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The Aluffi algebra of an ideal

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Dedication

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Introduction

In a remarkable paper ([2]) Paolo Aluffi introduced a graded algebra which he called quasi-symmetric. His purpose was to introduce the characteristic cycle of a hypersurface that would stand for the characteristic classes à la Schwartz–MacPherson the same way the conormal cycle is known to stand for the characteristic classes à la Chern–Mather.

The procedure he employed, in a nutshell, can be explained as follows. Let $\mathcal{X}$ denote a scheme of finite type over a field and let $\mathcal{E}$ stand for a locally free $\mathcal{O}_{\mathcal{X}}$-module of rank $e+1$, with $\mathbb{P}(\mathcal{E})$ designating the corresponding $\mathbb{P}$-bundle of rank e over $\mathcal{X}$ and the structural map $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathcal{X}$.

Letting $\mathcal{A}$ indicate the Chow group, there is a well known isomorphism that reads on the graded pieces of the group as

$$\mathcal{A}_r\, \mathbb{P}(\mathcal{E}) \xleftarrow[\simeq]{\pi^*} \mathcal{A}_{r-e}\, \mathcal{X},$$

given by the Segre pullback operator. Thus, any cycle C in the left hand side has a unique expression of the form

$$C = \sum_{j=0}^{e} c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^j \cap \pi^*(C_{r-e+j}).$$

Aluffi calls the sum $C_{r-e} + C_{r-e+1} + \cdots + C_r$ the *shadow* of C. The main result can be briefly stated in the following way:

Theorem. Let $\mathcal{X} \subset \mathcal{M}$, where $\mathcal{X}$ is a hypersurface and $\mathcal{M}$ is a smooth variety. Let $\mathcal{Y} \subset \mathcal{X}$ stand for the singular variety of $\mathcal{X}$. Then

$$\begin{aligned}
(-1)^{\dim \mathcal{X}} \check{c}_{\mathrm{Ma}}(\mathcal{X}) &= \text{shadow of the conormal cycle } [\mathrm{Bl}_{\mathcal{Y}}(\mathcal{X})] \subset \mathbb{P}(T^*\mathcal{M}|\mathcal{X}) \\
(-1)^{\dim \mathcal{X}} \check{c}_{\mathrm{SM}}(\mathcal{X}) &= \text{shadow of the characteristic cycle } [\mathrm{q}\mathcal{S}_{\mathcal{Y}}(\mathcal{X})] \subset \mathbb{P}(T^*\mathcal{M}|\mathcal{X}).
\end{aligned}$$

Here, “Ma” stands for “Mather”, while “SM” stands for “Schwartz–Macpherson”. The algebra $\mathrm{q}\mathcal{S}_{\mathcal{Y}}(\mathcal{X})$ is what Aluffi called *quasi-symmetric*.

In this work we trade Aluffi’s original designation for the present one both to honor its creator and to indicate that the algebra itself has a more complex behavior for more general schemes than hypersurfaces and, as such, it will often tilt to the other end of the spectrum, namely, becoming a honest blowup algebra.

Grosso modo the material presented here encloses two sorts of results. First, one studies the properties of the Aluffi algebra in a quite general ring-theoretic setup, bringing in some of the typical objects and invariants of commutative algebra. This will take up most of the first chapter. The second chapter deals with the special case of a projectively embedded hypersurface and its singular locus ("gradient ideal") in an effort to close on the main guidelines for the deeper transit of the algebra amidst the prevailing notions in commutative algebra.

As a sidekick, one has a better view of how more structured is the algebra in the case of a homogeneous equation than that of its affine companion. In characteristic zero – or high enough characteristic – the intervenience of the Euler formula is rather crucial in order to obtain the specifics of the algebra. We will comment more on this in the next paragraph.

We now describe the content of the thesis in more detail.

We start with the definition of this algebra for the pair of ideals $J \subset I$ of a commutative Notherian ring R and call it *embedded Aluffi algebra*. The Aluffi algebra of I/J intermediates between the symmetric algebra of I/J and the corresponding Rees algebra, thus yielding first crude dimension bounds. In the next step we assume that the base ring R is local (respectively, standard graded), with maximal (respectively, irrelevant maximal) ideal $\mathfrak{m}$. In the *non-degenerate* case, this enables us to bring in the dimension of Aluffi algebra as related to the dimension of the base ring and its analytic spread. In particular, if I is generated by $\dim R$ analytically independent elements, then the dimension of Aluffi algebra is the dimension of R and the extension of $\mathfrak{m}$ to the Rees algebra of I over R is a minimal prime ideal of the Aluffi algebra. Equidimensionality of the Aluffi algebra may be quite rare. However, it is shown that for the pair of ideals $J \subset I \subset R$, where J is a principal ideal generated by a regular element, if the base ring R is catenary, equidimensional and equicodimensional, then the Aluffi algebra is equidimensinal of dimension $\dim R$.

In section 1.5 we proof that the torsion of the Aluffi algebra is the *module of Valabrega–Valla* [19]. Thus, provided I has regular elements modulo J, the Rees algebra is the symmetric algebra modulo torsion of the Aluffi algebra. The natural question arises as to what pairs of ideals $J \subset I$, the Aluffi algebra is torsion-free. One of the easy cases is when I/J is of linear type over R/J, thus retrieving a result of Valla ([21]. The module of Valabrega-Valla has a close relationship with standard bases in the sense of Hironaka ([4]), which we distill in relation to properties of the torsion. In particular, one finds necessary and sufficient conditions for the natural surjection from the Aluffi algebra to the corresponding Rees algebra of I/J to be an isomorphism. We show that a power of I annihilates the kernel of the above surjection and relates the exponent to the the Artin-Rees number of J relative to I. The precise relationship between the Aluffi algebra with this and the notion of relation type are collected in Sections 1.6 and 1.7.

In section 1.8, we study the minimal primes of the Aluffi algebra. Here, if P is a prime ideal of R associated to J then its extended-contracted ideal $\tilde{P} = PR[t] \cap \mathcal{R}_R(I)$ is a prime ideal of $\mathcal{R}_R(I)$ associated to the Aluffi algebra. It seems suggestive to call such primes *torsion primes*. The annihilation nature of the torsion allows us a glimpse of the remaining minimal primes. This gives that, provided R/J and the Aluffi algebra be equidimensional, the dimension of the latter is dimension of the Rees algebra of

I/J (i.e., $\dim R/J + 1$). Somewhat surprising is that, for a non-degenerate pair of ideals $J \subset I$ in a regular local ring R, if the Aluffi algebra is Cohen-Macaulay then J must be principal.

As a matter of illustrating the theory, we have selected examples of pairs of ideals for which the corresponding Aluffi algebra has no torsion. In this connection, a side question seems to impose itself naturally, to wit: let $J \subset R = k[\mathbf{x}]$ denote an ideal of codimension at least 2, generated by 2-forms, and let $I \subset R$ denote the ideal generated by the r–minors ($r \geq 2$) of the Jacobian matrix Θ of the generators of J. If I is $(\mathbf{x})$–primary, then $I = (\mathbf{x})^r$.

After the generalities on the Aluffi algebra, we examine more closely the case where J is a principal ideal in a polynomial ring over an infinite field. A source of motivation is the original work of Aluffi, where he has inquired into the structure of the algebra in the case that J is generated by the equation of a reduced hypersurface. Thus, let $J = (f) \subset R$ where f is a squarefree polynomial in the polynomial ring $R = k[X_1, \ldots, X_n]$. We shall focus on the Jacobian ideal $I = I_f = (f, \partial f/\partial X_1, \ldots, \partial f/\partial X_n)$.

Now, in general f will not be Eulerian, hence the local number of generators of I_f maybe an early obstruction for I_f being an ideal of linear type – examples of this sort are available with no difficulty. Of course, if k is algebraically closed of characteristic zero one knows that a power of f belongs to the *gradient* ideal $(\partial f/\partial X_1, \ldots, \partial f/\partial X_n)$ of f, so up to radical it does not make a difference. Alas, as pointed out by Aluffi, the radical of the singular locus is hardly of true interest in the present realm.

On the other hand, if f is Eulerian – e.g., if f homogeneous in the standard grading of the polynomial ring and and its degree is not a multiple of the characteristic – then it seems like a good bet to expect that I often be of linear type over R (of course $I/(f)$ over $R/(f)$ will never be of linear type – not even generated by analytically independent elements for that matter – as the defining equations of the dual variety to $V(f)$ will constitute a perpetual obstruction).

We will modestly consider only the homogeneous case and actually spend a good deal of effort looking at the case where the projective hypersurface f has isolated singularities – e.g., in the plane case.

Once for all, we will assume throughout that $\mathrm{char}(k) = 0$ or at least that the latter does not divide the degree of f. In this case, by the Euler formula, $f \in I_f$.

In the first section, we shall call the Aluffi algebra of $I_f/(f)$ the *Aluffi gradient* algebra. We suppose that the hypersurface is defined by f has isolated singularity and its partial derivation are algebraically independent, so we collect the central backstage for the Aluffi gradient algebra. Note that if I_f is of linear type, then the symmetric algebra of $I_f/(f)$ is isomorphic to the Aluffi gradient algebra. We prove that the converse is true. In fact, we retrieve a result of Aluffi in the projective hypersurface case, where he proved the converse only for weakly of linear type. Finally we show that the symmetric algebra of $I_f/(f)$ is Cohen-Macaulay, therefore if I_f is of linear type the Aluffi gradient algebra is Cohen-Macaulay.

As it will turn, even for projective plane curves, the fine structure of the gradient Aluffi algebra seems to be fairly intrincate and may depend on the nature of the singularities. In the above we said that if I_f is of linear type, then the symmetric algebra and the Aluffi gradient algebra are isomorph. This motivate us to study the Jacobian ideal of a projective hypersurface. In the second section, we study the linear type property of the jacobian ideal of f. We assume that f has isolated singularity, this means that the gradient ideal I_f is a strict almost complete intersection. Fortunately enough, this property is fairly manageable from the algebraic viewpoint. We prove that, the gradient ideal of f is of linear type if and only if the coordinates of vector fields of $\mathbb{P}^n_k$ vanishing on f generate an irrelevant ideal if and only if locally at each singular prime the gradient ideal is a complete intersection.

As it turns, the gradient ideal I_f is not of linear type in general. As a simple application of section 2.2, we show that, if f is a irreducible homogenous polynomial in $R = k[x, y, z]$ of degree ≤ 3, then the gradient ideal I_f is of linear type. Our guiding question is, what can we say about the linear type property of the gradient ideal of f when f defines a irreducible *quartic, quintic* and *sextic*. We show simple examples of rational plane quintics and sextics whose corresponding gradient ideals are not of linear type. Moreover, the associated Aluffi gradient algebras behaves quite erratically from the viewpoint of their associated primes.

Using a well-known classification of rational quartics, according to the nature of its singularities (see [23]), one is able to deal with the linear type property. For this, one considers families of singular curves with parameter space an affine space, with coordinates $\mathbf{a} = a_1, \ldots, a_m$, assuming that the general member of the family is a reduced singular plane curve. If $F \in S = k[x, y, z, \mathbf{a}]$ is the equation $F \in S = k[x, y, z, \mathbf{a}]$ which defines the family, we speak of a normal form of F depending on this singular locus. In examples 2.4.1 and 2.4.2, we write a normal form for families of irreducible singular quartic plane curve such that the general member has singularities of type node and cusp. We can apply the method given in such example for finding a normal form for an arbitrary family with simple singularities. Note that with more involved singularities, finding a normal form may be hard.

The normal form has degeneration – in the sense of evaluating the normal form F on tuples in k – making the singularities of the general member of the family degenerate to more special singularities (e.g., a node degenerating to a cusp, or a cusp to a tacnode,etc). In order to deal with this problem, one considers the the ideal $\mathcal{I}$ generated by the 1–minors of the a syzygy matrix of the ideal generated by the x, y, z–partial derivation of F. One shows that $\operatorname{codim} \mathcal{I} = 3$ if and only if the plane projective curve obtained by evaluating $(a_1, \ldots, a_m)$ at points in $\mathbb{A}^m$ off the zero set of $(\mathcal{I} : (x, y, z)S^{\infty}) \cap k[\mathbf{a}]$ has gradient ideal of linear type. This yields a handy purely algebraic criterion to verify the property of being of linear type for the gradient ideal of a plane curve.

The main result of this part is a proof that, for degree up to 4 (inclusive) the Jacobian ideal is of linear type, hence the Aluffi algebra coincides with the naive blowup. This is false in higher degrees - we treat some aspects of this in the Appendix.

Chapter 1

The embedded Aluffi algebra

1.1 Preliminaries

Let A be a ring and $\mathfrak{a}$ an ideal of A. The two most common and important commutative algebras related to these data are the symmetric algebra $\mathcal{S}_A(\mathfrak{a})$ and the *Rees algebra* $\mathcal{R}_A(\mathfrak{a})$. The latter is defined as

$$\mathcal{R}_A(\mathfrak{a}) := \bigoplus_{t\geq 0} \mathfrak{a}^t \simeq A[\mathfrak{a}u] \subset A[u],$$

which immediately implies that it is torsionfree over the base ring A. From the definition follows a natural surjection of standard A-graded algebras

$$\mathcal{S}_A(\mathfrak{a}) \twoheadrightarrow \mathcal{R}_A(\mathfrak{a}). \tag{1.1}$$

This map is injective locally on the primes of the base off the support of $A/\mathfrak{a}$. It follow by general arguments that, provided $\mathfrak{a}$ has some regular element, the kernel is the A-torsion submodule (ideal) of the symmetric algebra. If the map in (1.1) is injective one says that the ideal $\mathfrak{a}$ is of *linear type*, a rather non-negligible notion in parts of ideal syzygy theory.

While the dimension of the symmetric algebra lacks an easy closed expression in terms of its basic structural ingredients, there is a general result about dimension of the Rees Algebra of a ideal which is well-known (see, e.g., [6], also [22]).

Proposition 1.1.1 *Let R be a Notherian ring of finite Krull dimension and I be an ideal of R. Then:*

$$\dim \mathcal{R}_R(I) = \begin{cases} \dim R+1, & \textit{if } I \nsubseteq \bigcap_P P,\ P \textit{ is a prime ideal with } \dim R/P = \dim R; \\ \dim R, & \textit{otherwise.} \end{cases}$$

So much for this well-established modicum. Let us now turn to the main object of the work.

Given $\mathfrak{a} \subset A$ as above, let $R \twoheadrightarrow A$ denote a surjective ring homomorphism and let $I \subset R$ denote the inverse image of $\mathfrak{a}$ in R. By functoriality of the symmetric algebra and the above piece, there are surjective A-algebra maps

$$\begin{array}{ccc} \mathcal{S}_R(I) & \twoheadrightarrow & \mathcal{R}_R(I) \\ \downarrow & & \\ \mathcal{S}_A(\mathfrak{a}) & & \end{array}$$

This can be naturally completed to a commutative diagram by the obvious tensor product, thus yielding:

Definition 1.1.2 The *R-embedded Aluffi algebra* of $\mathfrak{a}$ is

$$\mathcal{A}_{R \twoheadrightarrow A}(\mathfrak{a}) := \mathcal{S}_A(\mathfrak{a}) \otimes_{\mathcal{S}_R(I)} \mathcal{R}_R(I).$$

It is rather easy to see that the R-embedded Aluffi algebra is functorial in the sense that any homomorphism of rings $R \to S$ compatible with ring surjections $R \twoheadrightarrow A$ and $S \twoheadrightarrow A$ induces an A-algebra surjection

$$\mathcal{A}_{R \twoheadrightarrow A}(\mathfrak{a}) \twoheadrightarrow \mathcal{A}_{S \twoheadrightarrow A}(\mathfrak{a}).$$

Having in mind a notion of the algebra that is independent of a particular map $R \twoheadrightarrow A$ – or, geometrically, independent of a chosen embedding of Spec (A) – Aluffi has used the above fact to bring out the inverse limit

$$\mathcal{A}_A(\mathfrak{a}) := \lim_{R \twoheadrightarrow A} \mathcal{A}_{R \twoheadrightarrow A}(\mathfrak{a}),$$

which takes in account all possible embeddings at once. This he dubbed the *quasi-symmetric algebra* of $\mathfrak{a} \subset A$.

However, in the sequel, he argues that $\mathcal{A}_A(\mathfrak{a})$ is actually independent of the choice of the source provided R is constrained to be regular, so that if R is indeed regular then $\mathcal{A}_A(\mathfrak{a}) \simeq \mathcal{A}_{R \twoheadrightarrow A}(\mathfrak{a})$ by the structural map ([2, Theorem 2.9]).

In this thesis an approach is taken to look at one single member of the above inverse limit, pretending that one is in the situation where R is regular, but not assuming this ab initio - hence, we deal solely with an R-embedded Aluffi algebra as defined above. Also, provided no confusion is caused, we usually omit "R-embedded" altogether.

With this change of viewpoint, we are dealing with a ring R and two ideals $J \subset I \subset R$. Thus, we refer to $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J)$ as the Aluffi algebra of the pair $J \subset I$ and even write $\mathcal{A}_{R/J}(R/J)$ to make it lighter on the reading.

At first, note that if the ideal I is of linear type – i.e., the natural surjection $\mathcal{S}_R(I) \twoheadrightarrow \mathcal{R}_R(I)$ is injective – then trivially $\mathcal{S}_{R/J}(I/J) = \mathcal{A}_{R/J}(I/J)$. The following example shows that, in general, there is no converse to this statement even when R is a hypersurface domain.

Example 1.1.3 Let $R = k[x, y, z] = k[X, Y, Z]/(XY - Z^2)$, with $J = (x, z)$ (the ideal of a ruling in the affine cone) and $I = (x, y, z)$. Then $R/J \simeq k[Y]$ and $I/J \simeq (Y)$. Therefore, I/J is of linear type, hence $\mathcal{S}_{R/J}(I/J) \simeq \mathcal{A}_{R/J}(I/J) \simeq \mathcal{R}_{R/J}(I/J)$.

More generally, one can take $(R, \mathfrak{m})$ to be a non-regular local ring – or a non-degenerate standard graded algebra over a field and its irrelevant ideal – with $J \subset I = \mathfrak{m}$ such that R/J is regular. Then I/J is generated by a regular sequence on R/J, hence is of linear type.

A simpler example dropping the integrality assumption on R is given by $R = k[x, y] = k[X, Y]/(XY)$, with $J = (x) \subset I = (x, y)$.

It would be interesting to find such examples with $(R, \mathfrak{m})$ a regular local ring and $J \subset \mathfrak{m}I$. We will see that no such examples exist in certain situations where J is principal – see Theorem 2.1.1.

As a preliminary, we have a useful presentation of the R-embedded Aluffi algebra. This has in fact been already observed in [2, Theorem 2.9] in the context of schemes. Our proof is in the purely algebraic context and perhaps slightly more illuminating.

Lemma 1.1.4 *Let $J \subset I$ be ideal of the ring R.There are natural A-algebra isomorphisms*

$$\mathcal{A}_{R/J}(I/J) \simeq \frac{\mathcal{R}_R(I)}{(J, \tilde{J})\,\mathcal{R}_R(I)}$$

where J is in degree 0 and $\tilde{J}$ is in degree 1. In particular, there is a surjective A-algebra homomorphism $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$.

