Homological dimensions and Cohen-Macaulay rings

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Abstract

Almost 50 years ago, Auslander, Buchsbaum and Serre used homological methods to characterize regular local rings as noetherian local rings with finite global dimension. This means that every module over such a ring has a finite projective dimension. Their characterization opened an active area of research in homological algebra. The main theme of the present thesis is in this line of thought.

This thesis is divided into two parts. In the first part we mainly study the homological dimension of a module which is not necessarily finitely generated. We prove a dual result of the well-known Auslander-Bridger formula. In his thesis [88], T. Sharif defined and studied the notion of complete intersection flat dimension. Here we continue Sharif's investigation and introduce the notion of Cohen-Macaulay flat dimension with the same approach. We derive some relations between these dimensions and other existing homological dimensions.

The subject of the second part is the semistar multiplicative ideal theory. More precisely, we deal with the going-down property of extensions of integral domains. To achieve this, we develop the theory of $\star$-GD domains and derive their fundamental properties. Then we give new characterizations of P$\star$MDs in terms of $\star$-GD domains.

Contents

Citations to Previously Published Work

Chapter 2 have appeared in the following paper:

> 1- P. Sahandi and T. Sharif, "Dual of the Auslander-Bridger formula and GF-perfectness", Math. Scand. **101**, (2007), 5-18.

Some part of Chapter 3 have appeared in the following paper:

> 2- P. Sahandi, T. Sharif, and S. Yassemi "Complete intersection flat dimension and intersection theorem", Submitted.

Chapter 4 have appeared in the following paper:

> 3- P. Sahandi, and S. Yassemi "Filter rings under flat base change", Algebra Colloq. **15** no. 3, (2008) 463–470.

Large portions of Chapter 5 have appeared in the following paper:

> 4- D. E. Dobbs, and P. Sahandi, "Going-down and semistar operations", to appear in J. Algebra Appl.

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Dedicated to my parents,

Chapter 1

Introduction

This thesis deals with two topics, which are relatively independent:

- Homological dimensions (Chapters 2, 3 and 4), and
- Multiplicative ideal theory (Chapter 5).

The first part forms the main body of the thesis, and is devoted to the study of homological dimensions of modules, which are not necessarily *finitely generated*, over commutative noetherian local rings with identity. The second part deals with multiplicative ideal theory, i.e., semistar operations.

The intertwining of these two subject areas had turned out to be one of the most beautiful surprises in this area of research.

The major motivation for studying homological dimensions goes back to 1956 when Auslander, Buchsbaum and Serre proved the following theorem: A commutative noetherian local ring R is regular if the residue field k has finite projective dimension and only if every R-module has finite projective dimension. This introduced the theme that finiteness of a homological dimension for modules singles out rings with special properties. Auslander and Bridger [4], introduced a homological dimension for finitely generated modules, designed to distinguish modules with properties similar to those of modules over Gorenstein rings. They called it G-dimension which is a refinement of the projective dimension. They showed that a local noetherian ring $(R, \mathfrak{m}, k)$ is Gorenstein if the residue field k has finite G-dimension and only if all finitely generated R-modules have finite G-dimension. More recently, other homological dimensions have been introduced to characterize complete intersection and Cohen-Macaulay rings (see [13] and [58]). Around twenty years after definition of G-dimension by Auslander and Bridger for finite modules, Enochs, Jenda and Torrecillas established a nice theory which included non-finite modules. They introduced new homological dimensions which generalized essential properties of Gorenstein dimension over special rings, namely, Gorenstein projective, injective and flat dimensions (see [39], [40] and [38]). We denote these dimensions by Gpd, Gid and Gfd, respectively.

In the first section of Chapter 2, we focus on a dual result of the well-known Aslander-Bridger formula. Recall that if a finite R-module M has finite Gorenstein dimension, then $\operatorname{G-dim}_R M + \operatorname{depth}_R M = \operatorname{depth} R$. We showed that:

Theorem. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring and let M be an Ext-finite R-module of infinite injective dimension. If $\operatorname{Gid}_R M < \infty$ and M has an Ext-finite GI-syzygy module, then

$$\operatorname{Gid}_R M + \operatorname{width}_R M = \operatorname{depth} R.$$

The notion of perfect modules is important in Commutative Algebra and Homological Algebra. In Section 2.2 we offer a notion of perfect modules for modules which are not necessarily finitely generated.

