \mathcal{H}_M^i \geq t-i-1$. As $\dim \mathcal{H}_M^i \leq t-i-1$ we have equality $\dim \mathcal{H}_M^i = t-i-1$. Note that \mathcal{H}_M^i is finitely generated and so $\mathfrak{m} \in \text{Assh } \mathcal{H}_M^i$ (see Theorem 1.2.4) from which it follows that $t-i-1 = 0$, i.e. $\mathcal{H}_M^{t-1} \neq 0$. \square

It is known that $\text{Att } H_{\mathfrak{m}}^d(M) = \text{Assh } M$ (Theorem 1.2.4). The following result provides some information about $\text{Att } H_{\mathfrak{m}}^t(M)$ for certain t , in particular for $t = d - 1$.

Proposition 3.1.3. *Assume that $C_R(M)$ is finite. Let t , $0 \leq t < d$, be an integer such that $\dim \mathcal{H}_M^i \leq t - i - 1$, for all $i \geq -1$. Then*

$$\text{Att } H_{\mathfrak{m}}^t(M) = \bigcup_{i=-1}^{t-1} \{\mathfrak{p} \in \text{Ass } \mathcal{H}_M^i : \dim R/\mathfrak{p} = t - i - 1\}.$$

Proof. Assume that $\mathfrak{p} \in \text{Att } H_{\mathfrak{m}}^t(M)$. Then $\mathfrak{p} \in \text{Att } H_{\mathfrak{m}}^{t-i-1}(\mathcal{H}_M^i)$ for some i , $-1 \leq i \leq t-1$ by Lemma 3.1.2(i). As $\dim \mathcal{H}_M^i \leq t-i-1$, we have the equality $\dim \mathcal{H}_M^i = t-i-1$ and so $\mathfrak{p} \in \text{Ass } \mathcal{H}_M^i$ and $\dim R/\mathfrak{p} = t-i-1$.

Conversely, assume that $-1 \leq i_0 \leq t-1$ and that $\mathfrak{p} \in \text{Ass } \mathcal{H}_M^{i_0}$ such that $\dim R/\mathfrak{p} = t - i_0 - 1$. Set $d' := \dim M_{\mathfrak{p}}$ and $t' := t - \dim R/\mathfrak{p}$.

As $\mathcal{C}_R(M)$ is finite, M is equidimensional and $\text{Supp } M$ is catenary by Proposition 2.2.2 and Corollary 2.2.6, we have $0 \leq t' < d'$. Note that $\mathcal{C}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cong (\mathcal{C}_A(M))_{\mathfrak{p}}$ so that $(\mathcal{H}_M^j)_{\mathfrak{p}} \cong \mathcal{H}_{M_{\mathfrak{p}}}^j$ for all j . As $\dim \mathcal{H}_{M_{\mathfrak{p}}}^j \leq \dim \mathcal{H}_M^j - \dim R/\mathfrak{p}$, we find that $\dim \mathcal{H}_{M_{\mathfrak{p}}}^j \leq t' - j - 1$ for all $j \geq -1$. On the other hand $\mathcal{H}_{M_{\mathfrak{p}}}^{t'-1} = (\mathcal{H}_M^{i_0})_{\mathfrak{p}} \neq 0$. Lemma 3.1.2(iii), replacing M by $M_{\mathfrak{p}}$ implies that $\mathfrak{p}R_{\mathfrak{p}} \in \text{Att } H_{\mathfrak{p}R_{\mathfrak{p}}}^{t'}(M_{\mathfrak{p}})$. Finally the Weak General Shifted Localization Principle (1.2.6), implies that $\mathfrak{p} \in \text{Att } H_{\mathfrak{m}}^{t'+\dim A/\mathfrak{p}}(M)$ that is $\mathfrak{p} \in \text{Att } H_{\mathfrak{m}}^t(M)$. \square

As a consequence of the above result, we may find a connection between vanishing of certain local cohomology modules and the dimensions of cohomologies of Cousin complex.

Corollary 3.1.4. *Assume that $\mathcal{C}_R(M)$ is finite. Let $l < d$ be an integer. The following statements are equivalent.*

- (i) $H_{\mathfrak{m}}^j(M) = 0$ for all j , $l < j < d$.
- (ii) $\dim \mathcal{H}_M^i \leq l - i - 1$ for all $i \geq -1$.

Proof. (i) \implies (ii). We prove it by descending induction on l . For $i = d - 1$ we have nothing to prove by Lemma 1.3.4. Assume that $l < d - 1$. We have, by induction hypothesis, that $\dim R/\mathfrak{p} \leq (l+1) - i - 1$ for all $\mathfrak{p} \in \text{Supp } \mathcal{H}_M^i$ and for all $i \geq -1$. If, for an ideal $\mathfrak{p} \in \text{Supp } \mathcal{H}_M^i$ and an integer i , $\dim R/\mathfrak{p} = (l+1) - i - 1$, then we get $\mathfrak{p} \in \text{Ass } \mathcal{H}_M^i$ and so $H_{\mathfrak{m}}^{l+1}(M) \neq 0$ by Proposition 3.1.3, which contradicts the assumption. Therefore $\dim R/\mathfrak{p} \neq (l+1) - i - 1$ for any $\mathfrak{p} \in \text{Supp } \mathcal{H}_M^i$ and all $i \geq -1$. That is $\dim \mathcal{H}_M^i < (l+1) - i - 1$ for all $i \geq -1$. In other words, $\dim \mathcal{H}_M^i \leq l - i - 1$ for all $i \geq -1$.

(ii) \implies (i). By descending induction on l . For $i = d - 1$ we have nothing to prove. Assume that $l < d - 1$. As $\dim \mathcal{H}_M^i \leq l - i - 1 < (l+1) - i - 1$ for all $i \geq -1$, we have, by induction hypothesis, that $H_{\mathfrak{m}}^j(M) = 0$ for all j , $l+1 < j < d$. Moreover, Proposition 3.1.3 implies that $\text{Att } H_{\mathfrak{m}}^{l+1}(M)$ is empty so that $H_{\mathfrak{m}}^{l+1}(M) = 0$. \square

The following result is now a clear conclusion of the above corollary.

Corollary 3.1.5. *Assume that M is not Cohen-Macaulay and that $\mathcal{C}_R(M)$ is finite. Set $s = 1 + \sup\{\dim \mathcal{H}_M^i + i : i \geq -1\}$. Then $H_{\mathfrak{m}}^s(M) \neq 0$ and $H_{\mathfrak{m}}^i(M) = 0$ for all i , $s < i < d$.*

The following corollary gives us a non-vanishing criterion of $H_{\mathfrak{m}}^{d-1}(M)$ when $\mathcal{C}_R(M)$ is finite.

Corollary 3.1.6. *Assume that $\mathcal{C}_R(M)$ is finite. Then*

- (i) $\text{Att } H_{\mathfrak{m}}^{d-1}(M) = \bigcup_{i=-1}^{d-2} \{\mathfrak{p} \in \text{Ass } \mathcal{H}_M^i : \dim R/\mathfrak{p} = d - i - 2\}$.

(ii) $H_{\mathfrak{m}}^{d-1}(M) \neq 0$ if and only if $\dim \mathcal{H}_M^i = d - i - 2$ for some i , $-1 \leq i \leq d - 2$.

Proof. It is clear by Proposition 3.1.3. \square

3.2 Top local cohomology modules

In this section we assume that (R, \mathfrak{m}) is a local ring. We say that R is complete precisely when it is complete with respect to the \mathfrak{m} -adic topology.

As mentioned before, by Theorem 1.2.4, $\text{Att } H_{\mathfrak{m}}^d(M) = \text{Assh } M$. The following result is due to Dibaei and Yassemi.

Theorem 3.2.1. [12, Theorem A] *For any ideal \mathfrak{a} of R ,*

$$\text{Att } H_{\mathfrak{a}}^d(M) = \{\mathfrak{p} \in \text{Supp } M : \text{cd}(\mathfrak{a}, R/\mathfrak{p}) = d\},$$

where $\text{cd}(\mathfrak{a}, K)$ is the cohomological dimension of an R -module K with respect to \mathfrak{a} , that is $\text{cd}(\mathfrak{a}, K) = \sup\{i \in \mathbb{Z} : H_{\mathfrak{a}}^i(K) \neq 0\}$.

Note that if $\mathfrak{p} \in \text{Supp } M$ such that $\text{cd}(\mathfrak{a}, R/\mathfrak{p}) = d$, then $\dim R/\mathfrak{p} = d$. Therefore $\text{Att } H_{\mathfrak{a}}^d(M) \subseteq \text{Assh } M$. The number of subsets T of $\text{Assh } M$ with $\text{Att } H_{\mathfrak{a}}^d(M) = T$, for some ideal \mathfrak{a} of a complete ring R , is exactly the number of non-isomorphic top local cohomology modules of M with respect to all ideals of R , by the following theorem.

Theorem 3.2.2. [11, Theorem 1.6] *Assume that R is complete. Then for any pair of ideals \mathfrak{a} and \mathfrak{b} of R , if $\text{Att } H_{\mathfrak{a}}^d(M) = \text{Att } H_{\mathfrak{b}}^d(M)$, then $H_{\mathfrak{a}}^d(M) \cong H_{\mathfrak{b}}^d(M)$.*

Now it is natural interesting to ask,

Question 3.2.3. [10, Question 2.9] *For any subset T of $\text{Assh } M$, is there an ideal \mathfrak{a} of R such that $\text{Att } H_{\mathfrak{a}}^d(M) = T$?*

Note that if $T = \text{Assh } M$, then the maximal ideal is the answer. Thus through this section we always assume that T is a non-empty proper subset of $\text{Assh } M$.