Proof. We have an exact sequence of R/J-module

$$0 \to J/IJ \to I/JI \to I/J \to 0$$

Hence, by the functorial property of the symmetric algebra, we get an epimorphism

$$\mathcal{S}_{R/J}(I/JI) \twoheadrightarrow \mathcal{S}_{R/J}(I/J)$$

whose kernel is the ideal of $\mathcal{S}_{R/J}(I/JI)$ generated by J/JI as elements of degree 1. By the change of base ring property of the symmetric algebra we get $\mathcal{S}_{R/J}(I/JI) \simeq \mathcal{S}_R(I)/J\mathcal{S}_R(I)$. Hence

$$\mathcal{S}_{R/J}(I/J) \simeq \frac{\mathcal{S}_R(I)}{(J, \tilde{J})\,\mathcal{S}_R(I)}.$$

Now tensoring with $\mathcal{R}_R(I)$ gives the first isomorphism. The second one is now immediate from the definition of $\tilde{J}$. □

1.2 A propaedeutic example

The following example may serve as a useful guide through various aspects of the subsequent sections. We will accordingly analyze the intermediate steps, once at the time.

• Consider the rational map $\mathcal{F} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ defined by five sufficiently general quadrics $\mathbf{q} = \{q_0, q_1, q_2, q_3, q_4\} \subset R$. Then the base ideal $(\mathbf{q}) \subset R$ is a Gorenstein ideal of finite colength.

It is classically known, and easy to see, that the image of this map is a surface obtained as a general projection of the 2-Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. Therefore, the homogeneous coordinate ring of the image is not integrally closed, hence neither is the k-subalgebra $k[\mathbf{q}] \subset R$ as they are isomorphic as k-standard graded algebras (up to an obvious degree normalization). Write $P \subset k[\mathbf{T}]$ for the homogeneous defining ideal of the image of $\mathcal{F}$.

• By geometric considerations, one knows that the homogeneous defining ideal of this smooth surface is generated by cubic forms. Perhaps remarkable is that a set of minimal generators can be taken to be seven among the ten maximal minors of a well-structured 5×3 matrix.

• To explain this supplementary fact, let φ denote the 5×5 skew-symmetric matrix whose Pfaffians are the generators of I. Pick a new set of indeterminates $\mathbf{T} = \{T_0, T_1, T_2, T_3, T_4\}$ (think of them as the homogeneous coordinates of $\mathbb{P}^4$) and consider the entries of the matrix product $\mathbf{T} \cdot \varphi$. Next take the Jacobian matrix ψ of these bihomogeneous polynomials of bidegree $(1, 1)$ with respect to x, y, z – the so-called *Jacobian dual matrix* of φ ([16]). Note that this a 5×3 matrix whose entries are linear forms in $k[\mathbf{T}]$.

• By general argumentation as in [7], one can show that the maximal minors of ψ are polynomial relations of the 5 original quadrics. Therefore, from previous steps, the codimension of the ideal $I_3(\psi)$ is at most 2. The hard knuckle is to show that this is a prime ideal – here one can resort to the classical Northcott induction trick (inverting-localizing) or else to a computer calculation assuming general data. Altogether this yields that $I_3(\psi)$ is the homogenous defining ideal of the rational map defined by the 5 quadrics. Therefore, this is the homogenous defining ideal of a general projection of the 2-Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$, as mentioned above.

• Returning to the Rees algebra $\mathcal{R}_R(I)$, as a consequence of the above discussion, the ideal $(I_1(\mathbf{T}\cdot\varphi), I_3(\psi)) \subset R[\mathbf{T}]$ is contained in a presentation ideal of $\mathcal{R}_R(I)$ on $R[\mathbf{T}]$ and has codimension ≥ 4. Therefore, its codimension is exactly 4 by Proposition 1.1.1. The claim is that this ideal is prime, hence must coincide with the presentation ideal of $\mathcal{R}_R(I)$ on $R[\mathbf{T}]$.

• Of course, the previous step can be readily accomplished by computer calculation assuming general data. Else, there is a subtler – but more convoluted – way, consisting in determining the rank of ψ modulo $I_3(\psi)$. This rank is obviously at most 2. Since the rational map defined by the general quadrics is a birational map of $\mathbb{P}^2$ onto image, as is known by classical arguments, one must have $\text{rank}(\psi) \equiv 2 \pmod{\mathrm{I}_3(\psi)}$ ([15, Theorem 2.4]). Moreover, by [*loc.cit.*], the coordinates of any nonzero homogeneous syzygy of $\psi \pmod{I_3(\psi)}$ defines the inverse rational map. In particular, these forms are algebraically independent over k. Actually, they will generate an ideal of linear type $\pmod{I_3(\psi)}$. From this and from [15, Proposition 2.1] now follows that $(I_1(\mathbf{T} \cdot \varphi), I_3(\psi)) \subset R[\mathbf{T}]$ is a presentation ideal of $\mathcal{R}_R(I)$ on $R[\mathbf{T}]$

• Now, we go slightly more special. Namely, let $J \subset R$ stand for the defining ideal of four straight lines through the origin of k^3, in general position. Let $I \subset R$ denote the Jacobian ideal of R/J lifted to R, i.e., $I = (J, I_2(\Theta))$, where $I_2(\Theta)$ stands for the ideal of 2-minors of the Jacobian matrix of a set of minimal homogeneous generators of J.

• Thus, I defines the singular scheme of the four lines. Note that the defining ideal of the four lines in $\mathbb{A}^3$ is a complete intersection of two 2-forms, say, $J = (q_0, q_1)$. It is apparent that the 2-minors of the Jacobian matrix of q_0, q_1, together with these two quadrics, are five sufficiently general 2-forms. Thus, we can apply the previous situation.

• So let us now write down a presentation of the Aluffi algebra of the pair $J \subset I$ drawing on Lemma 1.1.4. We present the Rees algebra $\mathcal{R}_R(I)$ over $R[\mathbf{T}]$ based on the assignment $T_j \mapsto q_j$, with $J = (q_0, q_1)$. It follows immediately that

$$\begin{aligned} \mathcal{A}_{R/J}(I/J) &\simeq R[\mathbf{T}]/(J, T_0, T_1, I_1(\mathbf{T}\varphi), I_3(\psi)) \\ &\simeq (R/J)[T_2, T_3, T_4]/(I_1(\mathbf{T}\bar{\varphi}), I_3(\bar{\psi})), \end{aligned} \tag{1.2}$$

where $\bar{\varphi}$ denotes the submatrix of φ consisting of the three last rows and $\bar{\psi}$ is obtained from ψ by evaluating both T_0, T_1 at zero. The explicit format of the defining equations of the Aluffi algebra as in (1.2) is a bit involved. Its R/J-torsion is a nonzero ideal generated in degree 2, i.e, by elements of $J \cap I^2$ not belonging to JI (a better insight into the torsion is given in Proposition1.5.5).

Next is an example with concrete data.

Example 1.2.1 In order to have a familiar geometric transcription, we assume that k is algebraically closed and of characteristic zero. Let $J \subset R = k[x, y, z]$ denote the homogeneous defining ideal of the four points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$ and $(1 : 1 : 1)$ in the projective plane $\mathbb{P}^2_k$. An easy calculation gives $J = (x^2 - xz, y^2 - yz)$, while the Jacobian matrix of these 2-forms is:

$$\Theta = \begin{pmatrix} 2x - z & 0 & -x \\ 0 & 2y - z & -y \end{pmatrix}$$

Therefore, $I = (x^2 - xz, y^2 - yz, x(2y - z), y(2x - z), (2x - z)(2y - z))$. One can in principle reconstruct the skew-symmetric 5×5 matrix whose Pfaffians are these five 2-forms. However, a computer calculation will do everything for us quite readily. The defining equations of the Aluffi algebra are still intricate to write down explicitly. Instead, we can give the torsion ideal, which is fairly simple for these data (always by computer calculation). In terms of the defining variables and equations, the torsion of the Aluffi algebra is the ideal generated by the residues of $T_3(T_2 + T_3)$ and $T_4(T_2 - T_4)$. Or still, in terms of the internal grading of the algebra, using the description in Proposition1.5.5, the torsion is generated by the appropriate residues of $\{xz^2(x - z), yz^2(y - z)\} \subset J \cap I^2$.

Remark 1.2.2 In the above example one has the following values for dimensions:

$$\dim \mathcal{S}_{R/J}(I/J) = 3, \ \dim \mathcal{A}_{R/J}(I/J) = \dim \mathcal{R}_{R/J}(I/J) = 2.$$

Thus, it would look like that, at least from a dimension theoretic point of view, the Aluffi algebra is tilting more towards to the blowup algebra than to the naive blowup algebra. Aluffi's original terminology possibly comes from the case where J is a principal ideal, in which case the situation can tilt over in the other direction.

Moreover, if one starts with a set of points in $\mathbb{P}^n$ in general position, whose cardinality is at most $n+1$, the structural surjection $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ may turn out to be an isomorphism (see Example 1.9.7 for the case of the coordinate points of $\mathbb{P}^n$).

1.3 Dimension

From the definition and Lemma 1.1.4, the Aluffi algebra is squeezed as

$$\mathcal{S}_{R/J}(I/J) \twoheadrightarrow \mathcal{A}_{R/J}(I/J)(I/J) \twoheadrightarrow \mathcal{R}_{I/J}(I/J) \tag{1.3}$$

and is moreover a residue ring of the ambient Rees algebra $\mathcal{R}_R(I)$.

Therefore, one has right at the outset:

$$\dim \mathcal{R}_{I/J}(I/J) \leq \dim \mathcal{A}_{R/J}(I/J) \leq \min\{\mathcal{R}_R(I), \dim \mathcal{S}_{R/J}(I/J)\} \tag{1.4}$$

By general elementary reasons, there is an a priori upper bound:

Lemma 1.3.1 *Let $J \subsetneq I \subsetneq R$ be ideals of the Noetherian ring R. If J has a regular element then* $\dim \mathcal{A}_{R/J}(I/J) \leq \dim R$.

Proof. Since $\mathcal{R}_R(I)$ is R-torsionfree, one has $\mathrm{ht}\, J\mathcal{R}_R(I) \geq 1$. From this it follow immediately that $\dim \mathcal{A}_{R/J}(I/J) \leq \dim \mathcal{R}_R(I)/J\mathcal{R}_R(I) \leq \dim R + 1 - 1 = \dim R$. □

Note that this is all one can assert in such generality because if, e.g., a power of the ideal I is contained in J, then $\dim \mathcal{A}_{R/J}(I/J) = \dim R/J \leq \dim R - \mathrm{ht}\, J$ is arbitrarily smaller than $\dim R$. In order to obtain more precise dimensional results one needs at least that $\mathrm{ht}\,(J) < \mathrm{ht}\,(I)$.

Proposition 1.3.2 *Let $J \subsetneq I \subsetneq R$ be ideals of the Noetherian ring R. If*

(a) *If I/J has a regular element then*

$$\dim R/J + 1 \leq \dim \mathcal{A}_{R/J}(I/J) \leq \min\{\dim R + 1\,,\, \dim \mathcal{S}_{R/J}(I/J)\,\}.$$

(b) *If R is universally catenary, J is a prime ideal and I/J has a regular element then*

$$\min\{\dim R + 1\,,\, \dim R/J + 1 + \mathfrak{f}_{R/J}(I/J)\} \geq \dim \mathcal{A}_{R/J}(I/J) \geq \dim R/J + 1,$$

where $\mathfrak{f}_{R/J}(I/J) = \min\{k \geq 0 \mid \mu(I/J)_\wp \leq \dim(R/J)_\wp + k + 1,\ \forall\, \wp \in \mathrm{Spec}(R/J)\}$, *in particular, if $\mu(I/J)_\wp \leq \dim(R/J)_\wp + 1$ for every prime $\wp \subset R/J$ then* $\dim \mathcal{A}_{R/J}(I/J) = \dim R/J + 1$.

Proof. (a) This follows immediately from (1.3) by the well-known dimension formula for the Rees algebra of an ideal containing a regular element.

(b) The leftmost inequality follows from (b) by way of an explicit formula for the dimension of the symmetric algebra based on [18, Theorem 1.1.3]. □

Remark 1.3.3 The integer $\mathfrak{f}_{R/J}(I/J)$ has been called the *Fitting defect* of I/J ([13, Proposition 2.2]) and, with some additional hypotheses, it can also be given in terms of the zeroth local cohomology module on I/J.

Next is a reasonably important case where the inequality in Lemma 1.3.1 becomes an equality.

Theorem 1.3.4 *Let R be a catenary, equidimensional and equicodimensional Noetherian ring and let $J \subset I \subsetneq R$ be ideals, where J is a principal ideal generated by a regular element. Then $\mathcal{A}_{R/J}(I/J)$ is equidimensional and* $\dim \mathcal{A}_{R/J}(I/J) = \dim R$.

Proof. The value of the dimension follows from Lemma 1.3.1, and the right hand inequality of (b) by noting that, under the assumptions on R and J, $\dim R/J + 1 = \dim R - 1 + 1 = \dim R$.

Set $J = (f)$ with f a non-zero-divisor. To prove the equidimensionality part, we will show that $\mathcal{A}_{R/(f)}(I/(f))$ is equidimensional locally at every prime ideal $\mathcal{P} \subset \mathcal{R}_R(I)$ in its support. Localizing first at $\mathcal{P} \cap R$ in the base ring one can assume that $(R, \mathfrak{m})$ is local, with $\mathcal{P} \cap R = \mathfrak{m}$ and $I \subset \mathfrak{m}$. Now, $\mathcal{M} = (\mathfrak{m}, Iu) \subset \mathcal{R}_R(I)$ is not a minimal prime of $\mathcal{A}_{R/(f)}(I/(f))$. This is because the Aluffi algebra is graded, with grading induced from $\mathcal{R}_R(I)$, hence $\mathcal{M}$ would actually be its unique associated prime, which is impossible as $\dim R \geq 1$. Thus, for the purpose of showing equidimensionality, we may assume that $\mathcal{P}$ is a homogeneous ideal properly contained in $\mathcal{M}$.

Let $I = (f_1, \ldots, f_m)$. Note that in the present situation, one has by Lemma 1.1.4:

$$\mathcal{A}_{R/(f)}(I/(f)) \simeq \mathcal{R}_R(I)/(f, \tilde{f}).$$

Write $\mathcal{R}_R(I) = R[f_1u, \ldots, f_mu] \subset R[u]$, so that $\tilde{f} = \sum_{j=1}^m g_j f_j u$, for suitable $g_j \in R$.

Suppose first that $(Iu) \not\subset \mathcal{P}$. Say, $f_1u \notin \mathcal{P}$. Localizing at $\mathcal{P}$ yields

$$\begin{aligned}
\mathcal{A}_{R/(f)}(I/(f))_{\mathcal{P}} &\simeq R[Iu]_{\mathcal{P}}/(f, \tilde{f})_{\mathcal{P}} \simeq R\left[\frac{I}{f_1}, f_1u, (f_1u)^{-1}\right]_{\mathcal{P}'} \Big/ \left(f, g_1 + \sum_{j=2}^m g_j \frac{f_j}{f_1}\right)_{\mathcal{P}'} \\
&= R\left[\frac{I}{f_1}, f_1u, (f_1u)^{-1}\right]_{\mathcal{P}'} \Big/ \left(f, \frac{f}{f_1}\right)_{\mathcal{P}'} \\
&= R\left[\frac{I}{f_1}, f_1u, (f_1u)^{-1}\right]_{\mathcal{P}'} \Big/ \left(\frac{f}{f_1}\right)_{\mathcal{P}'},
\end{aligned}$$

where $\mathcal{P}'$ denotes the corresponding image of $\mathcal{P}$. The rightmost ring above is a factor ring of a catenary, equidimensional and equicodimensional ring by a principal ideal generated by a regular element, hence it is equidimensional and so is $\mathcal{A}_{R/(f)}(I/(f))_{\mathcal{P}}$ too.

Suppose now that $(Iu) \subset \mathcal{P}$. Then $\mathfrak{m} \not\subset \mathcal{P}$ since $\mathcal{P} \subsetneq \mathcal{M}$, hence $p := \mathcal{P} \cap R \subsetneq \mathfrak{m}$. Note that p is a prime containing f.

If $I \not\subset p$ then $\mathcal{A}_{R/(f)}(I/(f))_{\mathcal{P}}$ is a localization of the ring

$$R_p[I_p u]/(f, \tilde{f}) = R_p[u]/(f, fu) = R_p[u]/(f)$$

and we conclude as above. If $I \subset p$ then $\mathcal{A}_{R/(f)}(I/(f))_{\mathcal{P}}$ is a localization of the Aluffi algebra $\mathcal{A}_{R_p/(f)}(I_p/(f))$ and we conclude by induction on $\dim R$. □

1.4 Local or graded case

As above, the minimally interesting setup regarding dimensions requires that J have a regular element - or at least that J have positive codimension. Therefore we will assume throughout that this is the case.

Let us suppose in this part that $(R, \mathfrak{m})$ is a Noetherian local ring and its maximal ideal or a standard graded algebra over a field and its maximal irrelevant ideal. Let $J \subset I \subset \mathfrak{m}$ with J containing a regular element. We confront ourselves with two quite opposite situations, namely, when $J \subset \mathfrak{m}I$ and when J contains minimal generators of I. Note that the first alternative is a trivial obstruction for J being a reduction of I as together they would entail for $t >> 0$

$$I^t = JI^{t-1} \subset \mathfrak{m}I^t,$$

hence $I^t = \{0\}$, i.e., I would be nilpotent.

Assume as before that I is a proper ideal. Drawing on a terminology of geometric flavor, let us agree to say that the pair $J \subset I$ of ideals is *non-degenerate* if $J \subset \mathfrak{m}I$. If on the other extreme, $J \subset I$ is generated by a subset of minimal generators of J, we may call the pair $J \subset I$ *totally degenerate*. The latter case can usually be disposed of by a standard argument (see Proposition 1.5.2).

Definition 1.4.1 Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an ideal. The special fiber of the Rees algebra $\mathcal{R}_R(I)$ is the ring

$$\mathcal{F}(I) = \mathcal{R}_R(I)/\mathfrak{m}\mathcal{R}_R(I) = \bigoplus_{t \geq 0} I^t/mI^t.$$

Its Krull dimension is called the *analytic spread* of I, and is denoted by $\ell(I)$

Proposition 1.4.2 *Let $(R, \mathfrak{m})$ be as above with $R/\mathfrak{m}$ infinite. Suppose that $J \subset \mathfrak{m}I$ and J has a regular element. Write $\ell(I)$ for the analytic spread of the ideal I. Then*

(i) $\ell(I) \leq \dim \mathcal{A}_{R/J}(I/J) \leq \dim R$.

(ii) *If moreover I is generated by* $\dim R$ *analytically independent elements then*

$$\dim \mathcal{A}_{R/J}(I/J) = \dim R$$

and $\mathfrak{m}\mathcal{R}_R(I)$ is a minimal prime of $\mathcal{A}_{R/J}(I/J)$ of maximal dimension.

(iii) *If $J \subset I^2$ then* $\dim \mathcal{A}_{R/J}(I/J) = \dim R$.

Proof. (i) $J \subset \mathfrak{m}I$ implies $JI^{t-1} \subset \mathfrak{m}I^t$ for every $t \geq 0$. This yields a surjective homomorphism $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{F}(I)$, showing that $\ell(I) \leq \dim \mathcal{A}_{R/J}(I/J)$. The other inequality comes from Lemma 1.3.1.

(ii) The first statement is clear from (i).

For the second assertion note that the assumption on I being generated by $\dim R$ analytically independent elements implies that $\mathcal{F}(I)$ is a polynomial ring over $R/\mathfrak{m}$ and, in particular, that $\mathfrak{m}\mathcal{R}_R(I)$ is a prime ideal of $\mathcal{R}_R(I)$. Since $JI^{t-1} \subset \mathfrak{m}I^t \subset \mathfrak{m}$ then $(J, \tilde{J}) \subset \mathfrak{m}\mathcal{R}_R(I)$ as ideals of $\mathcal{R}_R(I)$. Therefore $\mathcal{A}_{R/J}(I/J)/\mathfrak{m}\,\mathcal{A}_{R/J}(I/J) \simeq \mathcal{R}_R(I)/\mathfrak{m}\mathcal{R}_R(I) \simeq \mathcal{F}(I)$, hence $\mathfrak{m}\mathcal{A}_{R/J}(I/J)$ is a prime ideal with

$$\dim \mathcal{A}_{R/J}(I/J)/\mathfrak{m}\,\mathcal{A}_{R/J}(I/J) = \dim R = \dim \mathcal{A}_{R/J}(I/J)$$

by the first part.