For any R-module M the flat dimension of M is denoted by $\operatorname{fd}_R M$. There is always an inequality $\operatorname{fd}_R M \leqslant \operatorname{pd}_R M$, and equality holds if M is finitely generated, where $\operatorname{pd}_R M$ denotes the projective dimension of M. A deep result, due to Gruson-Raynaud [81] and Jensen [67], says that every flat R-module has finite projective dimension. Hence the flat dimension and the projective dimension of a module are finite simultaneously. Therefore it seems that, the flat dimension is a suitable extension of the projective dimension for non-finite modules.

In [24] Christensen, Foxby, and Frankild introduced the *large restricted flat dimension* which is denoted by Rfd and is defined by the formula

$$\operatorname{Rfd}_R M = \sup\{i \mid \operatorname{Tor}_i^R(L, M) \neq 0 \text{ for some } R\text{-module } L \text{ with } \operatorname{fd}_R L < \infty\}.$$

They showed that for every R-module M, there is an inequality

$$\operatorname{Rfd}_R M \leq \operatorname{fd}_R M$$

with equality if $\operatorname{fd}_R M < \infty$.

In [40] and [39] Enochs and Jenda have introduced the Gorenstein flat dimension $\operatorname{Gfd}_R M$ for any R-module M. Holm studied this concept further in [61] and proved that $\operatorname{Gfd}_R M$ is a refinement of $\operatorname{fd}_R M$ and that $\operatorname{Rfd}_R M$ is a refinement of $\operatorname{Gfd}_R M$. In other words, for any R-module M there is a chain of inequalities

$$\operatorname{Rfd}_R M \leq \operatorname{Gfd}_R M \leq \operatorname{fd}_R M,$$

and if one of these quantities is finite then there is equality everywhere to its left.

The main goal of Chapter 3 is to introduce and study notions of complete intersection flat dimension (CIfd) and Cohen-Macaulay flat dimension (CMfd) as refinements of flat dimensions for every module M over a noetherian ring R. The main result of this chapter is the comparison of the Gorenstein flat and the complete intersection flat dimensions:

Theorem. Let M be an R-module. Then there is the following sequence of inequalities:

$$\mathrm{Rfd}_R\, M \leq \mathrm{CMfd}_R\, M \leq \mathrm{Gfd}_R\, M \leq \mathrm{CIfd}_R\, M \leq \mathrm{fd}_R\, M.$$

If one of these dimensions is finite, then it is equal to those on its left.

We also introduce and study a variety of refinements of flat dimension, namely, upper Cohen-Macaulay flat dimension ($\mathrm{CM}^*\,\mathrm{fd}$) and upper Gorentein flat dimension ($\mathrm{G}^*\,\mathrm{fd}$) for every module M over a noetherian ring R (see Sections 3 for definitions). These dimensions fit into the following scheme of inequalities:

$$\mathrm{Rfd}_R\, M \leq \mathrm{CM}^*\,\mathrm{fd}_R\, M \leq \mathrm{G}^*\,\mathrm{fd}_R\, M \leq \mathrm{CI}\,\mathrm{fd}_R\, M \leq \mathrm{fd}_R\, M,$$

with equality to the left of any finite number.

The new homological flat dimensions are in many aspects, similar to the classical ones. As a second example of what can be gained from our homological flat dimensions, we have the following result which maybe called Intersection Theorem for homological flat dimensions:

Theorem. Let M be an R-module with $\mathrm{H}\,\mathrm{fd}_R\, M < \infty$ and of finite depth. Suppose that R is an equicharacteristic zero ring, then:

$$\dim R \leq \dim_R M + \mathrm{H}\,\mathrm{fd}_R\, M,$$

for $\mathrm{H} = \mathrm{CI}$, G^*, and CM^*.

In [27] Cuong, Schenzel, and Trung introduced the notion of filter regular sequence as an extension of the well-known concept of regular sequence. Using this notion they defined the filter modules. An R-module is called *filter module* if every system of parameters is a filter regular sequence. Consider the following standard fact, cf. [17, Theorem 2.1.7]: Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings. Then S is a Cohen-Macaulay ring if and only if R and $S/\mathfrak{m}S$ are Cohen–Macaulay. It seems natural to ask: what parts of this result holds when we replace "a Cohen-Macaulay ring" with "a filter ring". In Chapter 4 we give a complete answer to this question.