In special case when $d = 1$, it is easy to deal Question 3.2.3.

Proposition 3.2.4. *If $\dim M = 1$, then any subset T of $\text{Assh } M$ is equal to the set $\text{Att } H_{\mathfrak{a}}^1(M)$ for some ideal \mathfrak{a} of R .*

Proof. Set $\mathfrak{a} := \bigcap_{\mathfrak{p} \in \text{Assh } M \setminus T} \mathfrak{p}$. Note that $\text{rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}$ for all $\mathfrak{p} \in T$ and $\mathfrak{a} + \mathfrak{p} = \mathfrak{p}$ for all $\mathfrak{p} \in \text{Assh } M \setminus T$. Thus $H_{\mathfrak{a}}^1(R/\mathfrak{p}) \neq 0$ if and only if $\mathfrak{p} \in T$. \square

In the following result we find a characterization for a subset of $\text{Assh } M$ to be the set of attached primes of the top local cohomology of M with respect to an ideal \mathfrak{a} .

Proposition 3.2.5. *Assume that R is complete, $d \geq 1$ and set $\text{Assh } M \setminus T = \{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$. The following statements are equivalent.*

- (i) *There exists an ideal \mathfrak{a} of R such that $\text{Att } H_{\mathfrak{a}}^d(M) = T$.*
- (ii) *For each i , $1 \leq i \leq r$, there exists $Q_i \in \text{Supp } M$ with $\dim R/Q_i = 1$ such that*

$$\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq Q_i \quad \text{and} \quad \mathfrak{q}_i \subseteq Q_i.$$

With Q_i , $1 \leq i \leq r$, as above, $\text{Att } H_{\mathfrak{a}}^d(M) = T$ where $\mathfrak{a} = \bigcap_{i=1}^r Q_i$.

Proof. (i) \Rightarrow (ii). By Theorem 3.2.1, $H_{\mathfrak{a}}^d(R/\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in T$, that is $\mathfrak{a} + \mathfrak{p}$ is \mathfrak{m} -primary for all $\mathfrak{p} \in T$, by the Lichtenbaum-Hartshorne Theorem(1.2.3). On the other hand, for $1 \leq i \leq r$, $\mathfrak{q}_i \notin T$ which is equivalent to say that $\mathfrak{a} + \mathfrak{q}_i$ is not an \mathfrak{m} -primary ideal. Hence there exists a prime ideal $Q_i \in \text{Supp } M$ such that $\dim R/Q_i = 1$ and $\mathfrak{a} + \mathfrak{q}_i \subseteq Q_i$. It follows that $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq Q_i$.

(ii) \Rightarrow (i). Set $\mathfrak{a} := \bigcap_{i=1}^r Q_i$. For each i , $1 \leq i \leq r$, $\mathfrak{a} + \mathfrak{q}_i \subseteq Q_i$ implies that $\mathfrak{a} + \mathfrak{q}_i$ is not \mathfrak{m} -primary and so $H_{\mathfrak{a}}^d(R/\mathfrak{q}_i) = 0$. Thus by Theorem 3.2.2 $\text{Att } H_{\mathfrak{a}}^d(M) \subseteq T$. Assume $\mathfrak{p} \in T$ and $Q \in \text{Supp } M$ such that $\mathfrak{a} + \mathfrak{p} \subseteq Q$. Then $Q_i \subseteq Q$ for some i , $1 \leq i \leq r$. Since $\mathfrak{p} \not\subseteq Q_i$, we have $Q_i \neq Q$, so $Q = \mathfrak{m}$. Hence $\mathfrak{a} + \mathfrak{p}$ is \mathfrak{m} -primary ideal. Now, by the Lichtenbaum-Hartshorne Theorem(1.2.3), and Theorem 3.2.1, it follows that $\mathfrak{p} \in \text{Att } H_{\mathfrak{a}}^d(M)$. \square

Corollary 3.2.6. *If $H_{\mathfrak{a}}^d(M) \neq 0$ and R is complete, then there is an ideal \mathfrak{b} of R such that $\dim R/\mathfrak{b} \leq 1$ and $H_{\mathfrak{a}}^d(M) \cong H_{\mathfrak{b}}^d(M)$.*

Proof. If $\text{Att } H_{\mathfrak{a}}^d(M) = \text{Assh } M$, then $H_{\mathfrak{a}}^d(M) \cong H_{\mathfrak{m}}^d(M)$ by Theorem 3.2.2. Otherwise $d \geq 1$ and $\text{Att } H_{\mathfrak{a}}^d(M)$ is a proper subset of $\text{Assh } M$. Set $\text{Assh } M \setminus \text{Att } H_{\mathfrak{a}}^d(M) := \{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$. By Proposition 3.2.5, there are $Q_i \in \text{Supp } M$ with $\dim R/Q_i = 1$, $i = 1, \dots, r$, such that $\text{Att } H_{\mathfrak{a}}^d(M) = \text{Att } H_{\mathfrak{b}}^d(M)$ with $\mathfrak{b} = \bigcap_{i=1}^r Q_i$. Now, by Theorem 3.2.2, we have $H_{\mathfrak{a}}^d(M) \cong H_{\mathfrak{b}}^d(M)$. As $\dim R/\mathfrak{b} = 1$, the proof is complete. \square

Proposition 3.2.5 provides a useful method to find examples of top local cohomology modules with specified attached primes.

Example 3.2.7. Set $R = k[[X, Y, Z, W]]$, where k is a field and X, Y, Z, W are independent indeterminates. Then R is a complete Noetherian local ring with maximal ideal $\mathfrak{m} =$

(X, Y, Z, W) . Consider prime ideals

$$\mathfrak{p}_1 = (X, Y) \quad , \quad \mathfrak{p}_2 = (Z, W) \quad , \quad \mathfrak{p}_3 = (Y, Z) \quad , \quad \mathfrak{p}_4 = (X, W)$$

and set $M = \frac{R}{\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4}$ as an R -module, so that we have $\text{Assh } M = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4\}$ and $\dim M = 2$. We get $\{\mathfrak{p}_i\} = \text{Att } H_{\mathfrak{a}_i}^2(M)$, where $\mathfrak{a}_1 = \mathfrak{p}_2, \mathfrak{a}_2 = \mathfrak{p}_1, \mathfrak{a}_3 = \mathfrak{p}_4, \mathfrak{a}_4 = \mathfrak{p}_3$, and $\{\mathfrak{p}_i, \mathfrak{p}_j\} = \text{Att } H_{\mathfrak{a}_{ij}}^2(M)$, where

$$\begin{aligned} \mathfrak{a}_{12} &= (Y^2 + YZ, Z^2 + YZ, X^2 + XW, W^2 + WX), \\ \mathfrak{a}_{34} &= (Z^2 + ZW, X^2 + YX, Y^2 + YX, W^2 + WZ), \\ \mathfrak{a}_{13} &= (Z^2 + XZ, W^2 + WY, X^2 + XZ), \\ \mathfrak{a}_{14} &= (W^2 + WY, Z^2 + ZY, Y^2 + YW), \\ \mathfrak{a}_{23} &= (X^2 + XZ, Y^2 + WY, W^2 + ZW), \\ \mathfrak{a}_{24} &= (X^2 + XZ, Y^2 + WY, Z^2 + ZW). \end{aligned}$$

Finally, we have $\{\mathfrak{p}_i, \mathfrak{p}_j, \mathfrak{p}_k\} = \text{Att } H_{\mathfrak{a}_{ijk}}^2(M)$, where $\mathfrak{a}_{123} = (X, W, Y+Z), \mathfrak{a}_{234} = (X, Y, W+Z), \mathfrak{a}_{134} = (Z, W, Y+X)$.

Lemma 3.2.8. *Assume that R is complete, $d \geq 2$, and $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq \bigcap_{\substack{\mathfrak{q} \in \text{Assh } R/ \sum_{\mathfrak{p} \in T'} \mathfrak{p}}} \mathfrak{q}$, where*

$T' = \text{Assh } M \setminus T$. *Then there exists a prime ideal $Q \in \text{Supp } M$ with $\dim R/Q = 1$ and $\text{Att } H_Q^d(M) = T$.*

Proof. Set $s := \text{ht}_M(\sum_{\mathfrak{p} \in T'} \mathfrak{p})$. We have $s \leq d-1$, otherwise $\text{Assh}(R/\sum_{\mathfrak{p} \in T'} \mathfrak{p}) = \{\mathfrak{m}\}$ which contradicts the condition $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq \bigcap_{\substack{\mathfrak{q} \in \text{Assh}(R/ \sum_{\mathfrak{p} \in T'} \mathfrak{p})}} \mathfrak{q}$.

As R is catenary, we have $\dim(R/\sum_{\mathfrak{p} \in T'} \mathfrak{p}) = n - s$. We first prove, by induction on j , $0 \leq j \leq d - s - 1$, that there exists a chain of prime ideals

$$Q_0 \subset Q_1 \subset \cdots \subset Q_j \subset \mathfrak{m},$$

such that $Q_0 \in \text{Assh}(R/\sum_{\mathfrak{p} \in T'} \mathfrak{p})$, $\dim R/Q_j = d - s - j$ and $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq Q_j$.