(iii) Since $J \subset I^2$, one has $JI^{t-1} \subset I^{t+1}$ for every $t \geq 0$. This yields a surjective homomorphism $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \operatorname{gr}_I(R)$, showing that $\dim \mathcal{A}_{R/J}(I/J) \geq \dim R$. The reverse inequality follows from Lemma 1.3.1. $\square$

Example 1.4.3 Let $f = x_0^2x_2 + x_0x_1x_3 + x_1^2x_4$ denote the celebrated Gordan–Noether cubic whose Hessian vanishes, i.e., the partial derivatives of f are algebraically dependent over k – in other words, the image of the corresponding polar map is a proper set of $\mathbb{P}^4$ (a quadric cone). On the other hand, $\dim \mathcal{A}_{R/(f)}(I_f/(f)) = \dim R = 5 > 4 = \ell(I)$, where I_f denotes the gradient ideal of f.

We wrap up with a comment on the last result. Namely, we actually have

$$\dim \mathcal{A}_{R/J}(I/J) \geq \max\{\ell(I), \dim R/J + 1\}.$$

The interesting case is when $\ell(I) \geq \dim R/J + 1$. Say, R is catenary and equidimensional. Then it would entail

$$\dim \mathcal{A}_{R/J}(I/J) \geq \frac{\dim R + 1}{2}.$$

We will have more to say about the dimension in Proposition 1.6.4.

1.5 Torsion: the Valabrega–Valla module

Let $J \subset I \subset R$ be ideals. By Lemma 1.1.4 one has

$$\mathcal{A}_{R/J}(I/J) \simeq \mathcal{R}_R(I)/(J, \tilde{J})\mathcal{R}_R(I) = \bigoplus_{t\geq 0} I^t/JI^{t-1}.$$

As was already observed, from this follows immediately that the Aluffi algebra surjects onto the Rees algebra

$$\mathcal{R}_{R/J}(I/J) = \bigoplus_{t\geq 0}(I^t, J)/J \simeq \bigoplus_{t\geq 0} I^t/J\cap I^t$$

Note that both algebras are standard graded over R/J. The kernel of this surjection is the so-called *module of Valabrega–Valla* (see [19], also [22, 5.1]):

$$\mathcal{W}_{J\subset I} = \bigoplus_{t\geq 2} \frac{J\cap I^t}{JI^{t-1}}. \tag{1.5}$$

Of course, as an ideal this kernel is generated by finitely many homogeneous elements, but as graded R/J-module it is conceivable that it it may fail to be so.

In the present framework we are mainly interested in the case where $\mathrm{ht}\,(J) < \mathrm{ht}\,(I)$ (strict inequality), while the Valabrega–Valla module has been mainly considered in connection to the situation in which J is a reduction of the ideal I. Of course in this case, $I^t \subset J$ for $t >> 0$, so $(\mathcal{A}_{R\twoheadrightarrow A}(I/J))_t = (\mathcal{R}_{R/J}(I/J))_t = \{0\}$ all $t >> 0$, hence the two algebras are finite R/J-modules, a case one can dismiss as of no interest for the present theory.

We retrieve a result of Valla ([21, Theorem 2.8]):

Corollary 1.5.1 *Let $J \subset I \subsetneq R$ be ideals of the local ring R. If I/J is of linear type over R/J (e.g., if I is generated by a regular sequence modulo J) then $J\cap I^t = JI^{t-1}$ for every positive integer t.*

Proof. This follows immediately from the structural “squeezing” (1.3). □

Note that the assumption in [6, Proposition 3.10] to the effect that I be of linear type over R does not intervene in the above statement.

Here is a useful explicit situation, where we write $I = (J, \mathfrak{a})$, with no particular care for minimal generation.

Proposition 1.5.2 *Let $I = (J, \mathfrak{a})$. If $J\cap\mathfrak{a}^t \subset J\mathfrak{a}^{t-1}$, for every $t \geq 0$ then $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is an isomorphism.*

Proof. One has:

$$J\cap I^t = J\cap(J,\mathfrak{a})^t = J\cap(J(J,\mathfrak{a})^{t-1}, \mathfrak{a}^t) = J(J,\mathfrak{a})^{t-1} + J\cap\mathfrak{a}^t \subset JI^{t-1} + J\mathfrak{a}^{t-1} \subset JI^{t-1}.$$

□

Remark 1.5.3 In the notation of the previous proposition, one of the main results of [8] is that if $\mathfrak{a}$ is generated by a d-sequence modulo J then the assumption of the proposition is fulfilled. Therefore, under the hypothesis of [*loc.cit.*], the surjection $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is an isomorphism. This result, however, is a special case of Corollary1.5.1 if one uses that an ideal generated by a d-sequence is of linear type. Of course, the proof of this fact requires some non-trivial work on itself and is previous to the later results, such as [6].

When $J = (f)$ is a principal ideal, one has a result somewhat subsumed in the spirit of [21].

Proposition 1.5.4 *Let $\mathfrak{a}$ be an ideal in the ring R and let $f \in R$ be an element such that $\mathfrak{a}^t : f = \mathfrak{a}^t$ for every integer $t \geq 0$. Then the inclusion $(f) \subset (f, \mathfrak{a})$ induces an isomorphism $\mathcal{A}_{R/(f)}((f, \mathfrak{a})/(f)) \simeq \mathcal{R}_{(f,\mathfrak{a})/(f)}((f, \mathfrak{a})/(f))$.*

Proof. The assumption means that $(f) \cap \mathfrak{a}^t = f\mathfrak{a}^t$ for every $t \geq 0$, hence $(f) \cap (f, \mathfrak{a})^t = f(f, \mathfrak{a})^{t-1} + (f) \cap \mathfrak{a}^t = f(f, \mathfrak{a})^{t-1}$ for $t > 0$. □

The Valabrega–Valla module gives the torsion in as many cases as the ones in which the Rees algebra is the symmetric algebra modulo torsion.

Proposition 1.5.5 *Let $J \subset I \subsetneq R$ be ideals of the Noetherian ring R. If I/J has a regular element then the R/J-torsion of the embedded Aluffi algebra of I/J is the kernel of the natural surjection $\mathcal{A}_{R/J}(I/J)(I/J) \twoheadrightarrow \mathcal{R}_{I/J}(I/J)$. If J is besides a prime ideal then the R/J-torsion of $\mathcal{A}_{R/J}(I/J)(I/J)$ is a minimal prime ideal.*

Proof. Consider the general elementary observation: given a ring A and A-modules

$$N \twoheadrightarrow M \twoheadrightarrow K$$

such that the A-torsion of N is the kernel of the composite $N \twoheadrightarrow K$ then the A-torsion of M is the kernel of $M \twoheadrightarrow K$. We apply this to the situation in (1.3), by recalling that if $I/J \subset R/J$ has a regular element in the ring R/J then the R/J-torsion of the symmetric algebra $\mathcal{S}_{R/J}(I/J)$ is the kernel of the natural surjection $\mathcal{S}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$.

If J is a prime ideal then $\mathcal{R}_{I/J}(I/J)$ is a domain and it is well-known that the kernel of the surjection $\mathcal{S}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{I/J}(I/J)$ is a minimal prime ideal of $\mathcal{S}_{R/J}(I/J)$. Therefore it follows that the kernel of $\mathcal{A}_{R/J}(I/J)(I/J) \twoheadrightarrow \mathcal{R}_{I/J}(I/J)$ is a minimal prime of $\mathcal{A}_{R/J}(I/J)$.

□

Corollary 1.5.6 *Let $J \subset I \subsetneq R$ be ideals of the Noetherian ring R. If I/J has a regular element then $\mathcal{W}_{J\subset I} = \mathrm{H}^0_{I/J}(\mathcal{A})$, where $\mathcal{A}$ denotes the Aluffi algebra and $\mathcal{W}_{J\subset I}$ is the Valabrega–Valla module as introduced above.*

Proof. By Proposition 1.5.5, $\mathcal{W}_{J\subset I}$ is the R/J-torsion of $\mathcal{A}$. On the other hand, localizing at primes of the base R/J not containing I/J makes the surjection $\mathcal{S}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{I/J}(I/J)$ an isomorphism, hence also the surjection $\mathcal{A} \twoheadrightarrow \mathcal{R}_{I/J}(I/J)$. Therefore, $\mathcal{A}$ is torsionfree locally at those primes. Since I/J has regular elements, the result follows easily (see, e.g., [14, Lemma 5.2]). □

Remark 1.5.7 The last result says, in particular, that there exists an integer $k \geq 0$ such that $I^k\left(J \cap I^t\right) \subset JI^{t-1}$ for every $t \geq 1$. In the section 1.7 we relate such an exponent to the so-called Artin–Rees number.

In the section 1.9 we collect some interesting examples of pairs of ideals whose the corresponding Aluffi algebra no have torsion.

1.6 Connection to Standard Base

A close associate of the module of Valabrega-Valla $\mathcal{W}_{J\subset I}$ is the well-known ideal

$$\mathfrak{J} = \ker\left(\mathrm{gr}_I(R) \twoheadrightarrow \mathrm{gr}_{I/J}(R/J)\right) = \bigoplus_{t\geq 0}(I^{t+1} + J \cap I^t)/I^{t+1},$$

generated by the I-initial forms of elements of J. Recall that the I-initial form of an element $f \in R$ is the residue class f^* of f in $I^{\nu(f)}/I^{\nu(f)+1}$ where $\nu(f)$ is the Ith order of f (i.e., $f \in I^t \setminus I^{t+1}$, setting $\nu(f) = \infty$ and $f^* = 0$ if $f \in \cap_{t\geq 0} I^t$).

Definition 1.6.1 *A set of elements $\{f_1, \ldots, f_r\}$ of J is called an I-*standard base *of J if $f_1^*, \ldots, f_r^*$ generate $\mathfrak{J}$ in $\mathrm{gr}_I(R)$.*

If R is Noetherian local then an I-standard base of J is a generating set of J (see [4, Lemma 6]). We will shorten $\nu(f_i)$ to ν_i if f_i is sufficiently clear from the context.

The following basic result will be used throughout. We include a proof for completeness.

Theorem 1.6.2 ([19]) *Let $J = (f_1, \ldots, f_m)$ be an ideal of ring R. Then $\{f_1, \ldots, f_m\}$ is an I-standard base of J if and only if*

$$J \cap I^t = \sum_{i=1}^{m} I^{t-\nu_i} f_i$$

for every positive integer t.

Proof. It is clear that $(f_1^*, \ldots, f_m^*) = \left(\sum_{i=1}^{m} I^{t-\nu_i} f_i + I^{t+1}\right) \Big/ I^{t+1}$. Hence, if $J \cap I^t = \sum_{i=1}^{m} I^{t-\nu_i} f_i$ for all $t \geq 1$, we have: $\mathfrak{J} = (f_1^*, \ldots, f_m^*)$.

conversely, if $\mathfrak{J}$ is generated by the f_i^*'s, then we have: $J \cap I^t \subset \sum_{i=1}^{m} I^{t-\nu_i} f_i + I^{t+1}$. for all $t \geq 1$. It follows that

$$J \cap I^t = \bigcap_{n \geq 1} \left(\sum_{i=1}^{m} I^{t-\nu_i} f_i + I^{n+t} \cap J \right)$$

By the Artin–Rees lemma there exists an integer q such that $J \cap I^{t+n} = I^{n+t-q}(J \cap I^q)$ for all $n + t \geq q$. Then if r is an integer such that $r \geq n - \nu_i$ for $i = 1, \ldots, m$, we get the following equality:

$$J \cap I^t = \bigcap_{n \geq q-t+r} \left(\sum_{i=1}^{m} I^{t-\nu_i} f_i + I^{n+t-q}(J \cap I^q) \right) = \sum_{i=1}^{m} I^{t-\nu_i} f_i,$$

since, if $n \geq q - t + r$, then $I^{n+t-q}(J \cap I^q) \subset JI^r \subset \sum_{i=1}^{m} I^{t-\nu_i} f_i$

□

For the sake of shortness we denote by $\mathfrak{R}$ and $\mathfrak{A}$ the kernel of the natural epimorphisms:

$$\mathcal{R}_R(I) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$$
$$\mathcal{R}_R(I) \twoheadrightarrow \mathcal{A}_{R \twoheadrightarrow A}(I/J)$$

The first is described as

$$\mathfrak{R} \;=\; \bigoplus_{n \geq 0} J \cap I^n = \{\sum_{r=0}^{s} c_r t^r \in \mathcal{R}_R(I) \mid c_r \in J \cap I^r\} \tag{1.6}$$

while the second on was already explained in Lemma 1.1.4

$$\mathfrak{A} \;=\; (J, \tilde{J})\mathcal{R}_R(I) = \bigoplus_{t \geq 0} JI^{n-1} = \{\sum_{r=0}^{s} c_r t^r \in \mathcal{R}_R(I) \mid c_r \in JI^{r-1}\} \tag{1.7}$$

The ideal $\mathfrak{R} = JR[t] \cap \mathcal{R}_R(I)$ has been studied in [21].

Lemma 1.6.3 *Let R be a local ring and $J \subset I$ be ideals of R. Let $\{f_1, \ldots, f_m\}$ be an I-standard base of J such that $1 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$. Then*

$$\mathfrak{R} \cap I^{\nu_m - 1}\mathcal{R}_R(I) \subset \mathfrak{A} \subset \mathfrak{R} \cap I^{\nu_1 - 1}\mathcal{R}_R(I).$$

Proof. By definition, we want to show the two inclusions

$$J \cap I^{n+\nu_m-1} \subset JI^{n-1} \subset J \cap I^{n+\nu_1-1}$$

as subideals of $J \cap I^n$, for every $n \geq 1$.

This is however a straightforward consequence of Theorem 1.6.2 as one has thereof

$$J \cap I^{\nu_m+n-1} = \sum_{i=1}^{m} f_i I^{\nu_m+n-1-\nu_i} \subset \sum_{i=1}^{m} f_i I^{n-1} = JI^{n-1},$$

and similarly

$$J \cap I^{\nu_1+n-1} = \sum_{i=1}^{m} f_i I^{\nu_1+n-1-\nu_i} \supset \sum_{i=1}^{m} f_i I^{n-1} = JI^{n-1}.$$

□

Proposition 1.6.4 *Let $J \subset I$ be ideals of a Noetherian ring R such that J has positive height and I/J has a regular element in R/J. Let $\{f_1, \ldots, f_m\}$ be an I-standard base of J such that $1 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$. Then*

(i) *If $\nu_m = 1$* (i.e., $\nu_1 = \cdots = \nu_m = 1$), *then* $\dim \mathcal{A}_{R/J}(I/J) = \dim R/J + 1$.

(ii) *If R is besides local and if $\nu_1 > 1$, then* $\dim \mathcal{A}_{R/J}(I/J) = \dim R$.

Proof. (i) Lemma 1.6.3 implies the surjections

$$\mathcal{R}_R(I)/\mathfrak{R} \cap I^{\nu_m-1}\mathcal{R}_R(I) \twoheadrightarrow \mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_R(I)/\mathfrak{R} \cap I^{\nu_1-1}\mathcal{R}_R(I),$$

which are now isomorphisms throughout by the present assumption.

The rest is routine since we assume that I/J has a regular element.

(ii) Quite generally, for a positive integer m one has

$$\begin{aligned} \dim \mathcal{R}_R(I)/\mathfrak{R} \cap I^m\mathcal{R}_R(I) &\geq \max\{\dim \mathcal{R}_R(I)/\mathfrak{R}\ ,\ \dim \mathcal{R}_R(I)/I^m\mathcal{R}_R(I)\} \\ &= \max\{\dim \mathcal{R}_{R/J}(I/J),\ \dim \mathrm{gr}_I(R)\} = \dim R \end{aligned}$$

since J is assumed to have positive height. On the other hand, again since J has positive height, by Lemma 1.3.1 one has $\dim \mathcal{A}_{R/J}(I/J) \leq \dim R$. The result follows now immediately. □

Example 1.6.5 Let $J \subset R = k[x, y, z]$ be the defining ideal of the monomial space curve with parametric equation $x = t^3,\ y = t^4,\ z = t^5$. Then J is generated by $x^3 - yz,\ y^2 - xz,\ z^2 - x^2y$, let $I = (x, y, z)$, by result of Robbiano and Valla [11] we can easy verify that $x^3 - yz,\ y^2 - xz,\ z^2 - x^2y$ is a I standard base of J. Then $\nu_1 > 1$ and $\dim \mathcal{A}_{R/J}(I/J) = \dim \mathcal{S}_{R/J}(I/J) = \dim R = 3$

By Lemma 1.6.3, if $\nu_i = \nu$, for $i = 1, \ldots, m$, then $\mathfrak{A} = \mathfrak{R} \cap I^{\nu-1}\mathcal{R}_R(I)$. In general the following theorem gives necessary and sufficient conditions for $\mathfrak{A} = \mathfrak{R}$, i.e., for the natural surjection $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ to be an isomorphism.

Theorem 1.6.6 *Let $J \subset I \subsetneq R$ be ideals of the local ring R. Then the following conditions are equivalents:*

(a) *The natural surjection* $\mathcal{A}_{R\twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ *is an isomorphism;*

(b) $J \cap I^n = JI^{n-1}$ *for every positive integer* n*;*

(c) *For every minimal set of generators* $f_1, \ldots, f_m$ *of* J *their* I*-orders are all equal to* 1 *and* $\{f_1, \ldots, f_m\}$ *is a* I*-standard base of* J*;*

(d) *There exists a minimal set of generators* $f_1, \ldots, f_m$ *of* J *whose* I*-orders are all equal to* 1 *such that* $\{f_1, \ldots, f_m\}$ *is a* I*-standard base of* J*.*

Proof. Since the module of Valabrega–Valla is the R/J-torsion of the Aluffi algebra, conditions (a) and (b) are equivalent. It is clear that (c) implies (d), while (d) implies (b) by Lemma 1.6.3. Now assume that $J \cap I^n = JI^{n-1}$ for every positive integer n and let $f_1, \ldots, f_m$ be a minimal set of generators of J. If for some i, $f_i \in I^2$ then $f_i \in JI$, which clearly contradicts the minimality of $f_1, \ldots, f_m$. Now the implication (b) $\Rightarrow$ (c) is immediate by Theorem 1.6.2. □

1.7 Relation to the Artin–Rees number

We close with yet another condition for the surjection $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ to be an isomorphism.

For this recall that, pretty generally, given ideals $J, I \subset R$ the *Artin–Rees number* of J relative to I is the integer

$$\min\{k \geq 0 \mid J \cap I^t = (J \cap I^k)I^{t-k} \ \forall\, t \geq k\}.$$

We observe that if $J \subset I \subsetneq R$, where R is Noetherian and J has regular elements then the Artin–Rees number of J relative to I is ≥ 1.

Proposition 1.7.1 *Let* $J \subset I \subsetneq R$ *be ideals of a Noetherian ring* R *and let* $k \geq 1$ *be an upper bound for the Artin–Rees number of* J *relative to* I*, i.e., one is given that* $J \cap I^t = (J \cap I^k)I^{t-k} \ \forall\, t \geq k$*.*

Then I^{k-1} *annihilates the kernel of the surjection*$\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$*. Moreover, the latter is an isomorphism if and only if the Artin–Rees number of* J *relative to* I *is* 1*.*

Proof. One has $(J \cap I^t)I^{k-1} = (J \cap I^k)I^{t-k}I^{k-1} = (J \cap I^k)I^{t-1} \subset JI^{t-1}$ for $t \geq k$. On the other hand, for $t \leq k-1$, one has $I^{k-1} \subset I^{t-1}$, hence $(J \cap I^t)I^{k-1} \subset (J \cap I^t)I^{t-1} \subset JI^{t-1}$.