The subject of the second part of this thesis is the semistar multiplicative ideal theory. For several decades, star operations, as described in [58, Section 32], have proven to be an essential tool in multiplicative ideal theory, allowing one to study various classes of domains. Nearly 15 years ago, Okabe and Matsuda [77] introduced the concept of a semistar operation to extend the notion of a star operation. Since then, semistar operations have been much studied and, thanks to its greater flexibility compared to star operations, semistar operations have permitted a finer study and classification of integral domains. For instance, semistar-theoretic analogues of the classical notions of Noetherian and Prüfer domains have been introduced: see [37] and [36] for the basics on $\star$-Noetherian domains and $P\star MD$s, respectively. Among semistar-theoretic analogues of classical properties, we note that Chang and Fontana [21, Lemma 2.15] have recently found an instance where $\widetilde{\star}$-INC, $\widetilde{\star}$-LO and $\widetilde{\star}$-GU arise naturally. Given that INC, LO and GU are usually introduced together with the classical going-down property GD (cf. [69, page 28]), it seems reasonable to develop a semistar-theoretic analogue of GD and then, using it, to introduce a semistar-theoretic analogue of the going-down domains that was introduced in [29], [35]. Both of these goals are accomplished in this thesis.

Indeed, if $D \subseteq T$ is an extension of domains and $\star$ (resp., $\star'$) is a semistar operation on D (resp., T), we define in Section 5.1 what it means for $D \subseteq T$ to satisfy the $(\star, \star')$-GD property. Sufficient conditions are given for $(\star, \star')$-GD, generalizing classical sufficient conditions for GD such as flatness, openness of the contraction map of spectra and (to some extent) the hypotheses of the classical going-down theorem. Moreover, if $\star$ is a semistar operation on a domain D, we define in Section 5.2 what it means for D to be a $\star$-GD domain. Let $\widetilde{\star}$ be the canonical spectral semistar operation *of finite type* associated to $\star$ (whose definition is recalled below). In determining whether a domain D is a $\widetilde{\star}$-GD domain, the domain extensions T of D for which suitable $(\widetilde{\star}, \star')$-GD is tested can be the $\widetilde{\star}$-valuation overrings of D, the simple overrings of D, or all T (see Theorem 5.2.14), thus generalizing [35, Theorem 1]. In Theorem 5.2.11, $P\star MD$s are characterized as the $\widetilde{\star}$-treed (resp., $\widetilde{\star}$-GD) domains D which are $\widetilde{\star}$-finite conductor domains such that $D^{\widetilde{\star}}$ is integrally closed, thus generalizing [71, Theorem 1]. The final two results of this section give several characterizations of the $\widetilde{\star}$-Noetherian domains D of $\widetilde{\star}$-dimension 1 in terms of the behavior of the $(\star, \star')$-linked overrings of D and the $\star$-Nagata rings $Na(D, \star)$. In fact, Corollary 5.2.20 can be viewed as a semistar-theoretic generalization of the Krull-Akizuki Theorem, while Theorem 5.2.21 generalizes the fact [29, Corollary 2.3] that Noetherian going-down domains must have (Krull) dimension at most 1. In the last section we characterize going-down (treed) $\star$-Nagata domains.

1.1 Definitions and Notations

In this section we recall various definitions of homological dimensions.

Definition 1.1.1. *A finite R-module M has G-dimension 0 if the following conditions are satisfied:*

(i) $M \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$*,*

(ii) $\mathrm{Ext}^i_R(M,R) = 0$ *for all* $i > 0$*, and*

(iii) $\mathrm{Ext}^i_R(\mathrm{Hom}_R(M,R),R) = 0$ *for all* $i > 0$*.*

The Gorenstein dimension of M which is defined by Auslander and Bridger [4] and is denoted by $\text{G-dim}_R\, M$, is the least number n for which there exists an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0,$$

where G_i has G-dimension 0 for $i = 0, \cdots, n$.

A finite R-module M is called *perfect* (resp. *G-perfect*) if $\mathrm{pd}_R\, M = \mathrm{grade}_R\, M$ (resp. $\text{G-dim}_R\, M = \mathrm{grade}_R\, M$). Let Q be a local ring and J an ideal of Q. By abuse of language we say that J is *perfect* (resp. *G-perfect*) if the Q-module Q/J has the corresponding property.