There is $Q_0 \in \text{Assh}(R/\sum_{\mathfrak{p} \in T'} \mathfrak{p})$ such that $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq Q_0$. Note that $\dim R/Q_0 = \dim(R/\sum_{\mathfrak{p} \in T'} \mathfrak{p}) = d - s$. Now, assume that $0 < j \leq d - s - 1$ and that we have proved the existence of a chain $Q_0 \subset Q_1 \subset \cdots \subset Q_{j-1}$ of prime ideals such that $Q_0 \in \text{Assh}(R/\sum_{\mathfrak{p} \in T'} \mathfrak{p})$, $\dim R/Q_j = d - s - (j - 1)$ and that $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq Q_{j-1}$.

Since $d - s - (j - 1) = d - s + 1 - j \geq 2$, the set V defined as

$$V = \{\mathfrak{q} \in \text{Supp } M \mid Q_{j-1} \subset \mathfrak{q} \subset \mathfrak{q}' \subseteq \mathfrak{m}, \dim R/\mathfrak{q} = d - s - j, \\ \mathfrak{q}' \in \text{Spec } R \text{ and } \dim R/\mathfrak{q}' = d - s - j - 1\}$$

is non-empty and so, by Ratliff's weak existence theorem [24, Theorem 31.2], is not finite. As $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq Q_{j-1}$, we have $Q_{j-1} \subset Q_{j-1} + \bigcap_{\mathfrak{p} \in T} \mathfrak{p}$. If, for $\mathfrak{q} \in V$, $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \subseteq \mathfrak{q}$, then \mathfrak{q} is a minimal prime of $Q_{j-1} + \bigcap_{\mathfrak{p} \in T} \mathfrak{p}$. As V is an infinite set, there is $Q_j \in V$ such that $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq Q_j$. Thus the induction is complete. Now by taking $Q := Q_{d-s-1}$ and by Proposition 3.2.5, the claim follows. \square

Corollary 3.2.9. *Assume that R is complete and $|T| = |\text{Assh } M| - 1$. Then there is an ideal \mathfrak{a} of R such that $\text{Att } H_{\mathfrak{a}}^d(M) = T$.*

Proof. Note that $\text{Assh } M \setminus T$ is a singleton set $\{\mathfrak{q}\}$, say, and so $\text{ht}_M \mathfrak{q} = 0$ and $\bigcap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq \mathfrak{q}$. Therefore the result follows by Lemma 3.2.8. \square

Lemma 3.2.10. *Let \mathfrak{a}_1 and \mathfrak{a}_2 be ideals of a complete ring R . Then there exists an ideal \mathfrak{b} of R such that $\text{Att } H_{\mathfrak{b}}^d(M) = \text{Att } H_{\mathfrak{a}_1}^d(M) \cap \text{Att } H_{\mathfrak{a}_2}^d(M)$.*

Proof. Set $T_1 = \text{Att } H_{\mathfrak{a}_1}^d(M)$ and $T_2 = \text{Att } H_{\mathfrak{a}_2}^d(M)$. We may assume that $T_1 \cap T_2$ is a non-empty proper subset of $\text{Assh } M$. Assume that $\mathfrak{q} \in \text{Assh } M \setminus (T_1 \cap T_2) = (\text{Assh } M \setminus T_1) \cup (\text{Assh } M \setminus T_2)$. By Proposition 3.2.5, there exists $Q \in \text{Supp } M$ with $\dim R/Q = 1$ such that $\mathfrak{q} \subseteq Q$ and $\bigcap_{\mathfrak{p} \in T_1 \cap T_2} \mathfrak{p} \not\subseteq Q$. Now, by Proposition 3.2.5, again there exists an ideal \mathfrak{b} of R such that $\text{Att } H_{\mathfrak{b}}^d(M) = T_1 \cap T_2$. \square

Now we are ready to present our main result.

Theorem 3.2.11. *Assume that R is complete and $T \subseteq \text{Assh } M$, then there exists an ideal \mathfrak{a} of R such that $T = \text{Att } H_{\mathfrak{a}}^d(M)$.*

Proof. By Corollary 3.2.4, we may assume that $\dim M \geq 2$ and that T is a non-empty proper subset of $\text{Assh } M$. Set $T = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ and $\text{Assh } M \setminus T = \{\mathfrak{p}_{t+1}, \dots, \mathfrak{p}_{t+r}\}$. We use induction on r . For $r = 1$, Corollary 3.2.9 proves the first step of induction. Assume that $r > 1$ and that the case $r - 1$ is proved. Set $T_1 = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t, \mathfrak{p}_{t+1}\}$ and $T_2 = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t, \mathfrak{p}_{t+2}\}$. By induction assumption there exist ideals \mathfrak{a}_1 and \mathfrak{a}_2 of R such that $T_1 = \text{Att } H_{\mathfrak{a}_1}^d(M)$ and $T_2 = \text{Att } H_{\mathfrak{a}_2}^d(M)$. Now by the Lemma 3.2.10, there exists an ideal \mathfrak{a} of R such that $T = T_1 \cap T_2 = \text{Att } H_{\mathfrak{a}}^d(M)$. \square

Corollary 3.2.12. (See [3, Corollary 1.7]) *Assume that R is complete. Then the number of non-isomorphic top local cohomology modules of M with respect to all ideals of R is equal to $2^{|\text{Assh } M|}$.*

Proof. It follows from Theorem 3.2.11 and Theorem 3.2.2. \square

3.3 Applications to generalized Cohen-Macaulay modules

In this section we study some properties of a generalized Cohen-Macaulay modules in terms of certain prime ideals \mathfrak{p} which R/\mathfrak{p} has a uniform local cohomological annihilator and we give a new characterization of these modules.

Lemma 3.3.1. *Let (R, \mathfrak{m}) be a g.CM local ring. Then R/\mathfrak{p} has a uniform local cohomological annihilator for all $\mathfrak{p} \in \text{Spec } R$. In particular, any equidimensional R -module M has a uniform local cohomological annihilator.*

Proof. Note that if $\dim R = 0$, then there is nothing to prove. So we assume that $\dim R > 0$. Let $\mathfrak{p} \in \text{Spec } R$ with $\text{ht } \mathfrak{p} = 0$. As R is g.CM and $\mathfrak{m} \not\subseteq \cup_{\mathfrak{p} \in \text{Min } R} \mathfrak{p}$, R has a uniform local cohomological annihilator and thus R/\mathfrak{p} has a uniform local cohomological annihilator by Theorem 2.2.1. Assume that $\text{ht}_M \mathfrak{p} = t > 0$. There is a subset of system of parameters x_1, \dots, x_t of R contained in \mathfrak{p} . By Theorem 1.2.16(iii), $R/(x_1, \dots, x_t)$ is g.CM and so it has a uniform local cohomological annihilator. In particular R/\mathfrak{p} has a uniform local cohomological annihilator by Theorem 2.2.1. The final part follows immediately from the first part and Proposition 2.2.4. \square

The converse of the above result is not true in general, but we may get a positive answer in a special case where $M_{\mathfrak{p}}$ is a Cohen-macaulay $R_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in \text{Supp } M \setminus \{\mathfrak{m}\}$. We need the following lemma which is straightforward and we give a proof for completeness.

Lemma 3.3.2. *Assume that R is a noetherian local ring. Then*

- (a) *If $\mathfrak{Q} \in \text{Min } \widehat{M}$, then $\mathfrak{Q} \in \text{Min } \widehat{A}/\mathfrak{Q}^{ce}$.*
- (b) *If R is universally catenary and M is equidimensional, then \widehat{M} is equidimensional as \widehat{R} -module.*

Proof. (a). It follows that $\mathfrak{Q}^c \in \text{Min } M$ by the Going Down Theorem. Assume that $\mathfrak{Q}' \in \text{Min } \widehat{R}/\mathfrak{Q}^{ce}$ such that $\mathfrak{Q}' \subseteq \mathfrak{Q}$. The exact sequence $0 \rightarrow R/\mathfrak{Q}^c \rightarrow M$ implies the exact sequence $0 \rightarrow \widehat{R}/\mathfrak{Q}^{ce} \rightarrow \widehat{M}$. Therefore $\mathfrak{Q}' \in \text{Ass } \widehat{M}$ and so $\mathfrak{Q}' = \mathfrak{Q}$.

(b). Assume that $\mathfrak{Q} \in \text{Min } \widehat{M}$. By the Going Down Theorem, $\mathfrak{Q}^c \in \text{Min } M$ from which we have $\dim \widehat{A}/\mathfrak{Q}^{ce} = \dim R/\mathfrak{Q}^c = \dim M$. As, by part (a), $\mathfrak{Q} \in \text{Min } \widehat{R}/\mathfrak{Q}^{ce}$ and using the

fact that R/\mathfrak{Q}^c is formally equidimensional, we get $\dim \widehat{R}/\mathfrak{Q} = \dim \widehat{R}/\mathfrak{Q}^{ce}$ which implies that $\dim \widehat{R}/\mathfrak{Q} = \dim M$. \square

Lemma 3.3.3. *Assume that (R, \mathfrak{m}) is a local ring such that R/\mathfrak{p} has a uniform local cohomological annihilator for all $\mathfrak{p} \in \text{Spec } R$. Then the following statements are equivalent.*

- (i) M is equidimensional R -module and for all $\mathfrak{p} \in \text{Supp } M \setminus \{\mathfrak{m}\}$, $M_{\mathfrak{p}}$ is a Cohen-macaulay $R_{\mathfrak{p}}$ -module.
- (ii) M is a g.CM module.