The second assertion is clear. □

More generally:

Lemma 1.7.2 *Let* $J \subset I \subset R$ *be ideals of a ring* R*. Assume that* ℓ *is un upper bound for the Artin–Rees number of* J *relative to* I *such that* $J \cap I^t = JI^{t-1}$ *for every* $t \leq \ell$*. Then* $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ *is an isomorphism.*

Proof. By assumption $J \cap I^t = I^{t-\ell}(J \cap I^\ell)$ for $t \geq \ell$. Now use the assumed equality $J \cap I^\ell = JI^{\ell-1}$ to get $J \cap I^t = JI^{t-1}$ for $t \geq \ell$. □

Given a ring A and an ideal $\mathfrak{a} = (a_1, \ldots, a_m) \subset A$, one lets $R[T_1, \ldots, T_m] \to \mathcal{R}_R(\mathfrak{a}) = R[\mathfrak{a}T]$ be the graded map sending T_i to $a_i T$. The *relation type* of $\mathfrak{a}$ is the largest degree of any minimal system of homogeneous generators of the kernel $\mathcal{J}$. Since the isomorphism $R[T_1, \ldots, T_m]/\mathcal{J} \simeq \mathcal{R}_R(\mathfrak{a})$ is graded, an application of the Schanuel lemma to the graded pieces shows that the notion is independent of the set of generators of $\mathfrak{a}$.

Corollary 1.7.3 *Let R be a Notherian ring and $J \subset I$ be ideals in R such that*

(i) *I/J has relation type at most ℓ as an ideal of R/J.*

(ii) *$J \cap I^t = JI^{t-1}$ for every $t \leq \ell$.*

Then $\mathcal{A}_{R \twoheadrightarrow R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ is an isomorphism.

Proof. By Lemma 1.7.2, it suffices to show that ℓ is an upper bound for the Artin–Rees number of J relative to I. Thus, Let I be generated by elements $a_1, \ldots, a_m$ and let let $f \in J \cap I^t$, with $t \geq \ell$. Then there exists a polynomial F in $R[T_1, \ldots, T_m]$, homogeneous of degree t, such that $F(a_1, \ldots, a_m) = f$. Since $f \in J$, reducing modulo J shows that $\overline{F}$ is a relation on the $\overline{a_i}$'s. Then by assumption there are polynomials g_i of degree ℓ, and h_i of degree $t - \ell$, such that $\overline{F} = \sum \overline{g_i}\overline{h_i}$ in $R/J[T_1, \ldots, T_m]$ and $\overline{g_i}$ are relations on the $\overline{a_i}$. Hence $F = \sum g_i h_i + L$ for some polynomial $L \in R[T_1, \ldots, T_m]$ of degree t and coefficients in J. Since

$$L(a_1, \ldots, a_m) \in JI^t \subset I^{t-\ell}(J \cap I^\ell)$$

then $g_i(a_1, \ldots, a_m) \subset J \cap I^\ell$ and $h_i(a_1, \ldots, a_m) \in I^{t-\ell}$ for $t \geq \ell$. This shows that the element $f = F(a_1, \ldots, a_m) \in I^{t-\ell}(J \cap I^\ell)$, that is, $J \cap I^t = (J \cap I^\ell)I^{t-\ell}$ for $t \geq \ell$.

□

1.8 Torsion and minimal primes

In this section we look at the set of primes $p \in \operatorname{Spec}(R)$ containing J which are contractions of associated primes of $\mathcal{A}_{R/J}(I/J)$ as an $\mathcal{R}_R(I)$-module. Note that $\mathcal{A}_{R/J}(I/J)$ is R/J-torsion free if and only if each one of these contractions is contained in some prime in $\operatorname{Ass}_R(R/J)$.

To an ideal $\mathfrak{a} \subset R$, we associate its extended–contracted ideal $\tilde{\mathfrak{a}} := \mathfrak{a}R[t] \cap \mathcal{R}_R(I)$ in the Rees algebra $\mathcal{R}_R(I) \subset R[t]$ (t a variable over R). Note that this 'tilde' operation preserves primality. Moreover, it is known or easy to show that this operation preserves associated and minimal primes in the following sense:

$$p \in \operatorname{Ass}_R(R/J) \Rightarrow \tilde{p} \in \operatorname{Ass}_{\mathcal{R}_R(I)}\mathcal{R}_{R/J}(I/J) \quad (1.8)$$

$$p \in \operatorname{Min}_R(R/J) \Rightarrow \tilde{p} \in \operatorname{Min}_{\mathcal{R}_R(I)}\mathcal{R}_{R/J}(I/J) \quad (1.9)$$

We will prove a similar result for the Aluffi algebra.

Note that if $p \in \mathrm{Sup}_R(R/J)$ then

$$\tilde{p} = \oplus_{n\geq 0} p \cap I^n \supset \oplus_{n\geq 0} J \cap I^n \supset \oplus_{n\geq 0} J I^{n-1} = (J, \tilde{J})\mathcal{R}_R(I), \tag{1.10}$$

hence $\tilde{p}$ contains the defining ideal of the Aluffi algebra on $\mathcal{R}_R(I)$. We may often identify $\tilde{p}$ with its image in $\mathcal{A}_{R/J}(I/J)$; as such it contains the torsion of $\mathcal{A}_{R/J}(I/J)$.

In particular, if p is an associated prime of R/J then $\tilde{p}$ contains the torsion. The next result will show that such primes are actually associated primes of $\mathcal{A}_{R/J}(I/J)$ – which we call the *torsion primes*.

For the proof, we will use the notation $\mathfrak{A}, \mathfrak{R}$ for the defining ideals of the Aluffi algebra $\mathcal{A}_{R/J}(I/J)$ and of the relative Rees algebra $\mathcal{R}_{R/J}(I/J)$ on $\mathcal{R}_R(I)$, respectively.

Proposition 1.8.1 *Let $J \subset I \subsetneq R$ be ideals of a Noetherian ring R such that I/J has a regular element. Then:*

a) $p \in \mathrm{Ass}\,_R(R/J) \Rightarrow \tilde{p} \in \mathrm{Ass}\,_{\mathcal{R}_R(I)}(\mathcal{A}_{R/J}(I/J))$
   $p \in \mathrm{Min}_R(R/J) \Rightarrow \tilde{p} \in \mathrm{Min}_{\mathcal{R}_R(I)}(\mathcal{A}_{R/J}(I/J))$.

b) *If besides R/J is equidimensional then* $\dim \mathcal{A}_{R/J}(I/J)/\tilde{p} = \dim R/J + 1$ *for every* $p \in \mathrm{Min}_R(R/J)$.

Proof. (a) Let $p \in \mathrm{Ass}\,_R(R/J)$. Since I/J contains a regular element, one has $I \not\subset p$. Let $f_1, \ldots, f_n$ be an I-standard base of p, so that $p = (f_1, \ldots, f_m)$ and

$$\tilde{p} = \sum_{r\geq 0} p \cap I^r = \sum_{r\geq 0} \left(\sum_{i=1}^{m} f_i I^{r-\nu_i} \right),$$

where $\nu_i = \nu(f_i)$. Write $\nu = \max_i \nu_i$ for $i = 1, \ldots, m$ and take $b \in I^{v-1} \setminus P$.

Say, $p = J : a$. We claim that $\tilde{p} = \mathfrak{A} : ab$ which will prove that $\tilde{p}$ is an associated prime of $\mathcal{A}_{R/J}(I/J))$ on $\mathcal{R}_R(I)$.

For this, let $c_r \in J \cap I^r = \sum_{i=1}^m f_i I^{r-\nu_i}$, with $r \geq 0$. Then

$$bc_r \in \sum_{i=1}^{m} f_i\, b I^{r-\nu_i} \subset \sum_{i=1}^{m} f_i I^{r-\nu_i+\nu-1} \subset \sum_{i=1}^{m} f_i I^{r-1} = p\, I^{r-1} = (J : a) I^{r-1} \subset J I^{r-1} : a,$$

hence $b\tilde{p} \subset \mathfrak{A} : a$, hence $\tilde{p} \subset \mathfrak{A} : ab$.

For the inverse inclusion, since $\mathfrak{A} \subset \mathfrak{R}$, it follows that $\mathfrak{A} : ab \subset \mathfrak{R} : ab = (\mathfrak{R} : a) : b$. Note that $\widetilde{J : a} = (J : a)R[t] \cap \mathcal{R}_R(I) = (JR[t] : a) \cap \mathcal{R}_R(I) = (JR[t] \cap \mathcal{R}_R(I)) : a = \mathfrak{R} : a$.

Therefore $\mathfrak{A} : ba \subset \tilde{p} : b = \tilde{p}$. Thus, $\tilde{p} = \mathfrak{A} : ab$ as was to be shown.

To see that $\tilde{p}$ is actually a minimal prime thereof provided $p \in \mathrm{Min}_R(R/J)$, note that if $Q \subset \tilde{p}$ is a minimal prime of $\mathcal{A}_{R/J}(I/J)$ on $\mathcal{R}_R(I)$ then its contraction $q = Q \cap R$

is contained in $\tilde{p} \cap R = p$. Since $J \subset q$ and $p \in \mathrm{Min}_R(R/J)$ then $q = p$, hence $\tilde{q} \subset \tilde{p}$. But $\tilde{q} \subset Q$, so then $Q = \tilde{p}$.

(b) Using the same notation for $\tilde{p}$ both as an ideal of $\mathcal{R}_R(I)$ of $\mathcal{A}_{R/J}(I/J)$, one has

$$\mathcal{A}_{R/J}(I/J)/\tilde{p} \simeq \mathcal{R}_R(I)/\tilde{p} \simeq \oplus_{n\geq 0} I^n/p \cap I^n \simeq \oplus_{n\geq 0}(p, I^n)/p \simeq \mathcal{R}_{R/p}((p, I)/p).$$

Now, since R/p is a domain and $I \not\subset p$, one has $\dim \mathcal{R}_{R/p}((p, I)/p) = \dim R/p + 1$, and since R/J is equidimensional then finally $\dim \mathcal{A}_{R/J}(I/J)/\tilde{p} = \dim R/J + 1$.

□

Example 1.8.2 Let $J \subset I \subset R = k[x, y, z]$ be as example 1.2.1. Since J is a defining ideal of points, it is a codimension 2 perfect ideal. Thus the minimal primes of R/J are

$$\mathfrak{p}_1 = (x, y),\ \mathfrak{p}_2 = (x, y - z), \mathfrak{p}_3 = (y, x - z),\ \mathfrak{p}_4 = (y - z, x - z).$$

Then, the corresponding torsion primes on $R[\mathbf{T}]$ are:

$$\mathfrak{q}_1 = (x, y, T_1, T_2, T_3, T_4) \qquad \mathfrak{q}_2 = (x, y - z, T_1, T_2, T_3, T_4 + T_5)$$
$$\mathfrak{q}_3 = (y, x - z, T_1, T_2, T_4 + T_5, T_3 - T_5) \qquad \mathfrak{q}_4 = (y - z, x - z, T_1, T_2, T_4, T_3 - T_5)$$

One can get a hold of other minimal primes of the Aluffi Algebra according to the following result, pretty much modeled on the case of the symmetric algebra.

Proposition 1.8.3 *Let $J \subset I \subsetneq R$ be ideals of the Noetherian ring R. Any minimal prime $\wp$ of $\mathcal{A}_{R/J}(I/J)$ on $\mathcal{R}_R(I)$ is either of the form $\wp = \tilde{p}$ for some minimal prime of R/J on R, or else has the form $(q, \wp_+)$ where $q := \wp \cap R$ contains a minimal prime of R/I on R and $\wp_+ = \wp \cap (Iu)$.*

Proof. By Corollary 1.5.6 – rather by its proof – a power of I annihilates the kernel of $\mathcal{A}_{R/J}(I/J) \twoheadrightarrow \mathcal{R}_{R/J}(I/J)$ lifted to $\mathcal{R}_R(I)$ – call it $\mathcal{K}$. If $\wp \subset \mathcal{R}_R(I)$ is a minimal prime of $\mathcal{A}_{R/J}(I/J)$ it follows that $\wp$ contains either $\mathcal{K}$ or I. In the first case, it contains a minimal prime of $\mathcal{R}_R(I)/\mathcal{K} \simeq \mathcal{R}_{R/J}(I/J)$ hence must be a minimal prime of the latter on $\mathcal{R}_R(I)$. But, it is well known that the above extending-contracting operation induces a bijection between the minimal primes of R/J on R and the minimal primes of $\mathcal{R}_{R/J}(I/J)$ on $\mathcal{R}_R(I)$.

In the case $I \subset \wp$, since $\wp$ is homogeneous in the natural $\mathbb{N}$-grading of $\mathcal{R}_R(I)$, then it is clear that $\wp = (q, \wp_+)$, where $\wp_+ = \wp \cap \mathcal{R}_R(I)_+$; obviously, q contains a minimal prime of R/I on R. □

Remark 1.8.4 Note that if $\wp \subset \mathcal{R}_R(I)$ is a minimal prime of $\mathcal{A}_{R/J}(I/J)$ containing I then $\wp_+$ behaves erratically: it can actually be zero in certain cases (see, e.g., Proposition 1.4.2, (ii)). On the other hand, its contraction $q \subset R$ may turn out to be an embedded associated prime of R/I and not a minimal one (see Section 2).

Corollary 1.8.5 *Let $J \subset I \subsetneq R$ be ideals of a Noetherian ring R such that I/J has a regular element and R/J is equidimensional. If $\mathcal{A}_{R/J}(I/J)$ is equidimensional then*

$$\dim \mathcal{A}_{R/J}(I/J) = \dim R/J + 1.$$

Proof. By item (b) of the Proposition 1.8.1, one has $\dim \mathcal{A}_{R/J}(I/J)/\tilde{p} = \dim R/J+1$, for every $\tilde{p}$, with $p \in \mathrm{Spec}\,(R)$ a minimal prime of R/J. By (a) of the same proposition, we know that $\tilde{p}$ is a minimal prime of the Aluffi algebra. Since the latter is assumed to be equidimensional, the result follows. □

Equidimensionality of $\mathcal{A}_{R/J}(I/J)$ may be quite rare. The next result shows that, at least in the local or graded case, pure-dimensionality is really infrequent.

Theorem 1.8.6 *Let $(R, \mathfrak{m})$ be a Noetherian local ring and its maximal ideal $\mathfrak{m}$ or a graded algebra of finite type over a field and its maximal homogeneous ideal $\mathfrak{m}$. Suppose that:*

(a) *$J \subset \mathfrak{m}I$ and I/J has a regular element.*

(b) *Either $R/\mathfrak{m}$ is infinite and I has maximal analytic spread, or else $J \subset I^2$.*

(c) *$\mathcal{A}_{R\twoheadrightarrow A}(I/J)$ is pure-dimensional.*

Then J is a height one unmixed ideal. If R is besides regular then J must be principal.

Proof. Let $p \in \mathrm{Ass}_R(R/J)$ have height ≥ 2. By Proposition 1.8.1 and the the pure-dimensionality of $\mathcal{A}_{R/J}(I/J)$, the torsion prime $\tilde{p}$ satisfies

$$\begin{aligned} \dim \mathcal{A}_{R/J}(I/J) &= \dim \mathcal{A}_{R/J}(I/J)/\tilde{p} = \dim \mathcal{R}_R(I)/\tilde{p} = \dim \mathcal{R}_{R/p}((p, I)/p) \\ &= \dim R/p + 1 \leq \dim R - \mathrm{ht}\, p + 1 \leq \dim R - 2 + 1 = \dim R - 1. \end{aligned}$$

However, by Proposition 1.4.2 ((ii) or (iii) according to the case), $\dim \mathcal{A}_{R/J}(I/J) = \dim R$ – an absurd. □

Remark 1.8.7 Assume that J is a prime ideal. In the first alternative of item (b) of the previous result, regardless of whether or not $\mathcal{A}_{R\twoheadrightarrow A}(I/J)$ is pure-dimensional, one has two basic minimal primes of the Aluffi algebra, namely, $\mathfrak{m}\,\mathcal{A}_{R/J}(I/J)$ and its torsion (Proposition 1.5.5). While the first has maximal dimension (Proposition 1.4.2, (ii)), the latter has dimension $\dim R/J + 1$. Thus, solely requiring that the two have equal dimensions entails at least that J have codimension 1.

We wrap up with the following

Question 1.8.8 Suppose as above that $J = (f) \subset \mathfrak{m}I$, $I/(f)$ has a regular element and I has maximal analytic spread. To what extent can we assert that, conversely, $\mathcal{A}_{R\twoheadrightarrow A}(I/J)$ is pure-dimensional?

This seems to be the case in a variety of situations such as the one considered in Section 2.

1.9 Examples of Aluffi torsion-free pairs

Let us agree to call a pair of ideals $J \subset I$ an *$\mathcal{A}$-torsionfree pair* if the map $\mathcal{A}_{R\twoheadrightarrow A}(I/J) \twoheadrightarrow \mathcal{R}_{I/J}(I/J)$ is injective.

The examples we have in mind in this part are of the two sorts mentioned previously, namely, of totally degenerate or non-degenerate pairs. The first kind will be based on Proposition 1.5.2. For these, we let $R = k[\mathbf{X}]$ be an $\mathbb{N}$-graded polynomial ring over a field k, $J \subset R$ is a homogeneous ideal and $I \subset R$ is the Jacobian ideal of J, by which we always mean the ideal $(J, \mathcal{I}_r(\Theta))$ where $r = \mathrm{ht}\,(J)$ and Θ stands for the Jacobian matrix of a set of generators of J. One knows that this maybe a slack ideal, but it is well defined modulo J.

Example 1.9.1 Let $J \subset R = k[x,y,z]$ be the defining ideal of the monomial space curve with parametric equations $x = u^{n_1},\ y = u^{n_2},\ z = u^{n_3}$, where $\gcd(n_1,n_2,n_3) = 1$. Suppose that $n_1 = 2q+1,\ n_2 = 2q+p+1,\ n_3 = 2q+2p+1$, for non-negative integers p, q. If I is the Jacobian ideal of J then $J \subset I$ is an $\mathcal{A}$-torsionfree pair.