The ideal J is called Gorenstein if it is perfect and $\beta^Q_g(Q/J) = 1$ for $g = \mathrm{grade}_Q\, J$, where $\beta^Q_g(Q/J)$ stands for the g-th Betti number of Q/J. It is called complete intersection ideal, if J is generated by Q-regular elements.

We say that R has a CI-deformation (resp. G*-deformation, CM-deformation) if there exists a local ring Q and a complete intersection (resp. Gorenstein, G-perfect) ideal J in Q such that $R = Q/J$. A CI-quasi-deformation (resp. G*-quasi-deformation, CM-quasi-deformation) of R is a diagram of local homomorphisms $R \to R' \leftarrow Q$, with $R \to R'$ a flat extension and $R' \leftarrow Q$ a CI-deformation (resp. G*-deformation, CM-deformation). We set $M' = M \otimes_R R'$.

The *complete intersection dimension* of M as defined by Avramov, Gasharov, and Peeva [13] and denoted by $\text{CI-dim}_R M$ is

$$\text{CI-dim}_R M := \inf\{\text{pd}_Q M' - \text{pd}_Q R' \mid R \to R' \leftarrow Q \text{ is a CI-quasi-deformation}\}.$$

The *upper Gorenstein dimension* of M as defined by Veliche [94] and denoted by $\text{G}^*\text{- dim}_R M$ is

$$\text{G}^*\text{- dim}_R M := \inf\{\text{pd}_Q M' - \text{pd}_Q R' \mid R \to R' \leftarrow Q \text{ is a G}^*\text{-quasi-deformation}\}.$$

The *Cohen-Macaulay dimension* of M, as defined by Gerko [57] and denoted by $\text{CM-dim}_R M$ is

$$\text{CM-dim}_R M := \inf\{\text{G-dim}_Q M' - \text{G-dim}_Q R' \mid R \to R' \leftarrow Q \text{ is a CM-quasi-deformation}\}.$$

There is the following sequence of inequalities:

$$\text{CM-dim}_R M \le \text{G-dim}_R M \le \text{G}^*\text{- dim}_R M \le \text{CI- dim}_R M \le \text{pd}_R M,$$

with equality to the left of any finite number.

In [38] Enochs and Jenda introduced the Gorenstein injective dimension $\text{Gid}_R M$, and in [39] Enochs, Jenda and Torrecillas introduced the Gorenstein flat dimension $\text{Gfd}_R M$ of any R-module M as follows:

Definition 1.1.2. *An R-module M is said to be Gorenstein injective if there is an exact sequence*

$$\cdots \to E^{-1} \to E^0 \to E^1 \to \cdots$$

of injective R-modules such that $M = \ker(E^0 \to E^1)$ and for any injective R-module E, $\operatorname{Hom}_R(E, -)$ preserves exactness of the above complex. The Gorenstein injective dimension, denoted by $\operatorname{Gid}_R(-)$, is defined by using Gorenstein injective modules in a fashion similar to that of injective dimension.

It is known that $\operatorname{Gid}_R M \leq \operatorname{id}_R M$ with equality if $\operatorname{id}_R M$ is finite.

Definition 1.1.3. *An R-module M is said to be Gorenstein flat if there is an exact sequence*

$$\cdots \to F^{-1} \to F^0 \to F^1 \to \cdots$$

of flat R-modules such that $M = \ker(F^0 \to F^1)$ and such that for any injective R-module I, $I \otimes_R -$ preserves the exactness of the above complex. The Gorenstein flat dimension, denoted by $\operatorname{Gfd}_R(-)$, is defined by using Gorenstein flat modules in a fashion similar to that of flat dimension. Recall that for a finite R-module M we have $\operatorname{Gfd}_R M = \text{G-dim}_R M$; see [39].

For a notherian ring the following categories were introduced by Avramov and Foxby [8]:

Definition 1.1.4. *Let R be a ring with a dualizing complex D. Let $\mathcal{D}_b(R)$ denote the full subcategory of $\mathcal{D}(R)$ (the derived category of R-complexes) consisting of complexes X with $\mathrm{H}_n(X) = 0$ for $|n| \gg 0$. The* Auslander class $\mathbf{A}(R)$ *is defined as the full* *subcategory of* $\mathcal{D}_b(R)$*, consisting of those complexes* X *for which* $D \otimes_R^{\mathbf{L}} X \in \mathcal{D}_b(R)$*, and the canonical morphism*

$$\gamma_X : X \to \mathbf{R}\operatorname{Hom}_R(D, D \otimes_R^{\mathbf{L}} X),$$

ia an isomorphism. The Bass class $\mathbf{B}(R)$ *is defined as the full subcategory of* $\mathcal{D}_b(R)$*, consisting of those complexes* X *for which* $\mathbf{R}\operatorname{Hom}_R(D, X) \in \mathcal{D}_b(R)$ *and the canonical morphism*

$$\iota_X : D \otimes_R^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(D, X) \to X,$$

is an isomorphism.

