Extended Abstract

Let $S = k[\mathbf{x}] := k[x_1, \ldots, x_n]$ be a polynomial ring over a fixed field k. A monomial $x_1^{u_1} \cdots x_n^{u_n}$ in S is denoted by $\mathbf{x}^u$, where $u = (u_1, \ldots, u_n) \in \mathrm{N}^n$. A vector $u \in \mathrm{Z}^n$ can be written uniquely as $u = u^+ - u^-$, where u^+ and u^- are positive and negative parts of u, respectively. Let $B = (b_{ij})$ be an integer $n \times d$-matrix of rank d whose columns are vectors $b_1, \ldots, b_d$ in Z^n. For the lattice $\mathcal{L}_\mathcal{B}$ in Z^n which is spanned by the columns of B, the corresponding lattice ideal in S is the binomial ideal $I_{\mathcal{L}_\mathcal{B}} := \langle \mathbf{x}^{u^+} - \mathbf{x}^{u^-} \mid u \in \mathcal{L}_\mathcal{B} \rangle$. The matrix B is called a defining matrix of $I_{\mathcal{L}_\mathcal{B}}$. Such a matrix is of course not unique, but one can see easily that it is unique up to action of $SL_d(\mathrm{Z})$ (that is, if B' is a second integer $n \times d$-matrix of rank d, then $I_{\mathcal{L}_\mathcal{B}} = I_{\mathcal{L}_{\mathcal{B}'}}$ if and only if for a unimodular matrix $T \in SL_d(\mathrm{Z})$, we have $B' = BT$).

The relationships between the matrix B and the lattice ideal $I_{\mathcal{L}_\mathcal{B}}$ have been studied by many authors ([FMS1, FS, HS, PS1] and [PS2]). It is well-known that some numerical invariants and some algebraic properties of the lattice ideal $I_{\mathcal{L}_\mathcal{B}}$ can be read off directly from the matrix B. For example, the codimension of $I_{\mathcal{L}_\mathcal{B}}$ is equal to $\operatorname{rank}(B)$ and $I_{\mathcal{L}_\mathcal{B}}$ is a prime ideal if and only if B has content (the gcd of all $d \times d$ minors of B) equal to 1 [FS]. In general we can say that the matrix B has property $\wp$ (such as primeness, radicalness and so on) over a field k, if the lattice ideal $I_{\mathcal{L}_\mathcal{B}} \subset k[\mathbf{x}]$ has this property. Note that the statement "over a field k" in this definition is crucial, since a property $\wp$ may or may not depend upon k. For example, primeness is independent of k, but radicalness depends upon k. The major motivation of the study of matrices defining lattice ideals is that, not only a lot of information of the lattice ideals is encoded in these matrices, but also they may have interesting properties by themselves even when they do not involve integrality [FMS2].

In [FS] the authors show that if the matrix B is mixed, then it is complete intersection if and only if there exists a unimodular $d \times d$-matrix T such that the transposed matrix of $B' = BT$ is dominating. We recall from [FS] that an $r \times s$-matrix M is called mixed if every row of M has both a positive and a negative entry. The matrix M is called dominating if it does not contain a square mixed submatrix. Also in [FMS1] it has been shown that a mixed dominating matrix of an arbitrary size decomposes and will have a special format.

In this thesis we will study the class of Gorenstein matrices which is more general than the class of complete intersection matrices.

In Chapter 3, first we show that the Cohen-Macaulay type of an integer $(n+1) \times n$-matrix B is equal to the number of maximal lattice point free polytopes of fibers. Then as a consequence we will give a combinatorial characterization of Gorenstein matrices of size $(n+1) \times n$ in terms of maximal lattice point free polytopes. The geometric significance of this class of matrices is due to affine monomial curves. This is because these matrices define monomial curves when they are toric (prime) matrices [V, Chapter 10]. As an application of mentioned results, we solve some special cases of the Frobenius Problem, and give two different proofs for the fact that the Frobenius number of a numerical semigroup is the a-invariant of the semigroup algebra associated to it. These give rise to two different algorithms for computing the Frobenius number.

In Chapter 4, we will study Gorensteinness of a generic matrix which is defined in integer programming theory [BS]. We will show that Gorenstein generic matrices are precisely column matrices with some special properties.

References

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