Grading R by the exponents of the parameter u in the parametric equations, one knows ([5]) that J is a perfect codimension 2 ideal generated by the homogeneous polynomials

$$F_1 = x^{c_1} - y^{r_{12}}z^{r_{13}},\ F_2 = x^{r_{21}}z^{r_{23}} - y^{c_2},\ F_3 = x^{r_{31}}y^{r_{32}} - z^{c_3}$$

where $0 < r_{ij} < c_i$ $(i = 1,2,3, j \neq i)$. Note the relations

$$c_1 = r_{21} + r_{31},\ c_2 = r_{12} + r_{32},\ c_3 = r_{13} + r_{23}.$$

The Jacobian matrix of J is

$$\Theta = \left(\begin{array}{ccc} c_1x^{c_1-1} & -r_{12}y^{r_{12}-1}z^{r_{13}} & -r_{13}y^{r_{12}}z^{r_{13}-1} \\ r_{21}x^{r_{21}-1}z^{r_{23}} & -c_2y^{c_2-1} & r_{23}x^{r_{21}}z^{r_{23}-1} \\ r_{31}x^{r_{31}-1}y^{r_{32}} & r_{32}x^{r_{31}}y^{r_{32}-1} & -c_3z^{c_3-1} \end{array} \right)$$

The 2-minors of Θ are

$$\begin{array}{rcl} f_1 & = & -c_1c_2x^{c_1-1}y^{c_2-1} + r_{21}r_{12}x^{r_{21}-1}y^{r_{12}-1}z^{c_3} \\ f_2 & = & c_1r_{23}x^{c_1+r_{21}-1}z^{r_{23}-1} + r_{13}r_{21}x^{r_{21}-1}y^{r_{12}}z^{c_3-1} \\ f_3 & = & -r_{12}r_{23}x^{r_{21}}y^{r_{12}-1}z^{c_3-1} + c_2r_{13}y^{c_2+r_{12}-1}z^{r_{13}-1} \\ f_4 & = & c_1r_{32}x^{c_1+r_{31}-1}y^{r_{32}-1} + r_{31}r_{12}x^{r_{31}-1}y^{c_2-1}z^{r_{13}} \\ f_5 & = & -c_1c_3x^{c_1-1}z^{c_3-1} + r_{31}r_{13}x^{r_{31}-1}y^{c_2}z^{r_{13}-1} \\ f_6 & = & r_{32}r_{13}x^{r_{31}}y^{c_2-1}z^{r_{13}-1} + c_3r_{12}y^{r_{12}-1}z^{c_3+r_{13}-1} \\ f_7 & = & r_{21}r_{32}x^{c_1-1}y^{r_{32}-1}z^{r_{23}} + c_2r_{31}x^{r_{31}-1}y^{c_2+r_{32}-1} \\ f_8 & = & -r_{21}r_{31}x^{c_1-1}y^{r_{32}}z^{r_{23}-1} - c_3r_{21}x^{r_{21}-1}z^{c_3+r_{23}-1} \\ f_9 & = & -r_{32}r_{23}x^{c_1}y^{r_{32}-1}z^{r_{23}-1} + c_2c_3y^{c_2-1}z^{c_3-1} \end{array}$$

Write $\mathfrak{D}$ for the ideal generated by the following monomials

$$\begin{array}{lll} M_1 = x^{r_{21}-1}y^{r_{12}-1}z^{c_3} & M_2 = x^{r_{21}-1}y^{r_{12}}z^{c_3-1} & M_3 = y^{c_2+r_{12}-1}z^{r_{13}-1} \\ M_4 = x^{r_{31}-1}y^{c_2-1}z^{r_{13}} & M_5 = x^{r_{31}-1}y^{c_2}z^{r_{13}-1} & M_6 = y^{r_{12}-1}z^{c_3+r_{13}-1} \\ M_7 = x^{r_{31}-1}y^{c_2+r_{32}-1} & M_8 = x^{r_{21}-1}z^{c_3+r_{23}-1} & M_9 = y^{c_2-1}z^{c_3-1} \end{array}$$

The following relations come out

$$\begin{array}{rcl} f_1 & = & -c_1c_2x^{r_{21}-1}y^{r_{12}-1}F_3 + (r_{21}r_{12} - c_1c_2)M_1 \\ f_2 & = & c_1r_{23}x^{r_{21}-1}z^{r_{23}-1}F_1 + (r_{31}r_{21} - c_1r_{23})M_2 \\ f_3 & = & -r_{12}r_{23}y^{r_{12}-1}z^{r_{13}-1}F_2 - (r_{12}r_{23} + c_2r_{13})M_3 \\ f_4 & = & c_1r_{32}x^{r_{31}-1}y^{r_{32}-1}F_1 + (r_{31}r_{12} + c_1r_{32})M_4 \\ f_5 & = & -c_1c_3x^{r_{31}-1}z^{r_{13}-1}F_2 + (r_{31}r_{13} - c_1c_3)M_5 \\ f_6 & = & r_{32}r_{13}y^{r_{12}-1}z^{r_{13}-1}F_3 + (r_{21}c_3 + r_{32}r_{13})M_6 \\ f_7 & = & r_{21}r_{32}x^{r_{31}-1}y^{r_{32}-1}F_2 + (r_{31}c_2 + r_{21}r_{32})M_7 \\ f_8 & = & -r_{32}r_{23}x^{r_{21}-1}z^{r_{23}-1}F_3 - (r_{21}c_3 + r_{23}r_{31})M_8 \\ f_9 & = & r_{32}r_{23}y^{r_{32}-1}z^{r_{23}-1}F_1 + (c_2c_3 - r_{32}r_{23})M_9, \end{array}$$

By above, J is generated by

$$F_1 = x^{p+q+1} - yz^q, \quad F_2 = xz - y^2, \quad F_3 = x^{p+q}y - z^{q+1}$$

and

$$I = (J, \mathfrak{D}) = (xz - y^2, x^{p+q+1}, x^{p+q}y, x^{p+q-1}y^2, yz^q, y^2z^{q-1}, z^{q+1})$$

Set $\Delta = (x^{p+q+1}, x^{p+q}y, x^{p+q-1}y^2, yz^q, y^2z^{q-1}, z^{q+1})$. By a slight adaptation of Proposition 1.5.2 it suffices to show that $J \cap \Delta^t \subset J\Delta^{t-1}$ for every $t \geq 0$. Let $F = g_1F_1 + g_2F_2 + g_3F_3 \in J \cap \Delta^t$ with $g_i \in R$. Now, since $F_1, F_3 \in \Delta$, then $g_1, g_3 \in \Delta^{t-1}$. A calculation shows that, for $0 \leq k \leq n$,

$$z^{t(q+1)+1+k}y^tF_2, \; x^{p+q}y^{k+1}z^{(t-1)(q+1)-1+k}F_2, \; x^{p+q-1}y^2z^{(t-1)(q+1)-1}F_2 \in \Delta^t.$$

Then $g_2 \in \Delta^{t-1}$, thus proving that $F \in J\Delta^{t-1}$.

Conjecture 1.9.2 Let $J \subset R = k[x, y, z]$ be the defining ideal of the monomial space curve with parametric equations $x = t^{n_1}, \; y = t^{n_2}, \; z = t^{n_3}$, where $\gcd(n_1, n_2, n_3) = 1$. Let I be the Jacobian ideal of J. If J is almost complete intersection then the pair $J \subset I$ is $\mathcal{A}$–torsionfree.

Example 1.9.3 Let R be a Noetherian ring, let $a_1, \ldots, a_r$ be regular sequence on R and let $I = (a_1, \ldots, a_r)$. Then $J = (a_1^n, \ldots, a_i^n) \subset I^n$ for $i = 1, \ldots, r$ is $\mathcal{A}$-torsionfree.

By induction on t we show that $J \cap I^{nt} = JI^{n(t-1)}$ for every $t \geq 1$. For $t = 1$ the conclusion is obvious. Let $t > 1$, then by inductive assumption

$$J \cap I^{nt} = I^{nt} \cap J \cap I^{nt-1} = I^{nt} \cap JI^{n(t-1)-1} = I^{nt} \cap \Delta$$

Where Δ is the ideal generated by the elements $a_1^{s_1} \ldots a_r^{s_r}$ such that $s_1 + \ldots + s_r = nt - 1$ and $s_j \geq n$ for $1 \leq j \leq i$. Since $a_1, \ldots, a_r$ are regular sequence, if $F(X_1, \ldots, X_i)$ is a homogenous polynomial of degree $nt - 1$ over R such that $F(a_1, \ldots, a_i) \in I^{nt}$, hence all coefficients of F are in I. Then

$$J \cap (I^n)^t = J \cap I^{nt} = I^{nt} \cap \Delta = I\Delta = JI^{n(t-1)}$$

Example 1.9.4 Let $J \subset R = k[\mathbf{x}]$ be an ideal generated by forms of the same degree $d \geq 1$. If $I = (J, \mathfrak{m}^r)$, where $\mathfrak{m} = (\mathbf{x})$ and $r \geq d$, then the pair $J \subset I$ is $\mathcal{A}$-torsionfree.

To see why one uses Proposition 1.5.2. It suffices to show that $J \cap \mathfrak{m}^{rt} \subset J\mathfrak{m}^{r(t-1)}$ for every $t \geq 0$. Let $f_1, \ldots, f_m$ be generators of J of degree d and let F be a form in the f_i's such that $F \in \mathfrak{m}^{rt}$. Then $F = \sum_{i=1}^m g_i f_i$ where $g_i = \sum a_{\underline{\alpha}} \mathbf{x}^{\underline{\alpha}} \in R_{rt-d+\delta}$ for $\delta \geq 0$, since $R_{rt-d+\delta} = R_{r-d+\delta}.R_{rt-r}$, so we can rewrite g_i as

$$g_i = \sum_{\substack{|\underline{\alpha}|=r-d+\delta \\ |\underline{\beta}|=rt-r}} a_{\underline{\alpha},\underline{\beta}}\, \mathbf{x}^{\underline{\alpha}+\underline{\beta}}, \quad \text{hence} \quad F = \sum_{|\underline{\alpha}|=r-d+\delta} \mathbf{x}^{\underline{\alpha}} \left(\sum_{\substack{i=1 \\ |\underline{\beta}|=rt-r}}^{s} (\mathbf{x}^{\underline{\beta}}) f_i \right)$$

Therefore $F \in J\mathfrak{m}^{rt-r} = J\mathfrak{m}^{r(t-1)}$, as required.

Conjecture 1.9.5 Let $J \subset R = k[\mathbf{x}]$ denote an ideal generated by quadrics such that $\operatorname{ht} J \geq 2$, and let $I \subset R$ denote the ideal generated by the r–minors of the Jacobian matrix Θ of the generators of J. If $I = I_r(\Theta)$ is $(\mathbf{x})$–primary, then $I = (\mathbf{x})^r$.

If the conjecture is true then, along with the above result, it implies that the pair $J \subset I$ is torsionfree, where J stands for the homogeneous defining ideal J of any of the following varieties, and I denotes its Jacobian ideal.

(a) A rational normal curve;

(b) The Segre embedding of $\mathbb{P}^r \times \mathbb{P}^s$, with $r > 1$ or $s > 1$;

(c) The 2–Veronese embedding of a projective space;

(d) Generic 4×4 Pfaffians.

Example 1.9.6 Consider the monomial $x_1 \cdots x_n \in R = k[x_1, \ldots, x_n]$ $(n \geq 3)$ and let $J \subset R$ be the ideal generated by its partial derivatives $f_i =: x_1 x_2 \cdots \widehat{x_i} \cdots x_n$, for $i = 1, ..., n$. If I is the Jacobian ideal of J the pair $J \subset I$ is $\mathcal{A}$-torsionfree.

Proof. Its well-known and easy that J is a codimension 2 perfect ideal with Hilbert–Burch matrix

$$\varphi = \left(\begin{array}{ccccc} x_1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & x_2 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & x_{n-2} & \ldots & 0 \\ 0 & x_{n-1} & 0 & \ldots & 0 \\ -x_n & -x_n & -x_n & \ldots & -x_n \end{array} \right).$$

Setting $\Delta_{i,j} := \frac{\partial f_j}{\partial x_i} = x_1 x_2 ... \widehat{x_i} ... \widehat{x_j} ... x_n$, and inspecting the Hessian matrix Θ of $x_1 \cdots x_n$ – a symmetric matrix:

$$\mathcal{J} = \left(\begin{array}{ccccc} \Delta_{1,n} & \Delta_{2,n} & \ldots & \Delta_{n-1,n} & 0 \\ \Delta_{1,n-1} & \Delta_{2,n-1} & \ldots & 0 & \Delta_{n,n-1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \Delta_{1,2} & 0 & \ldots & \Delta_{n-1,2} & \Delta_{n,2} \\ 0 & \Delta_{2,1} & \ldots & \Delta_{n-1,1} & \Delta_{n,1} \end{array} \right)$$

one finds three basic types of 2×2 minors, namely

- Principal minors:
$$\det \left(\begin{array}{cc} 0 & \Delta_{i,j} \\ \Delta_{i,j} & 0 \end{array} \right) = \Delta_{i,j}^2,$$
one for each pair $i < j$;

- vanishing minors:
$$\det \left(\begin{array}{cc} \Delta_{i,j} & \Delta_{i,j'} \\ \Delta_{i',j} & \Delta_{i',j'} \end{array} \right) = 0,$$
one for each choice of row indices $1 \leq i, i' \leq n$ and column indices $1 \leq j, j' \leq n$;

- semidiagonal minors of typical form
$$\Lambda_j := \det \left(\begin{array}{cc} \Delta_{i,j} & 0 \\ * & \Delta_{i',i} \end{array} \right).$$

Since clearly, $\Lambda_j \in J$, we get that the Jacobian ideal I of J is generated by J and the squares of the second partial derivatives of $x_1 \cdots x_n$, i.e., $I = (J, \Delta_{i,j}^2)$ for $1 \leq i < j \leq n$.

As a side curiosity we note that $I = (J, \mathcal{I}_2(\Theta)) = (J, \mathcal{I}_{n-2}(\varphi)^2)$, hence $\sqrt{I} = \mathcal{I}_{n-2}(\varphi)$, so in particular I/J has codimension one. This example will therefore yield a case of a height one ideal in R/J which is $\mathcal{A}$-torsionfree, but clearly not of linear type because its number of generators on R/J is too large.

Setting $\Delta = (\Delta_{i,j}^2 \mid 1 \leq i < j \leq n)$, the usual algorithmic procedure to find generators of the intersection of monomial ideals yields for any t

$$J \cap \Delta^t = ((x_i, x_j)\Delta_{i,j}^{2t}, (\mathfrak{F}))$$

where $\mathfrak{F}$ is the set of all monomials in Δ^t excluding the monomials $\Delta_{i,j}^{2t}$ for $1 \leq i < j \leq n$. Another calculation shows that $(x_i, x_j)\Delta_{i,j}^{2t} \in J\Delta^{t-1}$, for $1 \leq i < j \leq n$, and that $\mathfrak{F} \subset J^2\Delta^{t-3}$. This proves that $J \cap \Delta^t \subset JI^{t-1}$. . $\square$

Example 1.9.7 Let $J \subset R = k[x_1, \ldots, x_n]$ denote the ideal of the coordinate points in projective space $\mathbb{P}^{n-1}$, i.e., $J = (x_i x_j \mid 1 \leq i < j \leq n)$. If I is the Jacobian ideal of J the pair $J \subset I$ is $\mathcal{A}$-torsionfree.

Proof. Since J contains all square-free monomials of degree 2, it is rather transparent that the Jacobian ideal I of J is generated by J and pure powers of the variables. Moreover, a closer inspection shows that, more precisely,

$$I = (J,\ x_1^{n-1},\ \ldots,\ x_n^{n-1}).$$

Setting $\Delta := (x_1^{n-1},\ \ldots,\ x_n^{n-1})$, the usual algorithmic procedure to find generators of the intersection of monomial ideals yields for any $t \geq 2$

$$J \cap \Delta^t = J \cap ((x_1^{\alpha_1} \ldots x_n^{\alpha_n})^{n-1} \,|\, \alpha_1 + \ldots + \alpha_n = t, \alpha_i \geq 0) = J\Delta^{t-1}.$$

$\square$

Question 1.9.8 (k algebraically closed) Let $J \subset R = k[x_1, \ldots, x_n]$ denote a radical homogeneous ideal of codimension $n-1$ (i.e., the ideal of a reduced set of points). If I is the Jacobian ideal of J, when is $J \subset I$ an $\mathcal{A}$-torsionfree pair?

Chapter 2

The Aluffi algebra of a projective hypersurface

As seen in the first chapter, (Proposition 1.8.6 and Remark 1.8.7), in order that the Aluffi algebra have a certain expected behavior over a regular ambient, the ideal J had better be principal. This may be taken as a motivation for this chapter. Another source for motivation is the original work of Aluffi, where he has inquired into the structure of the algebra in the case that ideal $J = (f)$ is generated by the equation of a reduced hypersurface.

2.1 The Aluffi gradient algebra

By Proposition 1.8.5, we know that $\dim \mathcal{A}_{R/(f)}(I_f/(f)) = \dim R = n$. One may dub the Aluffi embedded algebra in this case as the *Aluffi gradient algebra* of f. The following result collects the central backstage for the Aluffi gradient algebra.

Proposition 2.1.1 *Let k denote an infinite field, let $f \in R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a reduced homogeneous polynomial whose degree is not a multiple of the characteristic of k and let $I_f \subset R$ denote the corresponding gradient ideal. Assume that*

(i) *The singular locus of $V(f) \subset \mathbb{P}^{n-1}$ consists of a nonempty set of points (equivalently,* $\dim R/I_f = 1$)

(ii) *The partial derivatives of f are algebraically independent over k.*

Then:

(a) *There is a presentation of the Aluffi gradient algebra*

$$\mathcal{A}_{R/(f)}(I_f/(f)) \simeq R[\mathbf{T}]/(\mathcal{J}_f, f, \sum_{i=1}^{n} x_i T_i), \tag{2.1}$$

where $\mathcal{J}_f$ denotes the defining ideal of the Rees algebra $\mathcal{R}_R(I_f)$ on $R[\mathbf{T}] = R[T_1, \ldots, T_n]$.

(b) *The minimal primes of the gradient Aluffi algebra on* $\mathcal{R}_R(I_f)$ *are*

- *The minimal prime ideals of* $\mathcal{R}_{R/(f)}(I_f/(f))$*, all of the form* $\sum_{t\geq 0}(p)\cap I^t$ *for a prime factor* p *of* f
- *The extended ideal* $(\mathbf{x})\mathcal{R}_R(I_f)$
- *Prime ideals whose lifting to* $R[\mathbf{T}] = R[T_1,\ldots,T_n]$ *from a presentation* $R[\mathbf{T}]/\mathcal{A}\simeq\mathcal{R}_R(I)$ *have the form* $(P,\mathfrak{f})$*, where* $P\subset R$ *is a minimal prime of* R/I_f *and* $\mathfrak{f}$ *is an irreducible homogeneous polynomial in* $k[\mathbf{T}]$.

(c) *The Aluffi gradient algebra has non-trivial torsion.*

Proof. (a) Since the partials are homogeneous of the same degree algebraic independence over k is tantamount to analytic independence (i.e., the relations of the generators of I_f have coefficients in the ideal $(\mathbf{x})$) this follows immediately from Lemma 1.1.4 and the Euler formula taking in account the hypothesis on the characteristic.

(b) We apply Proposition 1.8.3, from which the first set of minimal primes is clear.

By Euler formula, $f\in(\mathbf{x})I_f$. Therefore, Proposition 1.4.2, (ii) implies that $(\mathbf{x})R[\mathbf{T}]$ is a minimal prime of the Aluffi gradient algebra over $R[\mathbf{T}]$.

Let $\mathcal{P}$ be a minimal prime of $\mathcal{A}_{R/(f)}(I_f/(f))$ whose contraction P to R contains I_f and is properly contained in $(\mathbf{x})$. By Proposition 1.8.3, $\mathcal{P}=(P,\mathcal{P}_+)$. By assumption (i) it follows that P is a minimal prime of R/I_f, hence has height $n-1$. By Proposition 1.3.4, $\mathcal{A}_{R/(f)}(I_f/(f))$ is equidimensional. Therefore the lifting of $\mathcal{P}$ to $R[\mathbf{T}]$ has height n. Since the lifting of any minimal generator of $(\mathcal{P}_+)$ is irreducible in $k[\mathbf{T}]$ it follows immediately that the lifting of $\mathcal{P}$ to $R[\mathbf{T}]$ has the required form.

(c) The defining equation of the dual curve to f belongs to the presentation ideal of $\mathcal{R}_{R/(f)}(I_f/(f))$ on $R[\mathbf{T}]$ and not to $(\mathbf{x})R[\mathbf{T}]$, hence by (c) it does not belong to the defining ideal of $\mathcal{A}_{R/(f)}(I_f/(f))$ on $R[\mathbf{T}]$.