Remark 1.1.5. *It is proved in [25, (4.1) and (4.4)] that if* R *admits a dualizing complex then for an* R*-module* M *we have:*

(a) $M \in \mathbf{A}(R)$ if and only if $\operatorname{Gfd}_R M < \infty$, *and*

(b) $M \in \mathbf{B}(R)$ if and only if $\operatorname{Gid}_R M < \infty$.

See also [44] and [45] for an interesting extension of this result.

The *large restricted flat dimension*, introduced and studied by Christensen, Foxby and Frankild in [24]. It is defined by the formula

$$\operatorname{Rfd}_R M := \sup\{i | \operatorname{Tor}_i^R(L, M) \neq 0 \text{ for some } R\text{-module } L \text{ with } \operatorname{fd}_R L < \infty\}.$$

This number is finite, as long as M is nonzero and the Krull dimension of R is finite; see [24, (2.2)]. They proved that, see [24, (2.4)]

$$\operatorname{Rfd}_R M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Part I

Homological dimensions

Chapter 2

Dual of the Auslander-Bridger formula and GF-perfectness

2.1 Dual of the Auslander-Bridger formula

Throughout this chapter all rings are commutative and Noetherian with nonzero identity. In [38] and [41] Enochs and Jenda introduced and studied mock finite Gorenstein injective modules. As an extension they introduced and studied the Ext-finite modules of finite Gorenstein injective dimension in [42]. We recall that an R-module M is called Ext-finite if $\mathrm{Ext}_R^i(N, M)$ is finite for each finite R-module N and for $i \geq 1$. Therefore, every finite R-module M is Ext-finite and it is also easy to see that every cosyzygy of an Ext-finite module is also Ext-finite [42, (4.7)]. Following Enochs and Jenda in [42] we prove a dual result for the Auslander-Bridger formula [4] for Ext-finite modules of finite Gorenstein injective dimension. Our approach to obtain a dual result is fundamentally different from the method of Enochs and Jenda in [42]. For the rest of this section let:

$\mathcal{P}_0 = \{M | M$ is an R-module of finite projective dimension$\}$.

$\mathcal{I}_0 = \{M | M$ is an R-module of finite injective dimension$\}$.

Let $(R, \mathfrak{m}, k)$ be a local ring. Recall that the depth of an R-module M, denoted by $\operatorname{depth}_R M$ is defined as

$$\operatorname{depth}_R M = \inf\{i | \operatorname{Ext}^i_R(k, M) \neq 0\}.$$

Lemma 2.1.1. *Let $(R,\mathfrak{m},k)$ be a local ring and let N be an* Ext*-finite R-module of finite injective dimension and of finite* depth*. Then*

$$\operatorname{id}_R N = \sup\{i | \operatorname{Ext}^i_R(T, N) \neq 0 \text{ for some } T \in \mathcal{P}_0 \text{ with } \ell_R(T) < \infty\},$$

if and only if R is a Cohen-Macaulay ring.

Proof. First of all suppose that the equality holds. Then there is an R-module T of finite length and of finite projective dimension. Hence R is Cohen-Macaulay by the Intersection Theorem cf. [85]. Conversely, suppose that R is a Cohen-Macaulay ring. By [89, (1.4)], $n = \operatorname{id}_R N = \sup\{i | \operatorname{Ext}^i_R(k, N) \neq 0\}$. Then $\operatorname{Ext}^n_R(k, N) \neq 0$. Let $x_1, \cdots, x_t$ be a maximal R-sequence in $\mathfrak{m}$. Since R is Cohen-Macaulay $\mathfrak{m} \in \operatorname{Ass}(R/(x_1, \cdots, x_t))$. Let $T = R/(x_1, \cdots, x_t)$. So we have the exact sequence $0 \to k \to T \to L \to 0$, which induces the exact sequence

$$\operatorname{Ext}^n_R(T, N) \to \operatorname{Ext}^n_R(k, N) \to 0.$$

So that $\operatorname{Ext}^n_R(T, N) \neq 0$, and this completes the proof. $\square$

Let $\mathcal{X}$ be a class of R-modules for some ring R. If $\phi : X \to M$ is linear where $X \in \mathcal{X}$ and M is an R-module, then $\phi : X \to M$ is called an $\mathcal{X}-$precover of M if

$$\operatorname{Hom}_R(Y, X) \to \operatorname{Hom}_R(Y, M) \to 0$$

is exact for all $Y \in \mathcal{X}$.