Proof. (i) \Rightarrow (ii). Since $H_{\mathfrak{m}}^i(M) \cong H_{\mathfrak{m}}^i(M/\Gamma_{\mathfrak{m}}(M))$ for $i > 0$, we may assume that $\Gamma_{\mathfrak{m}}(M) = 0$ and so $\mathfrak{m} \notin \text{Ass } M$. As, for each $\mathfrak{p} \in \text{Ass } M$, $M_{\mathfrak{p}}$ is Cohen-Macaulay, so $\text{Ass } M = \text{Min } M$. Thus $\mathcal{H}_M^{-1} = 0$, by Lemma 1.3.11. Since M is equidimensional and R/\mathfrak{p} has a uniform local cohomological annihilator for all $\mathfrak{p} \in \text{Min } M$, M has a uniform local cohomological annihilator by Proposition 2.2.4, and so $R/(0 :_R M)$ is universally catenary by Corollary 2.2.6. As a result, considering M as an $R/0 :_R M$ -module, Lemma 3.3.2 implies that \widehat{M} is equidimensional. Hence $\mathcal{C}_{\widehat{R}}(\widehat{M})$ is finite by Corollary 2.1.6.

Now, we prove the statement by using induction on $d = \dim M$. For $d = 2$, we have, by Corollary 3.1.6, that

$$\text{Att } H_{\mathfrak{m}}^1(\widehat{M}) = \{\mathfrak{p} \in \text{Ass } \mathcal{H}_{\widehat{M}}^{-1} : \text{ht}_{\widehat{M}}(\mathfrak{p}) = 1\} \cup \{\mathfrak{p} \in \text{Ass } \mathcal{H}_{\widehat{M}}^0 : \text{ht}_{\widehat{M}}(\mathfrak{p}) = 2\}.$$

If $\mathfrak{p} \in \text{Ass } \mathcal{H}_{\widehat{M}}^{-1}$ with $\text{ht}_{\widehat{M}}(\mathfrak{p}) = 1$, then $\mathfrak{p} \in \text{Ass } \widehat{M}$ and so $\mathfrak{p}^c \in \text{Ass } M = \text{Min } M$. On the other hand, since $\mathfrak{p} \in \text{Att } H_{\mathfrak{m}}^1(\widehat{M})$, $\mathfrak{p}^c \in \text{Att } H_{\mathfrak{m}}^1(M)$ by Theorem 1.1.2 which contradicts, with Lemma 2.2.7. Hence $\text{Att } H_{\mathfrak{m}}^1(\widehat{M}) \subseteq \{\widehat{\mathfrak{m}}\}$. Now, Corollary 1.1.1(i) implies that $H_{\mathfrak{m}}^1(M) \otimes_R \widehat{R}$ is finitely generated \widehat{R} -module and it is the first step of the induction.

Now assume that $d > 2$ and the statement holds up to $d - 1$. Let x be a uniform local cohomological annihilator of M . Since $\text{Min } M = \text{Ass } M$, x is a nonzero divisor on M by using its definition. On the other hand, as $R/0 :_R M$ is catenary, it is straightforward to see that M/xM satisfies the induction hypothesis for $d - 1$. Therefore, $H_{\mathfrak{m}}^i(M/xM)$ is finitely generated for all $i < d - 1$. The exact sequence $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ implies the long exact sequence

$$\dots \rightarrow H_{\mathfrak{m}}^i(M) \xrightarrow{x} H_{\mathfrak{m}}^i(M) \rightarrow H_{\mathfrak{m}}^i(M/xM) \rightarrow H_{\mathfrak{m}}^{i+1}(M) \xrightarrow{x} H_{\mathfrak{m}}^{i+1}(M) \rightarrow \dots$$

Since $xH_{\mathfrak{m}}^j(M) = 0$ for $j < d$, we get the exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^i(M) \rightarrow H_{\mathfrak{m}}^i(M/xM) \rightarrow H_{\mathfrak{m}}^{i+1}(M) \rightarrow 0,$$

for $i = 0, \dots, d - 2$. Now the result follows.

(ii) \Rightarrow (i) is clear by Theorem 1.2.16(i). \square

Now we can state a criterion for an equidimensional local ring to be a g.CM ring in terms of uniform local cohomological annihilators.

Corollary 3.3.4. *Assume that R is an equidimensional noetherian local ring. The following statements are equivalent.*

(i) R is g.CM.

(ii) For all $\mathfrak{p} \in \text{Spec } R \setminus \{\mathfrak{m}\}$, $R_{\mathfrak{p}}$ is a Cohen-Macaulay ring and R/\mathfrak{p} has a uniform local cohomological annihilator.

Proof. (i) \Rightarrow (ii). We know that $R_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \text{Spec } R \setminus \{\mathfrak{m}\}$, by Theorem 1.2.16(i). The rest is the subject of Lemma 3.3.1.

(ii) \Rightarrow (i) is immediate from Lemma 3.3.3. \square

The following remark is easy but we bring it here for completeness and future reference.

Remark 3.3.5. *Assume that (R, \mathfrak{m}) is local.*

(i) A finitely generated R -module M is g.CM if and only if all cohomology modules of $\mathcal{C}_R(M)$ are of finite lengths.

(ii) A finitely generated R -module M is quasi-Buchsbaum module if and only if $\mathcal{C}_R(M)$ is finite and $\mathfrak{m}\mathcal{H}_M^i = 0$ for all i .

Proof. (i). Assume that M is g.CM. By Theorem 1.2.16(i), we have $M_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \text{Supp } M \setminus \{\mathfrak{m}\}$. So that $\text{Supp } \mathcal{H}_M^i \subseteq \{\mathfrak{m}\}$ and, by Lemma 3.3.3, the result follows. The converse is clear by Lemma 3.3.3

(ii). It is similar to (i). \square

Chapter 4

Cohen-Macaulay loci of modules

Throughout this chapter M is a finitely generated R -module. In the case (R, \mathfrak{m}) is local, we use \hat{M} as the completion of M with respect to \mathfrak{m} . The main objective of this chapter is to study the Cohen-Macaulay locus of a module. We show that it is a Zariski-open subset of $\text{Spec } R$ in certain cases. Our results are also related to Cohen-Macaulayness of formal fibres over certain prime ideals.

4.1 Openness of Cohen-Macaulay locus

The Cohen-Macaulay locus of M is denoted by

$$\text{CM}(M) := \{\mathfrak{p} \in \text{Spec } R : M_{\mathfrak{p}} \text{ is Cohen-Macaulay as } R_{\mathfrak{p}}\text{-module}\}.$$

Let $\text{non-CM}(M) = \text{Spec } R \setminus \text{CM}(M)$. Trivially the Cohen-Macaulay locus of a Cohen-Macaulay module is $\text{Spec } R$ and of a generalized Cohen-Macaulay module M over a local ring (R, \mathfrak{m}) contains $\text{Spec } R \setminus \{\mathfrak{m}\}$ by Theorem 1.2.16(i). In these cases $\text{CM}(M)$ are Zariski-open subsets of $\text{Spec } R$.

The objective of this section is to study the Cohen-Macaulay locus of M and find out when it is a Zariski-open subset of $\text{Spec } R$. We first mention a remark for future references.

Remark 4.1.1. *For an R -module M of finite dimension, if the Cousin complex of M is finite, then $\text{non-CM}(M) = V(\prod_i (0 :_R \mathcal{H}_M^i))$ so that $\text{CM}(M)$ is open.*

Proof. It is clear, by Theorem 1.3.7 and Theorem 1.3.6, that

$$\text{CM}(M) = \text{Spec } (R) \setminus \bigcup_{i \geq -1} \text{Supp}_R(\mathcal{H}_M^i).$$

□

As we have seen in Corollary 2.1.6, the Cousin complex of every equidimensional module over a complete local ring is finite. So it is natural to ask, over which rings the

openness of Cohen-Macaulay locus property is heritable from the completion ring. As an example of such rings one may consider the rings whose formal fibres are Cohen-Macaulay. To see this, recall first the standard dimension and depth formula.

Lemma 4.1.2. [14, Chapitre IV, (6.1.2), (6.1.3),(6.3.3)] *Assume that (R, \mathfrak{m}) and (S, \mathfrak{n}) are local rings with $k = R/\mathfrak{m}$ and $f : R \rightarrow S$ is a flat local homomorphism. For a finitely generated R -module M , $M \otimes_R S$ is a finitely generated S -module and the following statements hold true.*

- (i) $\dim M \otimes_R S = \dim M + \dim S \otimes_R k$;
- (ii) $\text{depth } M \otimes_R S = \text{depth } M + \text{depth } S \otimes_R k$;
- (iii) $M \otimes_R S$ is Cohen-Macaulay S -module if and only if M is a Cohen-Macaulay R -module and $S \otimes_R k$ is a Cohen-Macaulay ring.

As an immediate corollary of the above lemma, we have the following result.