□

Remark 2.1.2 For $n\leq 4$, if the partial derivatives are k-linearly dependent then the result of Gordan–Hesse–Noether implies that they are algebraically independent over k (see [3, Proposition 2.7] for a proof based on an observation of Zak). Thus, the assumption in this range is just linear independence.

As to (c), it's valid with no restriction when f is reduced since the defining equations of the dual variety to the hypersurface $V(f)$ belong to the presentation ideal $\mathcal{A}_f$ and, moreover, the ideal generated by these contains properly the defining ideal of the polar map of $V(f)$ (see [3, Remark 2.4]).

Example 2.1.3 Here is a simple illustration. Let $f = x^2y^2+x^2z^2+y^2z^2$, the equation of a plane quartic with 3 ordinary nodes. The minimal primes of the corresponding gradient Aluffi algebra, lifted to $k[x,y,z,T,U,V]$ are its lifted torsion, $(x,y,z)k[x,y,z,T,U,V]$, (x,y,V), (x,z,U) and (y,z,T). Since I_f is of linear type (see next section), these are of course the minimal primes of the symmetric algebra $\mathcal{S}_{R/(f)}(I_f/(f))$.

The next result deals with the linear type property for the gradient ideal. Without extra effort one can include the *weakly of linear type* companion property, as defined in [2, 3.11], which means the equality of the symmetric and Rees algebra in all high degrees and is sufficient in order that the characteristic cycle be given in terms of the naive blowup ([2, Corollary 3.5]).

Theorem 2.1.4 *Let k denote an infinite field, let $f \in R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a reduced homogeneous polynomial whose degree is not a multiple of the characteristic of k and let $I_f \subset R$ denote the corresponding gradient ideal. Assume that the partial derivatives of f are algebraically independent over k. Then the following are equivalents:*

(a) *I_f is an ideal of linear type (respectively, weakly of linear type).*

(b) *The natural surjection*

$$\mathcal{S}_{R/(f)}(I_f/(f)) \twoheadrightarrow \mathcal{A}_{R \twoheadrightarrow R/(f)}(I_f/(f))$$

is an isomorphism (respectively, an isomorphism in all high degrees).

Proof. One implication is easy and pretty general. Conversely, write $\mathcal{J}_f = \bigoplus_{i \geq 1} \mathcal{J}_i$ for the defining ideal of the Rees algebra $\mathcal{R}_R(I_f)$ on $S := R[T]$ where $\mathcal{J}_i$ is its homogenous part of degree i in the standard grading of $S := R[\mathbf{T}]$. Note that $\mathcal{J}_1 S$ is defining ideal of symmetric algebra of I_f on R, so we need to show that for any $r \geq 0$, $\mathcal{J}_r \subset \mathcal{J}_1 S$. We induct on r, the result being trivial if $r = 1$. Thus, let $r \geq 2$. By hypothesis and by (2.1) one has

$$\mathcal{J}_f \subset (\mathcal{J}_1, \sum_{i=1}^{n} x_i T_i, f)S.$$

Let $F = F(\mathbf{T}) \in \mathcal{J}_r$. Then $F = L + (\sum_{i=1}^{n} x_i T_i)G + fH$ where $L \in \mathcal{J}_1 S_{r-1}$, $G \in S_{r-1}$ and $H \in S_r$. Let d denote the degree of f. Write $f_{x_i} = \partial f / \partial x_i$, for $1 \leq i \leq n$. Then

$$F(f_{x_1}, \ldots, f_{x_n}) = L(f_{x_1}, \ldots, f_{x_n}) + (\sum_{i=1}^{n} x_i f_{x_i}) G(f_{x_1}, \ldots, f_{x_n}) + fH(f_{x_1}, \ldots, f_{x_n}) = 0.$$

Since $L \in \mathcal{J}_f$, by Euler's formula, one has $f(dG + H)(f_{x_1}, \ldots, f_{x_n}) = 0$, hence $dG + H \in \mathcal{J}_f$ and, by homogeneity, $G \in \mathcal{J}_{r-1}$ and $H \in \mathcal{J}_r$.

By the inductive hypothesis, $G \in \mathcal{J}_1 S_{r-2}$, hence $(\sum_{i=1}^{n} x_i T_i)G \in \mathcal{J}_1 S_{r-1}$. Passing to the corresponding ideals, it follows that $\mathcal{J}_r S = (\mathcal{J}_1 S_{r-1})S + f\, \mathcal{J}_r S$. By the graded version of Nakayama's lemma, this implies that $\mathcal{J}_r S = (\mathcal{J}_1 S_{r-1})S$. Therefore I_f is of linear type.

The argument for the weak version of the property of being of linear type is exactly the same.

□

Theorem 2.1.5 *Let k denote an infinite field, let $f \in R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a reduced homogeneous polynomial whose degree is not a multiple of the characteristic of k and let $I_f \subset R$ denote the corresponding gradient ideal. Assume that, the singular locus of $V(f) \subset \mathbb{P}^{n-1}$ consists of a nonempty set of points (equivalently, $\dim R/I_f = 1$). The symmetric algebra $\mathcal{S}_{R/(f)}(I_f/(f))$ is Cohen–Macaulay; in particular, if I_f is of linear type then the Aluffi algebra is Cohen–Macaulay.*

Proof. We apply the criterion of [6, Theorem 10.1]. Namely, we have to verify the following conditions:

(A) $\mu(I_f/(f)_P) \leq \mathrm{ht}\,(P/(f)) + 1 = \mathrm{ht}\, P$, for every prime ideal $P \supset I_f$ of R.

(B) $\mathrm{depth}\,(H_i)_{P/(f)} \geq \mathrm{ht}\,(P/(f)) - \mu(I_f/(f)_{P/(f)}) + i = \mathrm{ht}\, P - \mu(I_f/(f)_{P/(f)}) + i - 1$, for every prime ideal $P \supset I_f$ of R and every i such that $0 \leq i \leq \mu(I_f/(f)_{P/(f)}) - \mathrm{ht}\,(I_f/(f)_{P/(f)})$, and where H_i denotes the ith Koszul homology module of the partial derivatives on $R/(f)$.

Note that the primes containing I_f are $\mathfrak{m} = (x_1, \ldots, x_n)$ and its minimal primes, the latter all of height $n-1$.

(A) Since I_f itself is generated by n elements, it suffices to check the minimal primes. Thus, let $P \subset R$ be such a prime. Say, without lost of generality, that $x_n \notin P$. Because of the Euler relation, $\partial f/\partial x_n = f_{x_n} \in I_f$ locally at P and module (f). Therefore, locally at P and module (f), I_f is generated by $n - 1 = \mathrm{ht}\,(P)$ elements.

(B) If P is a minimal prime of I_f we saw in (A) that $\mu(I_f/(f)_{P/(f)}) = n-1$. Since $\mathrm{ht}\, P = n - 1$, the condition is trivially verified as $i = 0, 1$.

Thus, let $P = \mathfrak{m}$. Again, an easy inspection of the numbers tell us that only the case where $i = 2$ needs an argument. Write $H_2 = (H_2)_{P/(f)}$, so one has to prove that $\mathrm{depth}\,(H_2) \geq 1$.

It is well know that

$$H_2 = \frac{(J,f)/(f) : I_f/(f)}{(J,f)/(f)} \simeq \mathrm{Hom}_{R/(f)}\left(\frac{R/(f)}{I_f/(f)}, \frac{R/(f)}{(J,f)/(f)}\right)$$

where $J \subset I_f$ is an ideal generated by a regular sequence of length $n-2$ modulo f. We have $\mathrm{Ass}_{R/(f)}(H_2) = \mathrm{Supp}_{R/(f)}(R/I_f) \cap \mathrm{Ass}_{R/(f)}(R/(J,f)) \subset \mathrm{Ass}_{R/(f)}(R/(J,f))$ Since (J,f) is generated by a regular sequence of length $n-1$, hence $\mathfrak{m}/(f) \notin \mathrm{Ass}_{R/(f)}(R/(J,f))$, thus $\mathfrak{m}/(f) \notin \mathrm{Ass}_{R/(f)}(H_2)$, this proofs that $\mathrm{depth}\, H_2 \geq 1$.

□

2.2 Linear type property of the Jacobian ideal

If f defines a smooth hypersurface then I_f is irrelevant, i.e., is generated by a regular sequence, hence is of linear type. We regard this case as uninteresting and assume that f has singularities. This entails $\mathrm{ht}\,(I_f/(f)) \leq n-2$. If moreover f is reduced then $\mathrm{ht}\,(I_f/(f)) \geq 1$. For $n = 3$ we therefore find $\mathrm{ht}\,(I_f/(f)) = 1$. Ideals of height 1 in non-regular rings of dimension 2 are of course a tall order. Of course these typically involve a non-trivial primary decomposition,

Again for $n = 3$, we note that I_f is complete intersection. Since we regard the linear type case as uninteresting we would better stay away from the natural conditions under which I_f is of linear type. Fortunately, in the almost complete intersection case this is fairly known. For convenience we file the following general result, which collects in a more detailed version several known facts about an almost complete intersection (see [17, Proposition 3.7], also [6, Proposition 8.4, Proposition 10.4, Remark 10.5]).

Proposition 2.2.1 *Let R denote a Cohen–Macaulay local ring and let $I \subset R$ denote a proper ideal of height $h \geq 0$. Assume that*

(i) *I is a strict almost complete intersection (i.e., minimally generated by $h+1$ elements)*

(ii) *R/I is equidimensional (i.e., $\dim R/I = \dim R/P$ for every minimal prime P of R/I)*

(iii) *I satisfies the so-called sliding depth inequality* $\operatorname{depth} R/I \geq \dim R/I - 1$.

Let $R^m \xrightarrow{\varphi} R^{h+1} \longrightarrow I \longrightarrow 0$ stand for a minimal free presentation of I as an R-module. The following conditions are equivalent:

(a) $\operatorname{ht} I_1(\varphi) \geq \operatorname{ht} I + 1$

(b) *I_P is a complete intersection for every minimal prime P of R/I*

(c) *I is of linear type.*

Proof. (a) $\Rightarrow$ (b) Localizing at such a prime leaves some element of $I_1(\varphi)$ invertible, so up to an elementary transformation on φ_P the local presentation has the form

$$R_P^{m-1} \oplus R_P \xrightarrow{\varphi_P} R_P^h \oplus R_P \longrightarrow I_P \longrightarrow 0,$$

with

$$\varphi_P = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \psi \end{array}\right)$$

Therefore, we get a free presentation $R_P^{m-1} \xrightarrow{\psi} R_P^h \longrightarrow I_P \longrightarrow 0$, thus showing that I_P is generated by (at most) h elements.

(b) $\Rightarrow$ (c) By [6, Proposition 10.4] The symmetric algebra of I is a Cohen–Macaulay ring. Therefore, by [6, Proposition 8.4] it suffices to show that $\operatorname{ht} I_Q \leq \operatorname{ht} Q$ for every prime $Q \subset R$ containing I. Let P be a minimal prime of R/I contained in Q. If $P = Q$ the hypothesis guarantees the inequality. Otherwise $\operatorname{ht}(Q) \geq h + 1$. But $\operatorname{ht}(I_Q) = \operatorname{ht} I_P = h$ because R/I is equidimensional, hence we are through.

(c) $\Rightarrow$ (a) By [6, Lemma 8.2 and Proposition 8.4] one has $\operatorname{ht} I_t(\varphi) \geq \operatorname{rank}(\varphi) - \mathrm{t} + 2$ for every $1 \leq t \leq \operatorname{rank}(\varphi) = \mathrm{h}$. In particular, $\operatorname{ht} I_1(\varphi) \geq h - 1 + 2 = \operatorname{ht} I + 1$.

□

We get the following Lemma for linear type property of gradient ideal.

Lemma 2.2.2 *Let $f \in R = k[x_1, \ldots, x_n]$ stand for a reduced homogeneous polynomial. Assume that the singular locus of $V(f) \subset \mathbb{P}^{n-1}$ consists of a nonempty set of points. The following are equivalent:*

(1) *The coordinates of the vector fields of $\mathbb{P}^{n-1}$ vanishing on f generate an irrelevant ideal.*

(2) *Locally at each singular point of $V(f)$ the gradient ideal is a complete intersection.*

(3) *The gradient ideal of f is of linear type.*

Proof. A vector field $v = \sum_{i=1}^{n} a_i \partial/\partial x_i$ vanishes on f if and only if $\sum_{i=1}^{n} a_i \partial f/\partial x_i = 0$. Therefore the coordinates of all such vector fields generate the ideal of 1-minors of a syzygy matrix of the gradient ideal. The result then follows immediately from Lemma 2.2.1 once its hypotheses are verified in this setup,as we proceed to see now.

Since f is assumed to be reduced, whose singular locus is a nonempty set of isolated singularities, its gradient ideal is a (homogeneous) ideal of codimension $n-1$, hence can only have minimal primes of codimension $n-1$ in R. Therefore it is equidimensional.

Finally, the depth condition is trivially verified for the numbers in question.

□

So much for the linear type property. Clearly, this property implies that the partial derivatives are algebraically independent over k. The latter property in turn reads geometrically to the effect that the polar map associated to the hypersurface $V(f) \subset \mathbb{P}^{n-1}$ is dominant. characteristic zero this is tantamount to saying that the Hessian of f does not vanish (cf. [3] for a detailed account on this).

As a simple application of th Lemma 2.2.1 for the case $n = 3$, we drive the following corollary:

Corollary 2.2.3 *Let $f \in R = k[x, y, z]$ denote an irreducible homogeneous polynomial over an algebraically closed field k of characteristic zero. If $\deg(f) \leq 3$ then the gradient ideal of f is an ideal of linear type.*

Proof. If f is a linear form then its gradient is the ring R which is R-free, hence trivially of linear type. An irreducible quadric is smooth, hence the partial derivatives constitute a regular sequence. It is well known that a complete intersection is of linear type.

Let f be a irreducible cubic, then f admit a singular point of multiplicity 2, say P, then either P is a simple node or a simple cusp. By projectivities we may assume that the possible singularity is located at the coordinate point $P = (0 : 0 : 1)$. Then f is a linear combination of

$$x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, z^2y, zy^2, xyz$$

and passes through P if and only if coefficient of z^3 vanishes. Since P is singular point, $\partial f/\partial x, \partial f/\partial y, \partial f/\partial z$ vanish at P. Note that $\partial f/\partial x, \partial f/\partial y$ vanish at P if and only if the coefficients of z^2x, z^2y vanish. If P is a simple cusp, one can assume that the tangent cone at $z \neq 0$ has equation y^2, then the coefficients of x^2z, xyz vanish. Then

$$f = y^2z + a_1x^3 + a_2y^3 + a_3x^2y + a_4xy^2$$

Note that $a_1 \neq 0$, otherwise f reduces, we can assume that $a_1 = 1$ by scaling x. Now we replace x by $z - \frac{a_3}{3}y$ one has

$$f = y^2z + x^3 + a_1y^3 + a_2xy^2$$

Finally, replace z by $z - a_1y - a_2x$ and renaming parameters, we get $f = y^2z + x^3$.

If P is a node, one can assume that the tangent cone at $z \neq 0$ has equation xy, the the coefficients of x^2z, y^2z vanish. Then

$$f = xyz + a_1x^3 + a_2y^3 + a_3x^2y + a_4xy^2$$

Note that $a_1.a_2 \neq 0$, otherwise f reduces, we can assume that $a_1 = a_2 = 1$ by scaling x and y. Finally replace z by $z - a_3x - a_4y$ and renaming parameters we get $f = xyz + x^3 + y^3$.

Therefore, If f is a non-smooth irreducible cubic then it is projectively equivalents to either $f = y^2z - x^3$ or $f = xyz + x^3 + y^3$. The respective gradient ideals are (x^2, yz, y^2) and $(3x^2 + yz, 3y^2 + xz, xy)$ and in both cases the ideal of the unique singular point is (x, y). An easy calculation with the generators shows the first one locally at (x, y) is generated by x^2, y, for the second 1–minors of the syzygy matrix is (x, y, z)–primary. Then by Lemma 2.2.1, the gradient ideal I_f is of linear type

□

2.3 The Aluffi gradient algebra of a plane curve

In general, the gradient ideal I_f will not be of linear type. We have seen in the corollary 2.2.3, if f is a irreducible homogenous polynomial in $R = k[x, y, z]$ of degree ≤ 3, then the gradient ideal I_f is of linear type. We can immediately show simple cases of rational plane quintics and sextic whose corresponding gradient ideals are not of linear type. Moreover, the associated Aluffi gradient algebras behaves quite erratically from the viewpoint of their associated primes. One can show that these examples lie on a suitable family with similar behavior.

Example 2.3.1 Let $f = y^4z + x^5 + x^3y^2$. Then $I_f = (x^2(5x^2 + 3y^2), y(2x^3 + 4y^2z), y^4)$. Canceling the common factor among the last two generators, gives rise to the obvious Koszul relation. From this it immediately follows that the radical of the ideal generated by the coordinates of the syzygies of I_f has x, y among its minimal generators. The rest follows by inspection, as it is not difficult to verify that no syzygy coordinate has as term a pure z-power. By Corollary 2.2.2, I_f is not of linear type.

Of course everything in this example is easily obtained by machine computation (e.g,. *Macaulay*). The three algebras $\mathcal{S}_{R/(f)}(I_f/(f)) \twoheadrightarrow \mathcal{A}_{R/(f)}(I_f/(f)) \twoheadrightarrow$

$\mathcal{R}_{R/(f)}(I_f/(f))$ are all distinct, but of the same dimension. By Theorem 2.1.5 the leftmost is Cohen–Macaulay, while the Aluffi algebra has no embedded primes though it is not Cohen–Macaulay.

Now let $f = zy^2(x^2+y^2)+x^5+y^5+x^3y^2$. Here the symmetric algebra is Cohen–Macaulay, while the Aluffi algebra has embedded primes.

Finally, let $f = xy^4z+x^6+y^6$. A computer calculation yields the following syzygy matrix of I_f:

$$\begin{pmatrix} 0 & -6y^3z & -6y^4 \\ xy & 9x^4+y^2z^2 & y^3z \\ -6y^2-4xz & -54x^3y-4yz^3 & 36x^4-4y^2z^2 \end{pmatrix}$$

Note that no syzygy coordinate contains a term which is a pure z-power. By Corollary 2.2.2, I_f is not of linear type. A computer calculation, monitored by the information on the format of the minimal primes of the corresponding Aluffi gradient algebra, shows that this algebra is puredimensional, though it is not Cohen-Macaulay.

2.4 Families of irreducible singular curves

In this part we study families of singular plane curves and a corresponding "relative" Jacobian ideal for the linear type property and the corresponding Aluffi gradient algebra. We start by making clear what we mean by a family for our purposes.

Let $k[\mathbf{a}] = k[a_1,\ldots,a_m]$ stand for a polynomial ring over the field k and let $F \in S := k[\mathbf{a}][x,y,z]$ denote a polynomial which is a form on x,y,z. We give S the structure of standard graded ring over k[**a**]. The basic assumption is that the content of F with respect to the **a**-coefficients is 1. Then F is a non-zero-divisor on $k[\mathbf{a}]/I$ for every ideal $I \subset k[\mathbf{a}]$, hence $\mathrm{Tor}^1_{k[\mathbf{a}]}(k[\mathbf{a}]/I, k[\mathbf{a}][x,y,z]/(F)) = \{0\}$ for any such ideal. This gives that the inclusion $k[\mathbf{a}] \subset k[\mathbf{a}][x,y,z]/(F)$ is flat, hence defines a family of curves in $\mathbb{P}^2$ over the parameters **a**.