Corollary 4.1.3. *Assume that $\mathfrak{P} \in \text{Supp } \widehat{M}$ and $\mathfrak{p} = R \cap \mathfrak{P}$. If the formal fibre $\widehat{R} \otimes_R k(\mathfrak{p})$ over \mathfrak{p} is Cohen-Macaulay, then $M_{\mathfrak{p}}$ is Cohen-Macaulay if and only if $\widehat{M}_{\mathfrak{P}}$ is Cohen-Macaulay.*

Proof. Note that the natural local homomorphism $R_{\mathfrak{p}} \rightarrow \widehat{R}_{\mathfrak{P}}$ is flat, so that the result follows by the above lemma. \square

Now we may easily see that the openness of Cohen-Macaulay locus property is heritable from the completion ring, if all formal fibres are Cohen-Macaulay.

Corollary 4.1.4. *Assume that all formal fibres of R are Cohen-Macaulay. If the Cohen-Macaulay locus of \widehat{M} is a Zariski-open subset of $\text{Spec } \widehat{R}$, then the Cohen-Macaulay locus of M is a Zariski-open subset of $\text{Spec } R$.*

Proof. Equivalently, we prove that $\text{Min}(\text{non-CM}(M))$ is a finite set. Choose $\mathfrak{p} \in \text{Min}(\text{non-CM}(M))$ and let \mathfrak{Q} be a minimal member of the non-empty set

$$\{\mathfrak{q} \in \text{Supp } \widehat{M} : \mathfrak{q} \cap R = \mathfrak{p}\}.$$

Since the formal fibre of R over \mathfrak{p} is Cohen-Macaulay, $\widehat{M}_{\mathfrak{Q}}$ is not Cohen-Macaulay by Corollary 4.1.3. On the other hand, for each $\mathfrak{q} \in \text{Supp } \widehat{M}$ with $\mathfrak{q} \subset \mathfrak{Q}$ we have $\mathfrak{q} \cap R \subset \mathfrak{p}$ and so $\widehat{M}_{\mathfrak{q}}$ is Cohen-Macaulay again by Corollary 4.1.3. Hence $\mathfrak{Q} \in \text{Min}(\text{non-CM}(\widehat{M}))$ which is a finite set. \square

The following lemma shows that the Cohen-Macaulay locus of M is open if it is true for certain submodules of M .

Lemma 4.1.5. *Let*

$S = \{T \subseteq \text{Min } M : \text{there exists } \mathfrak{q} \in \text{Supp } M \text{ such that } \text{ht}(\mathfrak{q}/\mathfrak{p}) \text{ is constant for all } \mathfrak{p} \in T\}$. For each $T \in S$, we assign a submodule M^T of M with $\text{Ass } M^T = T$ and $\text{Ass } M/M^T = \text{Ass } M \setminus T$. Then

$$\text{CM}(M) = \bigcup_{T \in S} (\text{CM}(M^T) \setminus \bigcup_{\mathfrak{p} \in \text{Ass } M \setminus T} V(\mathfrak{p})).$$

Proof. For each $T \in S$, there exists a submodule M^T of M with $\text{Ass } M^T = T$ and $\text{Ass } M/M^T = \text{Ass } M \setminus T$ (c.f. [1, Page 263, Proposition 4]). It is clear that

$$\text{Supp } M/M^T = \bigcup_{\mathfrak{p} \in \text{Ass } M \setminus T} V(\mathfrak{p}).$$

Let $\mathfrak{q} \in \text{CM}(M)$ and set $T' := \{\mathfrak{P} \cap A : \mathfrak{P} \in \text{Ass } M_{\mathfrak{q}}\}$. As $M_{\mathfrak{q}}$ is Cohen-Macaulay, $\text{ht}(\mathfrak{q}/\mathfrak{p}) = \dim M_{\mathfrak{q}}$ for all $\mathfrak{p} \in T'$ and so $T' \in S$. We claim that $\mathfrak{q} \notin \text{Supp } M/M^{T'}$. Assuming contrary, there is $\mathfrak{p} \in \text{Ass } M/M^{T'}$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Hence $\mathfrak{p}A_{\mathfrak{q}} \in \text{Ass } M_{\mathfrak{q}}$ which implies that $\mathfrak{p} \in T'$. This contradicts with the fact that $\text{Ass } M/M^{T'} = \text{Ass } M \setminus T'$. Therefore from the exact sequence

$$0 \longrightarrow M^{T'} \longrightarrow M \longrightarrow M/M^{T'} \longrightarrow 0 \quad (4.1.1)$$

we get $(M^{T'})_{\mathfrak{q}} \cong M_{\mathfrak{q}}$ so that $\mathfrak{q} \in \text{CM}(M^{T'})$.

Conversely, assume that $T \in S$ and that $\mathfrak{q} \in \text{CM}(M^T) \setminus \bigcup_{\mathfrak{p} \in \text{Ass } M \setminus T} V(\mathfrak{p})$. That is $(M^T)_{\mathfrak{q}}$ is Cohen-Macaulay and $\mathfrak{q} \notin \text{Supp } M/M^T$. Therefore $M_{\mathfrak{q}}$ is Cohen-Macaulay by (4.1.1), replacing T by T' . \square

Note that if $R/0 :_R M$ is catenary, then each module M^T in the above lemma is an equidimensional R -module. Therefore one can state the following remark.

Remark 4.1.6. If R is catenary and $\text{CM}(N)$ is open for all equidimensional submodules N of M , then $\text{CM}(M)$ is open.

It is now a routine check to see that, over a local ring R with Cohen-Macaulay formal fibres, the Cohen-Macaulay locus of any finitely generated R -module is open.

Remark 4.1.7. Assume that all formal fibres of R are Cohen-Macaulay. Then the Cohen-Macaulay locus of M is a Zariski-open subset of $\text{Spec } R$.

Proof. By Corollary 4.1.4, it is enough to show that $\text{CM}(\widehat{M})$ is a Zariski-open subset of $\text{Spec } \widehat{R}$. As \widehat{R} is catenary, we may assume that \widehat{M} is equidimensional \widehat{R} -module by Remark 4.1.6. Finally, Corollary 2.1.6 implies that $\mathcal{C}_{\widehat{A}}(\widehat{M})$ has finite cohomologies and so $\text{CM}(\widehat{M})$ is open by Remark 4.1.1. \square

We are now able to prove that any minimal element of $\text{non-CM}(M)$ is either an attached prime of $H_{\mathfrak{m}}^i(M)$ for some i or $R_{\mathfrak{p}}$ is not a Cohen-Macaulay ring.

Theorem 4.1.8. *Assume that (R, \mathfrak{m}) is a catenary local ring and that M is equidimensional R -module. Then*

$$\text{Min}(\text{non-CM}(M)) \subseteq \bigcup_{0 \leq i \leq \dim M} \text{Att } H_{\mathfrak{m}}^i(M) \cup \text{non-CM}(R).$$

Proof. Choose $\mathfrak{p} \in \text{Min}(\text{non-CM}(M))$. As R is catenary and M is equidimensional, $M_{\mathfrak{p}}$ is also equidimensional as $R_{\mathfrak{p}}$ -module. Assume that $R_{\mathfrak{p}}$ is a Cohen-Macaulay ring. For each $\mathfrak{q} \in \text{Spec } R$ with $\mathfrak{q} \subseteq \mathfrak{p}$, $R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}$ has a uniform local cohomological annihilator by Proposition 2.3.2. Therefore, by Lemma 3.3.3, $M_{\mathfrak{p}}$ is a g.CM $R_{\mathfrak{p}}$ -module. As $M_{\mathfrak{p}}$ is not Cohen-Macaulay, $H_{\mathfrak{p}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}) \neq 0$ for some integer i , $i < \dim M_{\mathfrak{p}}$. In particular, $H_{\mathfrak{p}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$ is a non-zero finite length $R_{\mathfrak{p}}$ -module so that $\text{Att } H_{\mathfrak{p}R_{\mathfrak{p}}}^i(M_{\mathfrak{p}}) = \{\mathfrak{p}A_{\mathfrak{p}}\}$. By Weak general shifted localization principle (Theorem 1.2.7), $\mathfrak{p} \in \text{Att } H_{\mathfrak{m}}^{i+t}(M)$, where $t = \dim(A/\mathfrak{p})$. Now the result follows. \square

Corollary 4.1.9. *Assume that (R, \mathfrak{m}) is a catenary local ring and that the $\text{non-CM}(R)$ is a finite set. Then the Cohen-Macaulay locus of M is open.*

Proof. By Lemma 4.1.5, we may assume that M is equidimensional. Now Theorem 4.1.8 implies that $\text{Min}(\text{non-CM}(M))$ is a finite set. In other words $\text{non-CM}(M)$ is a Zariski-closed subset of $\text{Spec } R$. \square

Here is an example of local rings which satisfy the above condition.

Example 4.1.10. Consider a local ring R satisfying Serre's condition (S_{d-2}) , $d := \dim R$, such that $\mathcal{C}_R(R)$ is finite. Then $\mathcal{H}_R^i = 0$ for $i \leq d-4$ and $i \geq d-1$, by Theorem 1.3.8 Lemma 1.3.11(ii). Now, $\dim \mathcal{H}_R^{d-3} \leq 1$ and $\dim \mathcal{H}_R^{d-2} \leq 0$, by Theorem 1.3.9(ii). Thus $\text{non-CM}(R) = \text{Supp } \mathcal{H}_R^{d-2} \cup \text{Supp } \mathcal{H}_R^{d-1}$ is a finite set.

By Remark 4.1.7, for a local ring R , if all formal fibres of R are Cohen-Macaulay, then the Cohen-Macaulay locus of each finitely generated R -module is open and in Corollary 4.1.9, we have seen that the same statement holds if R is a catenary local ring and that the $\text{non-CM}(R)$ is a finite set. The following examples, show that these two conditions are significant.

Example 4.1.11 gives a local ring S with Cohen-Macaulay formal fibres for which the set $\text{non-CM}(S)$ is infinite. Example 4.1.12 presents a local ring T which admits a non-Cohen-Macaulay formal fibre with finite $\text{non-CM}(T)$.

Example 4.1.11. Set $S = k[[X, Y, Z, U, V]]/(X) \cap (Y, Z)$, where k is a field. It is clear that S is a local ring with Cohen-Macaulay formal fibres. By Ratliff's weak existence theorem [24, Theorem 31.2], there are infinitely many prime ideals P of $k[[X, Y, Z, U, V]]$, with $(X, Y, Z) \subset P \subset (X, Y, Z, U, V)$. For any such prime ideal P , $S_{\overline{P}}$ is not equidimensional and so it is not Cohen-Macaulay. In other words, $\text{non-CM}(S)$ is infinite.

Example 4.1.12. It is shown in [13, Proposition 3.3] that there exists a local integral domain (R, \mathfrak{m}) of dimension 2 such that $\widehat{R} = \mathbb{C}[[X, Y, Z]]/(Z^2, tZ)$, where \mathbb{C} is the field of complex numbers and $t = X + Y + Y^2s$ for some $s \in \mathbb{C}[[Y]] \setminus \mathbb{C}\{Y\}$. As $\text{Ass } \widehat{R} = \{(Z), (Z, t)\}$, \widehat{R} does not satisfy (S_1) . Thus $\mathcal{H}_{\widehat{R}}^{-1} \neq 0$ while $\mathcal{H}_R^{-1} = 0$, by Theorem 1.3.8. Now Lemma 2.1.5 implies that there exists a formal fibre of R which is not Cohen-Macaulay. As R is an integral local domain, we have $\text{non-CM}(R) = \{\mathfrak{m}\}$.

4.2 Rings whose formal fibres are Cohen-Macaulay

It is shown in Corollary 4.1.4 that if all formal fibres of R are Cohen-Macaulay, then the Cohen-Macaulay locus of any finitely generated R -module M is a Zariski-open subset of $\text{Spec } R$. This result motivates us to determine rings whose formal fibres are Cohen-Macaulay. More precisely, we study the affect of certain formal fibres being Cohen-Macaulay on the structure of a module.

Throughout this section (R, \mathfrak{m}) is a local ring and M is a finitely generated R -module of dimension d .

We begin with the following result which is the heart of the proof of our main result Theorem 4.2.2

Proposition 4.2.1. *Assume that \mathfrak{p} is a prime ideal of $\text{Spec } R$ such that R/\mathfrak{p} has a uniform local cohomological annihilator. Then the formal fibre of R over \mathfrak{p} is Cohen-Macaulay.*

Proof. It is well known that $(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \otimes_R \widehat{R} \cong S^{-1}(\widehat{R}/\mathfrak{p}\widehat{R})$, where S is the image of $R \setminus \mathfrak{p}$ in \widehat{R} . Therefore we want to show that $(S^{-1}(\widehat{R}/\mathfrak{p}\widehat{R}))_{S^{-1}\mathfrak{q}}$ is Cohen-Macaulay for all $\mathfrak{q} \in \text{Spec } \widehat{R}$ with $S \cap \mathfrak{q} = \emptyset$. It is enough to show that $(\widehat{R}/\mathfrak{p}\widehat{R})_{\mathfrak{q}}$ is Cohen-Macaulay $\widehat{R}_{\mathfrak{q}}$ -module. Since R/\mathfrak{p} has a uniform local cohomological annihilator, $\widehat{R}/\mathfrak{p}\widehat{R}$ has a uniform local cohomological annihilator which, in particular, implies that $\widehat{R}/\mathfrak{p}\widehat{R}$ is equidimensional by Proposition 2.2.2. Assume, contrarily, $(\widehat{R}/\mathfrak{p}\widehat{R})_{\mathfrak{q}}$ is not Cohen-Macaulay. We may assume that $\mathfrak{q} \in \text{Min}(\text{non-CM}(\widehat{R}/\mathfrak{p}\widehat{R}))$ and that $(\mathfrak{q} \cap R) \cap (R \setminus \mathfrak{p}) = \emptyset$. In other words, $\text{non-CM}((\widehat{R}/\mathfrak{p}\widehat{R})_{\mathfrak{q}}) = \{\mathfrak{q}\widehat{R}_{\mathfrak{q}}\}$ and $\mathfrak{q} \cap R = \mathfrak{p}$.

Let us replace R and M in Lemma 4.1.6 by $\widehat{R}_{\mathfrak{q}}$ and $(\widehat{R}/\mathfrak{p}\widehat{R})_{\mathfrak{q}}$, respectively. Note that $\widehat{R}/\mathfrak{q}'$ is equidimensional for all $\mathfrak{q}' \in \text{Spec } \widehat{R}$, so that $\mathcal{C}_{\widehat{R}}(\widehat{R}/\mathfrak{q}')$ is finite by Proposition 2.3.2

and thus $\mathcal{C}_{\widehat{R}_q}(\widehat{R}_q/\mathfrak{q}'\widehat{R}_q)$ is finite. Therefore $\widehat{R}_q/\mathfrak{q}'\widehat{R}_q$ has a uniform local cohomological annihilator as \widehat{R}_q -module for all $\mathfrak{q}' \in \text{Spec } \widehat{R}$ with $\mathfrak{q}' \subseteq \mathfrak{q}$ by Theorem 2.2.13. As $(\widehat{R}/\mathfrak{p}\widehat{R})_{\mathfrak{q}}$ is equidimensional, we can apply Lemma 3.3.3 to deduce that $(\widehat{R}/\mathfrak{p}\widehat{R})_{\mathfrak{q}}$ is g.CM as \widehat{R}_q -module. In particular, $H_{\mathfrak{q}\widehat{R}_q}^i((\widehat{R}/\mathfrak{p}\widehat{R})_{\mathfrak{q}})$ is a non-zero \widehat{R}_q -module of finite length for some $i < \dim(\widehat{R}/\mathfrak{p}\widehat{R})_{\mathfrak{q}}$ for which we get $\text{Att } H_{\mathfrak{q}\widehat{R}_q}^i((\widehat{R}/\mathfrak{p}\widehat{R})_{\mathfrak{q}}) = \{\mathfrak{q}\widehat{R}_q\}$. Now, the weak general shifted localization principle (Theorem 1.2.7) implies that $\mathfrak{q} \in \text{Att } H_{\mathfrak{m}}^j(\widehat{R}/\mathfrak{p}\widehat{R})$ for some $j < \dim R/\mathfrak{p}$ which gives $\mathfrak{p} = \mathfrak{q} \cap R \in \text{Att } H_{\mathfrak{m}}^j(R/\mathfrak{p})$. This contradicts with Lemma 2.2.7. \square

The above proposition enables us to give a characterization of a finitely generated module which admits a uniform local cohomological annihilator in terms of a certain set of formal fibres of the ground ring.

Theorem 4.2.2. *The following statements are equivalent.*

- (i) \widehat{M} is equidimensional \widehat{R} -module and all formal fibres of R over minimal members of $\text{Supp } M$ are Cohen-Macaulay.
- (ii) M has a uniform local cohomological annihilator.

Proof. (i) \Rightarrow (ii). By Proposition 2.3.2, $\mathcal{C}_{\widehat{R}}(\widehat{M})$ is finite, which implies that the Cohen-Macaulay locus of M is open or equivalently $\text{Min}(\text{non-CM}(\widehat{M}))$ is a finite set (see Remark 4.1.1). Thus Lemma 1.3.10 implies that

$$\bigcap_{\mathfrak{q} \in \text{non-CM}(\widehat{M})} \mathfrak{q} \not\subseteq \bigcup_{\mathfrak{q} \in \text{Min } \widehat{M}} \mathfrak{q}. \quad (4.2.1)$$

Note that for an element $r \in (\bigcap_{\mathfrak{q} \in \text{non-CM}(\widehat{M})} \mathfrak{q}) \cap R$, Corollary 2.1.8 implies that $r^n H_{\mathfrak{m}}^i(\widehat{M}) = 0$ for some positive integer n and for all $0 \leq i < \dim M$. On the other hand $H_{\mathfrak{m}}^i(\widehat{M}) \cong H_{\mathfrak{m}}^i(M)$, so it is enough to show that

$$\left(\bigcap_{\mathfrak{q} \in \text{non-CM}(\widehat{M})} \mathfrak{q} \right) \cap R \not\subseteq \bigcup_{\mathfrak{p} \in \text{Min } M} \mathfrak{p}. \quad (4.2.2)$$

Assume contrarily that (4.2.2) does not hold. Then since $\text{Min}(\text{non-CM}(\widehat{M}))$ is a finite set, there is $\mathfrak{p} \in \text{Min } M$ such that $\mathfrak{p} = \mathfrak{q} \cap R$ for some $\mathfrak{q} \in \text{non-CM}(\widehat{M})$. Note that $R_{\mathfrak{p}} \rightarrow (\widehat{R})_{\mathfrak{q}}$ is a faithfully flat ring homomorphism and its fibre ring over $\mathfrak{p}R_{\mathfrak{p}}$ is $((R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \otimes_R \widehat{R})_{\mathfrak{q}}$ which is Cohen-Macaulay by our assumption. Therefore, by Corollary 4.1.3, $M_{\mathfrak{p}}$ is not Cohen-Macaulay. This contradicts with the fact that $\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$.