Thus, we speak of a *family of plane curves* over the *parameters* **a** when referring to this setup. We will of course adhere to the terminology of calling *general member* of the family the equation of the plane curve obtained by substituting general values in k for **a**. Moreover, our interest lies on the case where the general member of the family is a reduced singular plane curve. In this case we speak of a *family of plane singular curves.*

In the sequel we will assume moreover that $m \leq \binom{d+2}{2}-1$, where d is the (homogeneous) degree of F in x,y,z and that F has the form

$$F = \varphi_0(x,y,z) + \sum_{j=1}^{m} a_j\varphi_j(x,y,z), \tag{2.2}$$

where $\{\varphi_j(x,y,z)\,|\,0 \leq j \leq m\}$ is a set of monomials of degree d in x,y,z, and $\varphi_0(x,y,z) \neq 0$.

Note that the form of F depends on the singular points of the general member. Thus, it makes sense to speak about a *normal form* or *canonical form* of F depending on this singular locus. Our convention is that such a normal form is to be obtained through projective transformations applied to the x, y, z-coordinates allowing coefficients from $k[\mathbf{a}]$. Besides, in order to account for *degeneration of singularities* of the general member we need correspondingly to consider certain *degeneration ideals* in the parameter ring $k[\mathbf{a}]$.

Write

$$F \equiv \varphi_0(x, y, z) + \psi(x, y, z, a_1, \ldots, a_m),$$

as in (2.2), where $\varphi_0(x, y, z)$ involves the singularity type in terms of the projectivized tangent cones on suitable affine pieces.

Example 2.4.1 Let us write a normal form for the family Γ of irreducible singular quartic plane curves such that the singular locus of the general member consists of one simple node - note that at this point it is not totally clear that there exists at all such a family in the sense we established, since we must first obtain some $F \in S$ that works. By projectivities one can assume that the node is $P = (0 : 0 : 1)$ and the tangent cone at $z \neq 0$ has equation xy. Since the general member ought to vanish at P then we may omit the terms in z^4, z^3x and z^3y. Thus, an intermediate step towards a normal form is

$$F = xyz^2 + a_1x^3z + a_2x^2yz + a_3xy^2z + a_4y^3z + a_5x^4 + a_6x^3y + a_7x^2y^2 + a_8xy^3 + a_9y^4.$$

We can see that the specialization of F by k-values factors properly if both a_1 and a_5 have vanishing k-values; similarly, if both a_4 and a_9 have vanishing k-values. Thus, for writing a normal form we may incorporate x^4 and y^4 as terms of $\varphi_0(x, y, z)$. Finally, the projectivity $x = x, y = y, z = z - \frac{1}{2}(a_2x + a_3y)$ (characteristic $\neq 2$) allows to eliminate the terms in x^2yz and xy^2z. Up to renaming parameters, this yields the following normal form:

$$F = xyz^2 + x^4 + y^4 + a_1x^3z + a_2y^3z + a_3x^3y + a_4x^2y^2 + a_5xy^3$$

Example 2.4.2 In this example we would like to write a normal form for the family of irreducible singular quartic plane curve such that the singular locus of the general member admit two singular points of type node and cusp. By projectivities one can assume that the node is located at $P = (0 : 0 : 1$ and the tangent cone at $z \neq 0$ has equation $x^2 - y^2$ and the cusp is $q = (0 : 1 : 0)$ with the tangent cone at $y \neq 0$ has equation z^2. Thus, an first step towards a normal form is

$$F = (x^2 - y^2)z^2 + a_1x^4 + a_2x^3y + a_3x^3z + a_4x^2yz$$

we can see that the specialization of F by k–values factors if $a_1a_2 \neq 0$. Then we may incorporate x^4 as terms of $\varphi_0(x, y, z)$. Then up to renaming parameters, this yields the following normal form:

$$F = (x^2 - y^2)z^2 + x^4 + a_1x^3y + a_2x^3z + a_3x^2yz$$

2.5 Degeneration of singularity

The normal form has degenerations to other normal forms whose general member has more involved singularities or even acquires new singular points. The following example may illuminate this phenomenon.

Example 2.5.1 Consider the family of irreducible rational plane quartics with exactly three nodes. In [9, Lemma 11.3] a normal form is given of a family whose general member is an irreducible quartic with three double points, namely

$$F = \alpha x^2y^2 + \beta x^2z^2 + \gamma y^2z^2 + 2xyz(a_1x + a_2y + a_3z),\ \alpha\beta\gamma \neq 0.$$

To get a normal form whose general member is an irreducible quartic with three nodes, substitute $x = \alpha^{-1}x, \beta^{-1}y, \gamma^{-1}z$ and rename $(\alpha\beta\gamma)^{-1}a_1, (\alpha\beta\gamma)^{-1}a_2, (\alpha\beta\gamma)^{-1}a_3$ to a_1, a_2, a_3, respectively. One obtains the normal form

$$F = x^2y^2 + x^2z^2 + y^2z^2 + 2xyz(a_1x + a_2y + a_3z).$$

Note that for k-values $a_1 = \pm 1$, one of the nodes degenerates into a cusp, similarly, for $a_2 = \pm 1$ or $a_3 = \pm 1$, for k-values $a_1 = a_2 = \pm 1$, two nodes degenerate into two cusp, similarly $a_1 = a_3 = \pm 1$ or $a_2 = a_3 = \pm 1$ and finally for k-values $a_1 = a_2 = a_3 = \pm 1$ three nodes degenerate to three nodes. Thus the general member requires that the k-values of the triple (a_1, a_2, a_3) do not lie on the hypersurface $V((a_1^2 - 1)(a_2^2 - 1)(a_3^2 - 1))$ in order that it have exactly three nodes.

Requiring that the general member acquire no new singular points besides the three nodes imposes yet another obstruction. Of course, in the present low degree 4, because of genus reason there will be new singular points only if the general member properly factors. As we will see this obstruction is precisely given by the hypersurface whose equation is the discriminant $2a_1a_2a_3 + a_1^2 + a_2^2 + a_3^2 - 1$ of a suitable conic (see Section 2.6).

Remark 2.5.2 The last polynomial above can also be computed through resultants, as follows: first compute $g_1 := \mathrm{Res}\,(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial z}, z)$ and $g_2 := \mathrm{Res}\,(\frac{\partial F}{\partial y},\ \frac{\partial F}{\partial z}, z)$. These resultants admit the factors $8xy^2$ and $8x^2y$, respectively; then compute $\mathrm{Res}\,(g_1/8xy^2,\ g_2/8x^2y)$, which has $a_1^2 + a_2^2 + a_3^2 + 2a_1a_2a_3 - 1$ as a factor.

The following is our main result for this part. It would mostly suffice for it to assume that the **a**-coefficients of the terms of F be algebraically independent over k.

Theorem 2.5.3 *Let F denote a family of singular plane curves of degree d, on parameters $\mathbf{a} = a_1, ..., a_m$, whose general member is reduced and irreducible. Write $S = k[\mathbf{a}][x, y, z]$. Let $I_F \subset S$ denote the ideal generated by the x, y, z-partial derivatives of F and let $\mathcal{I} \in S$ stand for the ideal of 1–minors of the syzygy matrix of I_F. Then:*

(a) $\mathrm{codim}\,(I_F) = 2$

(b) $\operatorname{codim}(\mathcal{I}) \leq 3$

(c) *If k is algebraically closed of characteristic zero, the following are equivalent:*

   (i) $\operatorname{codim}(\mathcal{I}) = 3$

   (ii) *The contraction of the ideal $\mathcal{I} : (x, y, z)S^{\infty}$ to $k[\mathbf{a}]$ is a nonzero ideal.*

   (iii) *The plane projective curve obtained by evaluating $(a_1, \ldots, a_m)$ at points in $\mathbb{A}^m$ off the zero set of $(\mathcal{I} : (x, y, z)S^{\infty}) \cap k[\mathbf{a}]$ has gradient ideal of linear type.*

Proof. (a) Clearly, $\operatorname{codim}(I_F) \leq 3$. Since the general member is singular and reduced its gradient ideal has codimension 2. This forces $\operatorname{codim}(I_F) = 2$.

(b) We go more algebraic: the ring $S = k[\mathbf{a}][x, y, z]$ is standard graded with $S_0 = k[\mathbf{a}]$. Since F is a homogeneous polynomial in this grading, its partial derivatives are homogeneous of same degree. We claim that any syzygy of the partial derivatives has coefficients in $(x, y, z)S$. Indeed, since the partials are in the same degree, a syzygy outside $(x, y, z)S$ would have to be of degree 0, i.e., with all its coordinates in the zero degree part $k[\mathbf{a}]$. But this would force a syzygy among the coefficients of degree 0 of the partials, hence a polynomial relation of $\mathbf{a}$, which is nonsense since these are indeterminates over k.

Incidentally, note this argument breaks down for the second syzygies (and on) as the first syzygies may have different degrees in S.

(c) (i) $\Rightarrow$ (ii) Write $\mathfrak{m} = (x, y, z)S$. Clearly, $\mathcal{I} : \mathfrak{m}^{\infty} \not\subseteq (x, y, z)S$. This is because $\operatorname{codim}(\mathcal{I}) = 3$ and $\mathfrak{m}$ is a minimal prime therein by the proof of (b), hence the saturation picks up the remaining primary components of $\mathcal{I}$ (possibly empty), hence cannot be contained in $\mathfrak{m}$. This shows that $(\mathcal{I} : \mathfrak{m}^{\infty}) \cap k[\mathbf{a}] \neq \{0\}$.

(ii) $\Rightarrow$ (iii) Let $g = g(\mathbf{a}) \in (\mathcal{I} : \mathfrak{m}^{\infty}) \cap k[\mathbf{a}]$ be any nonzero element. By hypothesis, g conducts a power of $(x, y, z)S$ inside $\mathcal{I}$. Giving $\mathbf{a}$ k-values α off $V((\mathcal{I} : \mathfrak{m}^{\infty}) \cap k[\mathbf{a}])$ yields a power of the maximal ideal $(x, y, z) \subset k[x, y, z]$ inside the image $\mathcal{I}(\alpha)$ of $\mathcal{I}$ by this evaluation. Let $f = F(\alpha) \in k[x, y, z]$ denote the member of the family thus obtained. Then $\mathcal{I}(\alpha) \subset I_1(\varphi)$, where φ denotes the syzygy matrix of the partial derivatives of f. This shows that $I_1(\varphi)$ is (x, y, z)-primary. Therefore, the result follows from Corollary 2.2.2.

(iii) $\Rightarrow$ (i) The hypothesis is that the gradient ideal of the general member of the family is of linear type. Again by Corollary 2.2.2 this implies that the ideal of 1-minors of such a plane curve has codimension 3. On the other hand, for general value α of $\mathbf{a}$, the ideals $I_1(\varphi_\alpha)$ and $\mathcal{I}$ have the same codimension, where φ_α stands for the syzygy matrix of $F(\alpha)$.

□

Remark 2.5.4 In the above notation, if the image $\mathcal{I}(\mathbf{0}) \subset k[x, y, z]$ of $\mathcal{I}$ through the evaluation $\mathbf{a} \mapsto \mathbf{0}$ is (x, y, z)-primary then $\mathcal{I}$ has codimension 3. It follows that if

$f \in k[x,y,z]$ is a general fiber (by evaluation), the ideal generated by the entries of the syzygy matrix of I_f has codimension 3 as well.

Care has to be exercised: although $\mathcal{I}(\mathbf{0})$ is contained in the ideal of 1-minors of the syzygy matrix of the special member $F(\mathbf{0})$ obtained by evaluating F at $\mathbf{0}$, they may be different and, in fact, have different codimensions. Thus, e.g., the one-parameter family $F = y^4z + x^5 + ax^3y^2$ (see Example 2.3.1) is such that $\mathcal{I}$ has height 2 – hence the gradient ideal of the general member $F(\alpha)$ of the family is not of linear type – and nevertheless the special member $F(\mathbf{0})$ is easily seen to have gradient ideal of linear type. In particular, this shows a piece of difficulty of the theory in which, perhaps unexpectedly, the notion of being of linear type is neither kept by specialization nor by generization.

Remark 2.5.5 In the setting of elimination theory. Let $\pi : \mathbb{P}^2_k \times \mathbb{A}^2_k \to \mathbb{A}^m_k$ be the projection map to the second factor. Since $X = V(\mathcal{I}) \subset \mathbb{P}^2_k \times \mathbb{A}^m_k$ define a algebraic closed set, by the main theorem of the elimination theory, the image of X, say $\pi(X)$ is the zero set of $\mathcal{I} : (x,y,z)^\infty \cap k[\mathbf{a}]$. We can interpret $\pi(X)$ as the set of m–tuple α for which X has a non-trivial solution in x,y,z.

Conjecture 2.5.6 Under the hypotheses of the previous theorem, any of the conditions (i) through (iii) is equivalent to the following one: *the contraction of the ideal* $\mathcal{I} : (x,y,z)S^\infty$ *to* $k[\mathbf{a}]$ *is an ideal of codimension* 1.

2.6 Rational quartics curves

We review some preliminaries about rational quartics, the basic reference being [23].

An irreducible rational quartic having only double points can be obtained as a rational transform from a non-degenerate conic by means of one of the three basic plane quadratic Cremona maps:

(1) $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with defining coordinates $(yz : zx : xy)$

The base locus of this Cremona map consists of the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$, each with multiplicity one (in the classical terminology, three proper points – see [1]).

(2) $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with defining coordinates $(xz : yz : y^2)$

The base locus of this Cremona map consists of the points $(0:0:1)$ and $(1:0:0)$, with multiplicity 1 and 2, respectively (in the classical terminology, one proper point and another proper point with a point in its first neighborhood).

(3) $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with defining coordinates $(y^2 - xz : yz : z^2)$

The base locus of this Cremona map consists of the point $(1:0:0)$ with multiplicity 3, a so-called triple structure on a point (in the classical terminology, one proper point with a point in its first neighborhood and a point in its second neighborhood).

We will use Theorem 2.5.3 and Corollary 2.2.2, along with the above ideas, to prove the following curious phenomenon.

Theorem 2.6.1 (k algebraically closed of characteristic zero) *Let $f \in k[x, y, z]$ be an irreducible rational quartic plane curve. Then the gradient ideal of f is an ideal of linear type.*

Proof. We may assume that f is singular as otherwise the gradient ideal is generated by a regular sequence.

Then f has at least one double singular point and at most a triple point as it is assumed to be rational. Let us first consider the situation where f has a double point – hence has at most 3 such points and no triple point.

In this case, f comes as explained above from a conic by means of a Cremona map.

Let $F = a_1x^2 + a_2y^2 + a_3z^2 + 2a_4yz + 2a_5zx + 2a_6xy$ be the equation of the conic as above, assumed non-singular, i.e., the corresponding symmetric matrix has nonzero determinant $\Delta = a_1a_2a_3 + 2a_4a_5a_6 - a_1a_4^2 - a_2a_5^2 - a_3a_6^2$ (the *discriminant* of F).

Applying the above Cremona maps, we obtain, respectively:

(1) A quartic with exactly three double points at $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$ and $P_3 = (0 : 0 : 1)$, where P_1 (respectively, P_2, P_3) is a node except when the principal minor $a_1a_2 - a_6^2$ vanishes (respectively, except when the principal minors $a_1a_3 - a_5^2$, $a_2a_3 - a_4^2$ vanish).

Here we may harmlessly assume that all of them equal to $a_1 = a_2 = a_3 = 1$ provided they are all nonzero.

(2) A quartic with a double point at $P_1 = (0 : 0 : 1)$ (which is a node or a cusp according as to whether $a_2 \neq 0$ r $a_2 = 0$) and a double point at $P_2 = (1 : 0 : 0)$ (which is either a tacnode or a ramphoid cusp according as to whether the principal minor $a_1a_3 - a_5^2$ is nonzero or vanishes).

Here we may assume that $a_1 = a_3 = 1$ and $a_6 = 0$.

(3) A quartic with an oscnode at $P_1 = (1 : 0 : 0)$ if $a_2 \neq 0$; else, a singularity of type A_6.

Here we may assume that $a_1 = 1$ and $a_5 = a_6 = 0$.

An irreducible rational quartic having only double points – ordinary or not – falls within the following families up to coordinate change, according to the nature of its singularities. Below we have written $\tilde{f}$ instead of F to help us think of the general member instead of the family itself, thus making lighter to pinpoint the degeneration loci. To keep track of the parameters in each case we have maintained the original indices, however anaesthetical it may look. The requirement that $\Delta \neq 0$ will mean that the general member is irreducible.

(a) Three nodes:

$$\tilde{f}_a = y^2z^2 + x^2z^2 + x^2y^2 + 2xyz(a_4x + a_5y + a_6z), \qquad a_4, a_5, a_6 \neq \pm 1, \quad \Delta \neq 0$$

where $\Delta = 2a_4a_5a_6 + a_4^2 + a_5^2 + a_6^2 - 1$.

(b) Two nodes and one cusp:

$$\tilde{f}_b = y^2z^2 + x^2z^2 + x^2y^2 + 2xyz(a_4x + a_5y + z), \quad a_4, a_5 \neq \pm 1, \ \Delta = (a_4 + a_5)^2 \neq 0$$

(c) One node and two cusps:

$$\tilde{f}_c = y^2z^2 + x^2z^2 + x^2y^2 + 2xyz(a_4x + y + z), \quad a_4 \neq \pm 1$$

(d) Three cusps :

$$\tilde{f}_d = y^2z^2 + x^2z^2 + x^2y^2 - 2xyz(x + y + z)$$

(e) One tacnode and one cusp:

$$\tilde{f}_e = x^2z^2 + y^4 + 2y^3z + 2a_5xy^2z, \qquad a_5 \neq \pm 1$$

(f) One tacnode and one node :

$$\tilde{f}_f = z^2(x^2 + y^2) + y^4 + 2a_4y^3z + 2a_5xy^2z, \quad a_5 \neq \pm 1, \quad \Delta = a_4^2 + a_5^2 - 1 \neq 0$$

(g) One ramphoid cusp and one node:

$$\tilde{f}_g = x^2z^2 + y^4 + 2zy^3 + 2xy^2z + a_2z^2y^2, \quad a_2 \neq 0$$

(h) One ramphoid cusp and one cusp:

$$\tilde{f}_h = x^2z^2 + y^4 + 2zy^3 + 2xy^2z$$

(i) One oscnode :

$$\tilde{f}_i = (y^2 - xz)^2 + y^2z^2 + a_3z^4, \qquad a_3 \neq 0$$

(j) One singularity of type A_6:

$$\tilde{f}_j = (y^2 - xz)^2 + 2yz^3$$

We now consider the case where the quartic has a triple point, say, at $P = (0:0:1)$. In this case the equation of the curve can be taken in the form $\varphi(x, y)\, z + \psi(x, y) = 0$, where φ can be brought up to one among the forms $xy^2 - x^3$, xy^2 and y^3, and ψ may be further normalized in such a way that the resulting family has as few parameters as possible. After these reduction manipulations, up to projective equivalence, any irreducible plane quartic with a triple point falls within three basic families, according to the nature of the triple point:

(k) An ordinary triple point:

$$\tilde{f}_k = x(y^2 - x^2)z + y^4 + a_1x^2y^2 + a_2yx^3$$

(l) A triple point with double tangent:

$$\tilde{f}_l = xy^2z + x^4 + y^4 + a_1x^3y$$

(m) A higher cusp :

$$\tilde{f}_m = y^3z + x^4 + a_1x^2y^2.$$

We divide all above cases to four sets $A = \{a, c, e, f, k\}$, $B = \{i, g, b\}$, $C = \{d, h, j\}$. and $D = \{l, m\}$. We now use the Theorem 2.5.3 and corollary 2.2.2 in each one of the above cases.