(ii) \Rightarrow (i). As M has a uniform local cohomological annihilator, M is equidimensional and R/\mathfrak{p} has uniform local cohomological annihilator for all minimal prime \mathfrak{p} of M , by

Proposition 2.2.4. Thus the formal fibre over \mathfrak{p} is Cohen-Macaulay for $\mathfrak{p} \in \text{Min } M$ by Proposition 4.2.1. \square

The following result shows that if all formal fibres of a ring R are Cohen-Macaulay, then R is universally catenary if and only if $\mathcal{C}_R(R/\mathfrak{p})$ is finite for all $\mathfrak{p} \in \text{Spec } R$.

Corollary 4.2.3. *The following statements are equivalent.*

- (i) R is universally catenary ring and all of its formal fibres are Cohen-Macaulay.
- (ii) The Cousin complex $\mathcal{C}_R(R/\mathfrak{p})$ is finite for all $\mathfrak{p} \in \text{Spec } R$.
- (iii) R/\mathfrak{p} has a uniform local cohomological annihilator for all $\mathfrak{p} \in \text{Spec } R$.

Proof. (i) \Rightarrow (ii) is clear by Proposition 2.3.2.

(ii) \Rightarrow (iii) is clear by Theorem 2.2.13.

(iii) \Rightarrow (i). By Corollary 2.2.6, R/\mathfrak{p} is universally catenary for all primes \mathfrak{p} and so is R . The rest is clear by Proposition 4.2.1. \square

As we have seen in Lemma 2.2.8, $\text{non-CM}(M) \subseteq V(\mathfrak{a}(M))$. Our final goal is to investigate that when the equality holds. In 1982, Schenzel proved that the equality holds when M is equidimensional and R posses a dualizing complex [27, p. 52]. Recall that if a local R posses a dualizing complex then it is a homomorphic image of a Gorenstein local ring by [19, Theorem 6.1] and so R is universally catenary and all formal fibres of R are Cohen-Macaulay. Hence $\mathcal{C}_R(M)$ is finite for all equidimensional R -module M by Corollary 2.1.6 and Lemma 2.2.10.

In the following, we see that $\text{non-CM}(M) = V(\mathfrak{a}(M))$ whenever $\mathcal{C}_R(M)$ is finite.

Corollary 4.2.4. *Assume that M is a finitely generated R -module of dimension d and that $\mathcal{C}_R(M)$ is finite. Then*

$$V(\prod_{i=-1}^{d-1} (0 :_R \mathcal{H}_M^i)) = \text{non-CM}(M) = V(\mathfrak{a}(M)).$$

Proof. The first equality is in Remark 4.1.1. The second inequality is clear by Lemma 2.2.8 and Corollary 2.1.8. \square

As we have seen in Proposition 4.2.1, if R/\mathfrak{p} has a uniform local cohomological annihilator, then the formal fibre of R over \mathfrak{p} is Cohen-Macaulay. So to find the Cohen-Macaulay formal fibres, it is useful to study those prime ideals \mathfrak{p} such that R/\mathfrak{p} has a uniform local cohomological annihilator. Characterizing these ideals enables also us to characterize those modules M with $\text{non-CM}(M) = V(\mathfrak{a}(M))$ (see also Corollary 4.2.3).

Proposition 4.2.5. *Assume that $\mathfrak{p} \in \text{Spec } R$. A necessary and sufficient condition for R/\mathfrak{p} to have a uniform local cohomological annihilator is that there exists an equidimensional R -module M such that $\mathfrak{p} \in \text{Supp } M \setminus V(\mathfrak{a}(M))$.*

Proof. The necessary condition is clear by taking $M := R/\mathfrak{p}$. For the converse, assume that there is an equidimensional R -module M such that $\mathfrak{p} \in \text{Supp } M \setminus V(\mathfrak{a}(M))$. We prove the claim by induction on $h := \text{ht}_M \mathfrak{p}$. When $h = 0$, we have $\mathfrak{p} \in \text{Min } M$. Choose a submodule N of M with $\text{Ass } N = \{\mathfrak{p}\}$ and $\text{Ass } M/N = \text{Ass } M \setminus \{\mathfrak{p}\}$. It is clear that $(M/N)_{\mathfrak{p}} = 0$ so that $r(M/N) = 0$ for some $r \in R \setminus \mathfrak{p}$. On the other hand the fact that $\mathfrak{a}(M) \not\subseteq \mathfrak{p}$ implies that there is $s \in R \setminus \mathfrak{p}$ such that $sH_{\mathfrak{m}}^i(M) = 0$ for all $i < \dim M$. The exact sequence $H_{\mathfrak{m}}^{i-1}(M/N) \rightarrow H_{\mathfrak{m}}^i(N) \rightarrow H_{\mathfrak{m}}^i(M)$ implies $rsH_{\mathfrak{m}}^i(N) = 0$ for all $i < \dim N$. As $\mathfrak{p} \in \text{Min } N$, R/\mathfrak{p} has a uniform local cohomological annihilator by Proposition 2.2.4.

Now assume that $h > 0$. For any $\mathfrak{q} \in \text{Supp } M$ with $\mathfrak{q} \subset \mathfrak{p}$ we have $\mathfrak{q} \notin V(\mathfrak{a}(M))$ so that R/\mathfrak{q} has a uniform local annihilator by induction hypothesis. As $\mathfrak{p} \notin V(\mathfrak{a}(M))$, $M_{\mathfrak{p}}$ is Cohen-Macaulay, by Corollary 4.2.4. Choose a submodule K of M with $\text{Ass } K = \text{Min } M$ and $\text{Ass } M/K = \text{Ass } M \setminus \text{Min } M$. If $\mathfrak{p} \in \text{Supp } M/K$ then there is $\mathfrak{q} \in \text{Ass } M/K$ with $\mathfrak{q} \subseteq \mathfrak{p}$. Therefore $\mathfrak{q} \in \text{Ass } M$ and $\mathfrak{q} \notin \text{Min } M$. As $M_{\mathfrak{p}}$ is Cohen-Macaulay, so is $M_{\mathfrak{q}}$ which gives $\mathfrak{q} \in \text{Min } M$, which is a contradiction. Hence we have $\mathfrak{p} \notin \text{Supp } M/K$ which yields $r(M/K) = 0$ for some $r \in R \setminus \mathfrak{p}$ and so, by applying local cohomology to the exact sequence $0 \rightarrow K \rightarrow M \rightarrow M/K \rightarrow 0$, it follows that $\mathfrak{a}(K) \not\subseteq \mathfrak{p}$. As $M_{\mathfrak{p}} \cong K_{\mathfrak{p}}$, $K_{\mathfrak{p}}$ is Cohen-Macaulay and $\text{ht}_K \mathfrak{p} > 0$, there is $x \in \mathfrak{p}$ which is non-zero-divisor on K . The exact sequence $0 \rightarrow K \xrightarrow{x} K \rightarrow K/xK \rightarrow 0$ implies that $\mathfrak{a}(K)^2 \subseteq \mathfrak{a}(K/xK)$ which implies that $\mathfrak{a}(K/xK) \not\subseteq \mathfrak{p}$. As $\text{ht}_{K/xK}(\mathfrak{p}) < h$, R/\mathfrak{p} has a uniform local cohomological annihilator by the induction hypothesis. \square

As our first application of the above proposition, we have the following result.

Corollary 4.2.6. *Assume that M is a finitely generated R -module which satisfies the condition (S_n) . If $\mathcal{C}_R(M)$ is finite, then the formal fibres of R over all prime ideals $\mathfrak{p} \in \text{Supp } M$ with $\text{ht}_M \mathfrak{p} \leq n$ are Cohen-Macaulay.*

Proof. Let $\mathfrak{p} \in \text{Supp } M$ with $\text{ht}_M \mathfrak{p} \leq n$. Note that $V(\mathfrak{a}(N)) = \text{non-CM}(N)$ by Corollary 4.2.4. On the other hand $M_{\mathfrak{p}}$ is Cohen-Macaulay, so that the result follows by Proposition 4.2.5. \square

Now we may show that if $\mathcal{C}_R(M)$ is finite, then the formal fibres of R over some certain prime ideals are Cohen-Macaulay.

Corollary 4.2.7. *Assume that $\mathcal{C}_R(M)$ is finite. Then the formal fibres of R over all prime ideals $\mathfrak{p} \in \text{Supp } M$ with $\text{ht}_M \mathfrak{p} \leq 1$ are Cohen-Macaulay.*

Proof. By Corollary 2.1.3, there exists a finitely generated R -module N which satisfies (S_1) , $\text{Supp } N = \text{Supp } M$ and $\mathcal{C}_R(N)$ is finite. Now the result follows by Corollary 4.2.6. \square

In Corollary 4.2.4, it is shown that, for a finitely generated R -module M , $\text{non-CM}(M) = V(\mathfrak{a}(M))$ whenever $\mathcal{C}_R(M)$ is finite. In the following we characterize those modules M satisfying $\text{non-CM}(M) = V(\mathfrak{a}(M))$ without assuming that the Cousin complex of M to be finite.

Theorem 4.2.8. *For an equidimensional R -module M , the following statements are equivalent.*

- (i) R/\mathfrak{q} has a uniform local cohomological annihilator for all $\mathfrak{q} \in \text{CM}(M)$.
- (i') $\widehat{R}/\mathfrak{q}\widehat{R}$ is equidimensional \widehat{R} -module and the formal fibre ring $(R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}) \otimes_R \widehat{R}$ is Cohen-Macaulay for all $\mathfrak{q} \in \text{CM}(M)$.
- (ii) $\text{non-CM}(M) = V(\mathfrak{a}(M))$.
- (iii) $\text{non-CM}(M) \supseteq V(\mathfrak{a}(M))$.

Proof. The equivalence of (i) and (i') is the subject of Theorem 4.2.2.

(i) \implies (ii). The inclusion $\text{non-CM}(M) \subseteq V(\mathfrak{a}(M))$ is clear by Lemma 2.2.8.

Now assume that $\mathfrak{p} \supseteq \mathfrak{a}(M)$. Thus there is an integer i , $0 \leq i < d$, such that $\mathfrak{p} \supseteq 0 :_R H_{\mathfrak{m}}^i(M)$. There is $\mathfrak{Q} \in \text{Att } H_{\mathfrak{m}}^i(\widehat{M})$ with $\mathfrak{q} := R \cap \mathfrak{Q} \in \text{Att } H_{\mathfrak{m}}^i(M)$ and $\mathfrak{p} \supseteq \mathfrak{q}$. To show $\mathfrak{p} \in \text{non-CM}(M)$ it is enough to show that $\mathfrak{q} \in \text{non-CM}(M)$. Assuming contrarily, $\mathfrak{q} \in \text{CM}(M)$, R/\mathfrak{q} has a uniform local cohomological annihilator by our assumption and so the formal fibre $k(\mathfrak{q}) \otimes_R \widehat{R}$ is Cohen-Macaulay, by Theorem 4.2.2. As the map $R_{\mathfrak{q}} \rightarrow \widehat{R}_{\mathfrak{Q}}$ is faithfully flat ring homomorphism, we find that $k(\mathfrak{q}) \otimes_{R_{\mathfrak{q}}} \widehat{R}_{\mathfrak{Q}}$ is also Cohen-Macaulay. Therefore the standard dimension and depth formulas (Lemma 4.1.2), applied to the faithfully flat extension $R_{\mathfrak{q}} \rightarrow \widehat{R}_{\mathfrak{Q}}$, implies that $\widehat{M}_{\mathfrak{Q}}$ is Cohen-Macaulay. On the other hand, R/\mathfrak{r} has a uniform local cohomological annihilator for all $\mathfrak{r} \in \text{Min } M$ (simply because in this case $M_{\mathfrak{r}}$ has zero dimension and so $\mathfrak{r} \in \text{CM}(M)$). Thus, by Proposition 2.2.4, M has a uniform local cohomological annihilator and so does \widehat{M} . Therefore, by Proposition 2.2.2 \widehat{M} is equidimensional. Thus Corollary 2.1.6 implies that the Cousin complex $\mathcal{C}_{\widehat{R}}(\widehat{M})$ is finite. As $\mathfrak{Q} \in \text{Att } H_{\mathfrak{m}}^i(\widehat{M})$, we have, by Corollary 4.2.4, $\mathfrak{Q} \in \text{non-CM}(\widehat{M})$. This is a contradiction.

(iii) \implies (i). Assume that $\mathfrak{q} \in \text{CM}(M)$ so that $\mathfrak{q} \not\supseteq \mathfrak{a}(M)$ by our assumption. Now Proposition 4.2.5 implies that R/\mathfrak{q} has a uniform local cohomological annihilator. \square

Our last result in this section is a consequence of the above theorem, which improves Corollary 4.2.7 to a generalization of the fact that, all formal fibres of a Cohen-Macaulay ring are Cohen-Macaulay.

Corollary 4.2.9. *Assume that $\mathcal{C}_R(M)$ is finite. Then the formal fibres of R over all prime ideals $\mathfrak{p} \in \text{CM}(M) \cup \{\mathfrak{p} \in \text{Supp } M : \text{ht}_M \mathfrak{p} = 1\}$ are Cohen-Macaulay.*

Proof. By Corollary 4.2.7 all formal fibres of R over prime ideals \mathfrak{p} with $\text{ht} \mathfrak{p} = 1$, are Cohen-Macaulay. On the other hand $\text{non-CM}(M) = V(\mathfrak{a}(M))$ by Corollary 4.2.4. Now, Theorem 4.2.8 completes the proof. \square

4.3 Some comments

Throughout this section (R, \mathfrak{m}) is a local ring. As we have seen in Corollary 2.3.1, if $\mathcal{C}_R(M)$ is finite, then M is equidimensional and $R/0 :_R M$ is universally catenary. On the other hand, Proposition 2.3.2 shows that, if all formal fibres of a universally catenary local ring R are Cohen-Macaulay, then $\mathcal{C}_R(M)$ is finite for all equidimensional R -module M . These results lead us to the following natural question.

Question 4.3.1. *Assume that $\mathcal{C}_R(M)$ is finite. Are the formal fibres of R over all prime ideals $\mathfrak{p} \in \text{Supp } M$, Cohen-Macaulay?*

Corollary 4.2.7 shows that if $\mathcal{C}_R(M)$ is finite, then the formal fibres of R over all prime ideals $\mathfrak{p} \in \text{Supp } M$ with $\text{ht}_M \mathfrak{p} \leq 1$, are Cohen-Macaulay. As a consequence we may have a positive answer for the above question in the following special case.

Corollary 4.3.2. *Assume that $\mathcal{C}_R(M)$ is finite and $\dim M \leq 3$. Then the formal fibres of R over all prime ideals $\mathfrak{p} \in \text{Supp } M$ are Cohen-Macaulay.*

Proof. Let $\mathfrak{p} \in \text{Supp } M$. If $\text{ht}_M \mathfrak{p} \leq 1$, then the formal fibre over \mathfrak{p} is Cohen-Macaulay by Corollary 4.2.7. Now assume that $\text{ht}_M \mathfrak{p} > 1$. Thus $\dim R/\mathfrak{p} \leq 1$ and so R/\mathfrak{p} has a uniform local cohomological annihilator. Hence the formal fibre over \mathfrak{p} is Cohen-Macaulay by Proposition 4.2.1. \square

In Corollary 4.2.9, we have seen that if $\mathcal{C}_R(M)$ is finite, then the formal fibres of R over all prime ideals $\mathfrak{p} \in \text{CM}(M) \cup \{\mathfrak{p} \in \text{Supp } M : \text{ht}_M \mathfrak{p} = 1\}$ are Cohen-Macaulay. Thus we have the following result in which,

$$\dim(\text{non-CM}(M)) = \sup\{\dim R/\mathfrak{p} : \mathfrak{p} \in \text{non-CM}(M)\}.$$

Corollary 4.3.3. *Assume that $\mathcal{C}_R(M)$ is finite and $\dim(\text{non-CM}(M)) \leq 1$. Then the formal fibres of R over all prime ideals $\mathfrak{p} \in \text{Supp } M$ are Cohen-Macaulay.*

Proof. Let $\mathfrak{p} \in \text{Supp } M$. If $M_{\mathfrak{p}}$ is Cohen-Macaulay, the formal fibre of R over \mathfrak{p} is Cohen-Macaulay by Corollary 4.2.9. Now, assume that $\mathfrak{p} \in \text{non-CM}(M)$. Thus $\dim R/\mathfrak{p} \leq 1$ and so that R/\mathfrak{p} has a uniform local cohomological annihilator. Hence the formal fibre over \mathfrak{p} is Cohen-Macaulay by Proposition 4.2.1. \square

One way of dealing Question 4.3.1, is to find a reduction technique e.g. for dimension of the module. In this connection, we propose the following.

Problem 4.3.4. *Assume that $\mathcal{C}_R(M)$ is finite and x is a non-zero-divisor on M . Then $\mathcal{C}_R(M/xM)$ is finite.*

Assume that $\mathcal{C}_R(M)$ is finite and $\mathfrak{p} \in \text{Supp } M$. Consider the submodule N of M with $\text{Ass } N = \text{Ass } M \setminus \text{Assh } M$ and $\text{Ass } M/N = \text{Assh } M$ (c.f. [1, Page 263, Proposition 4]). As in the exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ we have $\text{ht}_M \mathfrak{p} \geq 1$ for all $\mathfrak{p} \in \text{Supp } N$, Lemma 2.1.1(b) implies that $\mathcal{C}_R(M/N)$ is finite. Note that $\text{Supp } M = \text{Supp } M/N$ and so $\mathfrak{p} \in \text{Supp } M/N$. Since $\text{Ass } M/N = \text{Min } M/N$, \mathfrak{p} contains a non-zero-divisor x on M/N . If we have positive answer for the above problem, the Cousin complex $\mathcal{C}_R(\frac{M/N}{xM/N})$ is finite. Now by an induction argument on $\dim M$ and using Corollary 4.3.2 one finds that the formal fibre of R over \mathfrak{p} is Cohen-Macaulay.

Note that if $\mathcal{C}_R(M)$ is finite, then M is equidimensional and $R/0 :_R M$ is universally catenary by Corollary 2.3.1 and so M/xM is equidimensional. Now, if the formal fibres over all $\mathfrak{p} \in \text{Supp } M$ are Cohen-Macaulay, then Proposition 2.3.2 follows that $\mathcal{C}_R(M/xM)$ is finite. Hence solving Problem 4.3.4 is equivalent to find an answer for Question 4.3.1.

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