Write $J_{\tilde{f}} = (\partial\widetilde{f}/\partial x, \partial\widetilde{f}/\partial y, \partial\widetilde{f}/\partial z)$, $\mathfrak{A} = (\mathcal{I} : (x, y, z)^\infty) \cap k[\mathbf{a}]$ and $\mathfrak{D} = \sqrt{\mathfrak{A}}$.

For $\tilde{f}_\xi$ with $\xi \in A$, inspecting the format of the partial derivatives gives that their syzygies are the syzygies of the special members $\tilde{f}_\xi(\mathbf{0})$ for $\xi \in A$. Therefore the codimension of $\mathcal{I}_\xi(\mathbf{0})$ for $\xi \in A$ is read off the ideal of 1-minors of the following matrices:

$$\mathcal{M}_a = \begin{pmatrix} xy^2 - xz^2 & -x^3 - xz^2 & 0 & -x^2z - y^2z & x^2y + yz^2 \\ -y^3 - yz^2 & x^2y - yz^2 & -xz^2 - y^2z & 0 & -xy^2 - xz^2 \\ y^2z + z^3 & x^2z + z^3 & x^2y + yz^2 & xy^2 + xz^2 & 0 \end{pmatrix},$$

$$\mathcal{M}_c = \begin{pmatrix} -xy + xz & 2x^2 - 3xy - 3xz + 2yz & 3xy^2 - 2y^2z - 3xz^2 + 2yz^2 \\ 3y2 - 2xz + 3yz & -2xy + 3y^2 - yz & -3y^3 - yz^2 \\ 2xy - 3yz - 3z^2 & -2xz - yz + 3z^2 & y^2z + 3z^3 \end{pmatrix},$$

$$\mathcal{M}_k = \begin{pmatrix} -12x^2 + 8y^2 - 2xz & 4x^2y - 4y^3 & x^3 - xy^2 & 0 \\ -2yz & -x^2z + y^2z & 0 & x^3 - xy^2 \\ 36xz + 6z^2 & -12xyz - 2yz^2 & -3x^2z + y^2z & 4y^3 + 2xyz \end{pmatrix}$$

$$\mathcal{M}_f = \begin{pmatrix} x^2 + y^2 & 2y^3 + yz^2 & 0 & 2xy^2 + xz^2 \\ 0 & -xz^2 & -xz^2 - y^2z & yz^2 \\ -xz & 0 & 2y^3 + yz^2 & -2y^2z - z^3 \end{pmatrix},$$

$$\mathcal{M}_e = \begin{pmatrix} 2xy + 3xz & 2x^2 - 3y^2 & 0 \\ yz & xz & -y^3 - x^2z \\ 2yz - 3z^2 & -2xz & 2y^3 + 3y^2z \end{pmatrix},$$

On the nose manipulation readily shows that $x^3, y^3, z^3 \in \mathcal{I}_\xi(\mathbf{0})$. Thus, $\mathcal{I}_\xi(\mathbf{0})$ has codimension 3 for $\xi \in A$. Now Remark 2.5.4 implies that $\mathcal{I}$ has codimension 3.

Then, by Theorem 2.5.3 $\tilde{f}_\xi(\alpha)$ has gradient ideal of linear type for every $\alpha \in \mathbb{A}_k^{r_\xi}$ (r_ξ is the number of the parameter depend on $\xi \in A$) off the zero sets of $\mathfrak{D}_a = (a_4^2-1)(a_5^2-1)(a_6^2-1)(2a_4a_5a_6+a_4^2+a_5^2+a_6^2-1)$, $\mathfrak{D}_c = (a_4^2-1), \mathfrak{D}_e = (a_5^2-1), \mathfrak{D}_f = (a_5^2-1)(a_4^2+a_5^2-1)$ and $\mathfrak{D}_k = a_1^2 - a_2^2 + 2a_1 - 1$, respectively.

For $\tilde{f}_\xi$ with $\xi \in B$, we denote by $\mathfrak{P}_\xi$ the set of minimal prime ideal of $\mathcal{I}_\xi$ for $\xi \in B$. It is easy to see that for $\xi = i, g$ the minimal prime ideal of $\mathcal{I}_\xi$ other than (x, y, z) are $\mathfrak{P}_g = \{(x, y, a_2)\}$ and $\mathfrak{P}_i = \{(x, y, a_3)\}$. For $\mathcal{I}_b$, we have the following minimal prime ideals: $\mathfrak{P}_b := \{(x, y, a_4+a_5),\ (xy+xz-yz, a_4-1, a_5+1),\ (x, z, a_5-1),\ (y, z, a_4-1),\ (xy+xz-yz, a_4+1, a_5-1),\ (y, z, a_4+1),\ (x, z, a_5+1)\}$. This implies that $\mathcal{I}_\xi$ for $\xi \in B$ has codimension 3. Then by Theorem 2.5.3, $\tilde{f}_\xi(\alpha)$ has gradient ideal of linear type for every $\alpha \in \mathbb{A}_k^{r_\xi}$ off the zero set of $\mathfrak{D}_g = a_2,\ \mathfrak{D}_i = a_3$ and $\mathfrak{D}_b = (a_4^2-1)(a_5^2-1)(a_4+a_5)$ respectively.

For $\tilde{f}_\xi$ with $\xi \in C$, we apply directly Lemma 2.2.2. Therefore the jacobian matrix of $\tilde{f}_\xi$ for $\xi \in C$ is:

$$\mathcal{M}_d = \begin{pmatrix} 3xy+3xz-2yz & 3xz-yz & 3x^2-xz \\ -3y^2+yz & 3xy+3y^2-3xz+5yz & -3xy+2xz-3yz \\ yz-3z^2 & 3xy-3xz-4yz-6z^2 & -xz+3z^2 \end{pmatrix},$$

$$\mathcal{M}_h = \begin{pmatrix} 2y^2-3xz & 10xy+9xz & 50x^2-27xz \\ -yz & -2xz+3yz & 10xy+10y^2+6xz-9yz \\ 3z^2 & -6yz-9z^2 & -20y^2-50xz-12yz+27z^2 \end{pmatrix},$$

$$\mathcal{M}_j = \begin{pmatrix} 6y^2+xz & 7xy-3z^2 & 7x^2+18yz \\ 3yz & 3xz & 3xy \\ -z2 & -yz & 6y^2-7xz \end{pmatrix}$$

An easy calculation shows that $x^2, y^2, z^2 \in I_1(\mathcal{M}_\xi)$ for $\xi \in C$. Therefore Lemma 2.2.2 implies that the gradient ideal is of linear type.

For $\tilde{f}_\xi$ with $\xi \in D$,. It is easy to check that the 1–minors of the syzygy matrix of ideal $J_{\tilde{f}_\xi}$ for $\xi = l, m$ is (x, y, z)-primary, that is, $x^3, y^3, z^3 \in \mathcal{I}_\xi$. Then the gradient ideal of $\tilde{f}_\xi(\alpha)$ for every $\alpha \in \mathbb{A}_k^1$ is of linear type.

Now we verify the linear type property for points in $V(\mathfrak{D}_\xi)$:

Case(a): since the polynomial $\Delta = 2a_4a_5a_6 + a_4^2 + a_5^2 + a_6^2 - 1$ is the discriminant of a conic, for $\alpha \in V(\Delta) \subset \mathbb{A}_k^3$, we have a reducible conic whose transformation by Cremona map (1) is a reducible rational quartic, but we are interested in irreducible rational quartic. For k-values $a_4 = \pm 1$ such that $a_5^2 + a_6^2 \pm 2a_5a_6 \neq 0$, one of the nodes degenerates into a cusp, then we have a irreducible rational quartic lies in the family, with two nodes (b). Similarly, for $a_5 = \pm 1$ or $a_6 = \pm 1$. For k-values $a_4 = a_5 = \pm 1$ such that $a_6 \neq \pm 1$, two nodes degenerate into two cusp and lies in the family (c). Similarly $a_4 = a_6 = \pm 1$ or $a_5 = a_6 = \pm 1$. Finally for k-values $a_1 = a_2 = a_3 = \pm 1$ three nodes degenerate to three nodes, we have quartic with three cusp.

Case (b): similar argument as case (a).

Case(c): for k-value $a_4 = \pm 1$, the node type singularity degenerate into cusp, then we get a quartic with three cusps (d). Case (e), for k-value $a_5 = \pm 1$, the Tacnode type singularity degenerate into Ramphoid cusp, hence we a quartic with one ramphoid cusp and one cusp

Case(f): for k-value in $\alpha \in V(a_4^2 + a_5^2 - 1)$ we get a reducible quartic. For k-value $a_5 = \pm 1$ such that $a_4 \neq 0$ we get a family lies in the family with one ramphoid cusp and one node.

Case (k): by Bezout's Theorem we note that it is impossible a quartic has two points of multiplicity 3 and also with multiplicity 2 and 3. Then for for k-value in $V(a_1^2 - a_2^2 + 2a_1 - 1)$ we obtain exactly a reducible quartic.

In case (g),(i), for k-value $a_2 = 0$ and $a_3 = 0$, we obtain a rational quartic in families (h) and (f), respectively. □

Chapter 3

Appendix

3.1 All quartic curves

In this part we will prove the following general result.

Theorem 3.1.1 *Let $f \in R = k[x, y, z]$ denote an irreducible homogeneous polynomial over an algebraically closed field k of characteristic zero. If $\deg(f) = 4$ then the gradient ideal of f is an ideal of linear type.*

Proof. An irreducible plane quartic has arithmetic genus 3, hence has at most 3 singular points including infinitely near ones. By projectivities we may assume that the possible singularities are located at the coordinate points. Then up to this change and possibly more, any irreducible plane quartic falls within the following families, according to the nature of its singularities [23]:

- $aA_1 + bA_2 \quad a + b \leq 3$
- $A_3, A_1 + A_3, A_2 + A_3$
- $A_4, A_1 + A_4, A_2 + A_4$
- A_5, A_6
- D_4, D_5, E_6

where we means by; A_1 (simple node), A_2 (simple node), A_3 (Tacnode), A_4 (Ramphiod cusp), A_5 (Oscnode) and D_4, $D_5 E_6$ (triple points).

We observe that families $3A_1$, $3A_2$, $2A_1 + A_2$, $A_1 + 2A_2$, $A_3 + A_1$, $A_3 + A_2$, $A_4 + A_1$, $A_4 + A_2$, A_5, A_6, D_4, D_5 and E_6 are irreducible rational quartics. In the Theorem 2.6 we proved any rational quartic has gradient ideal of linear type. For the rest families, we have the following normal forms:

(a) At least one simple node:

$$\tilde{f} = xyz^2 + x^4 + y^4 + a_1x^3y + a_2xy^3 + a_3x^2y^2 + a_4x^3z + a_5y^3z,$$

(a1) At least two nodes:

$$\tilde{f} = (y^2 + x^2)z^2 + x^2y^2 + a_1x^4 + a_2x^3y + a_3x^3z + a_4x^2yz,$$

(b) At least one simple cusp:

$$\tilde{f} = y^2z^2 + x^4 + a_1y^4 + a_2xy^3 + a_3x^2y^2 + a_4x^3z + a_5x^2yz,$$

(b1) At least two cusps:

$$\tilde{f} = y^2z^2 + x^4 + a_1x^3y + a_2x^3z + a_3x^2yz,$$

(c) At least one Tacnode:

$$\tilde{f} = y^2z^2 - x^4 + a_1xy^3 + a_2x^2y^2 + a_3y^4,$$

(d) At least one Ramphiod cusp:

$$\tilde{f} = x^2z^2 + 2xy^2z + y^4 + y^3z + a_1yz^3 + a_2z^4,$$

(g) One node and one cusp:

$$\tilde{f} = z^2(y^2 - x^2) + x^4 + a_1xy^2z + a_2x^3z + a_3x^3y$$

The ultimate goal, as the rational quartics, is to show that any irreducible quartic whose equation is obtained from any of the above cases, by evaluating the a's to elements of the ground field k, has gradient ideal of linear type.

We now apply the Theorem 2.5.3 in each one of the above cases. Write $J_{\tilde{f}} = (\partial\widetilde{f}/\partial x, \partial\widetilde{f}/\partial y, \partial\widetilde{f}/\partial z)$, $\mathfrak{A} = (\mathcal{I} : (x,y,z)^\infty) \cap k[\mathbf{a}]$ and $\mathfrak{D} = \sqrt{\mathfrak{A}}$.

Case (a), inspecting the format of the partial derivatives gives that their syzygies is the syzygies of the special members $\tilde{f}(\mathbf{0}) = xyz^2 + x^4 + y^4$. Therefore the codimension of $\mathcal{I}(\mathbf{0})$ is read off the ideal of 1-minors of the following matrix:

$$\mathcal{M}_a = \begin{pmatrix} 0 & -2xyz & 4y^3 + xz^2 & 48y^3z \\ -2xy & 0 & -4x^3 - yz^2 & -12yz^3 \\ 4y^3 + xz^2 & x^2y + yz^2 & 0 & 96x^2y^2 + 6z^4 \end{pmatrix}$$

On the nose manipulation readily shows that $x^4, y^4, z^5 \in \mathcal{I}(\mathbf{0})$. Thus, $\mathcal{I}(\mathbf{0})$ has codimension 3. Now Remark 2.5.4 implies that $\mathcal{I}$ has codimension 3. Then, by Theorem 2.5.3 $\tilde{f}(\alpha)$ has gradient ideal of linear type for every $\alpha \in \mathbb{A}^5_k$ off the zero set $\mathfrak{D}$.

For the reminded cases, a direct verification, inspired by (but not dependent upon) a calculation with *Macaualy* shows that ideal $\mathcal{I}$ has codimension 3. Then, by Theorem 2.5.3 $\tilde{f}(\alpha)$ has gradient ideal of linear type for every $\alpha \in \mathbb{A}^r_k$ (r depend on the number of parameteres) off the zero set $\mathfrak{D}$.

An easy calculation shows that $\mathfrak{D}$ for above cases is a messy irreducible polynomial in parameters $\mathbf{a}$. Since we have a complete classification of irreducible singular quartics then evaluation $\tilde{f}$ for points in $V(\mathfrak{D})$ is a quartics which is projectively equivalents with the general member of other families or a reducible quartic.

Then this prove that a quartic has gradient ideal of linear type.

□

3.2 Quintics and Sextics

In this section we study the linear type property of an irreducible quintic and sextic plane curve with only one simple singular point.

we have the following result about linear type property of the gradient ideal of a quintic plane curve.

Proposition 3.2.1 *Let $f \in R = k[x, y, z]$ denote an irreducible homogeneous polynomial of degree 5 over an algebraically closed field k of characteristic zero whose corresponding plane curve has a quadruple singular point P. Then the gradient ideal of f is of linear type if and only if P is ordinary (i.e., the curve has distinct tangents at P).*

Proof. By a projective change of coordinates, we may assume that $P = (0 : 0 : 1)$. In this case the equation of the quintic curve can be taken in the form $z\varphi(x, y) + \psi(x, y) = 0$ where φ and ψ are homogenous polynomials of degree 4 and 5 respectively, having no common multiple roots, hence φ can be brought up to one among the forms $x^3y - xy^3, y^4, xy^3, x^2y^2$ and $x^2y^2 - y^4$ and ψ may be further normalized in such a way that the resulting family has a few parameters as possible. After these reduction manipulations, up to projective equivalence, any irreducible plane quintic with a quadruple point falls within three basic families, according to the nature of the quadruple points:

(a) Distinct tangent:

$$\tilde{f} = zxy(x^2 - y^2) + x^5 + y^5 + a_1x^4y + a_2x^3y^2$$

(b) One single tangent of multiplicity 4:

$$\tilde{f} = zy^4 + x^5 + a_1x^3y^2 + a_2x^2y^3,$$

(c) One tangent of multiplicity 3:

$$\tilde{f} = xy^3z + x^5 + y^5 + a_1x^4y + a_2x^3y^2$$

(d) Two tangents of multiplicity 2 each:

$$\tilde{f} = zx^2y^2 + x^5 + y^5 + a_1x^4y + a_2xy^4$$

(e) One tangent of multiplicity 2 and two remaining distinct tangents:

$$\tilde{f} = zy^2(x^2 - y^2) + x^5 + a_1x^4y + a_2xy^4 + a_3y^5$$

Write $J_{\tilde{f}} = (\partial\widetilde{f}/\partial x, \partial\widetilde{f}/\partial y, \partial\widetilde{f}/\partial z)$, $\mathfrak{A} = (\mathcal{I} : (x, y, z)^\infty) \cap k[\mathbf{a}]$ and $\mathfrak{D} = \sqrt{\mathfrak{A}}$.

A direct verification inspired by(but not depended upon) a calculation with *Macaulay* shows that, in the case (a), ideal $\mathcal{I}$ generated by coordinates of all syzygies of the ideal $J(\tilde{f})$ has codimension 3. Then theorem 2.5.3 implies that $\tilde{f}(\alpha)$ has gradient ideal of linear type for every $\alpha \in \mathbb{A}_k^2$ off the zero set of $\mathfrak{D}$.

Since by Bezout Theorem, a quintic by a quadruple singular point can not admit other singular point, hence for k-value in $V(\mathfrak{D})$, we have exactly a reducible quintic.

In the other cases, an easy calculation shows that ideal $\mathcal{I}$ has codimension 2, hence the gradient ideal of the general member $\tilde{f}(\alpha)$ of the family is not of linear type.

□

Proposition 3.2.2 *Let $f \in R = k[x, y, z]$ be an irreducible homogeneous polynomial of degree* 6 *over an algebraically closed field k of characteristic zero whose corresponding plane curve has one singular point P with multiplicity* 5 . *Then the gradient ideal of f is of linear type if and only if P is ordinary quadruple point.*

Proof. By a projective change of coordinates, we may assume that $P = (0 : 0 : 1)$. In this case the equation of the sextic curve can be taken in the form $z\varphi(x, y)+\psi(x, y) = 0$ where φ and ψ are homogenous polynomials of degree 5 and 6 respectively, having no common multiple roots, hence φ can be brought up to one among the forms $xy(x - y)(x^2 + y^2), y^5, xy^4, x^2y^3, y^3(x^2 - y^2)$ and $xy^2(x^2 - y^2)$ and ψ may be further normalized in such a way that the resulting family has a few parameters as possible. After these reduction manipulations, up to projective equivalence, any irreducible plane sextic with a quintuple point falls within three basic families, according to the nature of the quintuple points:

(a) Distinct tangent:

$$\tilde{f} = xy(x - y)(x^2 - y^2)z + x^6 + y^6 + a_1x^4y^2 + a_2x^3y^3 + a_3x^2y^4$$

(b) One single tangent of multiplicity 5:

$$\tilde{f} = zy^5 + x^6 + a_1x^4y^2 + a_2x^3y^3 + a_3x^2y^4$$

(c) One tangent of multiplicity 4:

$$\tilde{f} = xy^4z + x^6 + y^6 + a_1x^5y + a_2x^3y^3 + a_3x^2y^4$$

(d) Two tangent of multiplicity 2 and 3 each:

$$\tilde{f} = x^2y^3z + x^6 + y^6 + a_1x^5y + a_2x^4y^2 + a_3xy^5$$

(e) One tangent of multiplicity 3 and two remaining distinct tangents:

$$\tilde{f} = y^3(x^2 - y^2)z + x^6 + a_1x^5y + a_2x^4y^2 + a_3x^3y^3 + a_4x^2y^4$$

(f) One tangent of multiplicity 2 and three remaining distinct tangents

$$\tilde{f} = xy^2(x^2 - y^2)z + x^6 + y^6 + a_1x^5y + a_2x^4y^2 + a_3x^3y^3$$

the rest of proof as similar argument in Proposition 3.2.1 □

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