Lemma 2.1.2. *Let R be a ring and let M be an R-module with $\operatorname{Gid}_R M < \infty$. Then there is a surjective $\mathcal{I}_0$-precover $\varphi : N \to M$ such that $\operatorname{id}_R N = \operatorname{Gid}_R M$ and $\ker \varphi$ is a Gorenstein injective R-module.*

Proof. We use an induction argument on $g = \operatorname{Gid}_R M$. If $g = 0$, then M is Gorenstein injective. So by definition of the Gorenstein injective modules, there is the exact sequence

$$0 \to H \to I \to M \to 0,$$

in which I is injective and H is Gorenstein injective. By [38, Proposition (2.4)] it is clear that I is an $\mathcal{I}_0$-precover of M. Now let $g \geqslant 1$. From [61, (2.15)] there is a Gorenstein injective module G and an R-module L with $\operatorname{id}_R L = g - 1$ such that the following sequence is exact

$$0 \to M \to G \to L \to 0.$$

Since G is Gorenstein injective we have the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & N & \rightarrow & E & \rightarrow & L & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & M & \rightarrow & G & \rightarrow & L & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where H is Gorenstein injective module and $\operatorname{id}_R N = g$. Now from [38, Proposition (2.4)] it follows that in the exact sequence

$$0 \rightarrow H \rightarrow N \rightarrow M \rightarrow 0,$$

N is an $\mathcal{I}_0$-precover of M such that $\operatorname{id}_R N = \operatorname{Gid}_R M$. $\square$

Let M be an R-module with $\operatorname{Gid}_R M < \infty$. From the above lemma we have the exact sequence

$$0 \rightarrow H_1 \rightarrow N \rightarrow M \rightarrow 0,$$

where H_1 is Gorenstein injective. From the definition of Gorenstein injective modules there is an injective R-module E_1 and a Gorenstein injective module H_2 such that the following sequence is exact.

$$0 \rightarrow H_2 \rightarrow E_1 \rightarrow H_1 \rightarrow 0.$$

By continuing in the same manner we can find Gorenstein injective modules H_i and injective modules E_i for $i \geqslant 1$ such that the sequence

$$\cdots \to E_2 \to E_1 \to N \to M \to 0$$

is exact. It is clear that $\mathrm{id}_R M < \infty$ if and only if the above sequence is finite.

Definition 2.1.3. *In the above construction we call the R-modules H_i for $i \geqslant 1$ the i-th GI-syzygy module of M.*

Now we are in the position of proving the first main result of this section. Recall that

$$\mathrm{width}_R M = \inf\{i | \operatorname{Tor}_i^R(k, M) \neq 0\}.$$

It is shown in [54, (14.17)] that $\mathrm{width}_R M$ is finite if and only if $\mathrm{depth}_R M$ is finite. Now we prove one of main results in this section.

Theorem 2.1.4. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring and let M be an* Ext*-finite R-module of infinite injective dimension. If $\mathrm{Gid}_R M < \infty$ and M has an* Ext*-finite GI-syzygy module, then*

$$\mathrm{Gid}_R M + \mathrm{width}_R M = \operatorname{depth} R.$$

Proof. Let $g = \mathrm{Gid}_R M$. If $g = 0$, then [42, (4.1)] gives the result. Now suppose that $g \geqslant 1$, so M has an $\mathcal{I}_0$-precover N, with $\mathrm{id}_R N = g$, and such that in the exact sequence

$$(*) \qquad 0 \to H \to N \to M \to 0$$

H is a Gorenstein injective module with $\mathrm{id}_R H = \infty$. By definition of GI-syzygy modules of M, it is easy to see that H is an Ext-finite module. So by [42, (4.1)]

$\operatorname{width}_R H = \operatorname{depth} R = \dim R = d$. If $\operatorname{width}_R M = \infty$, from the long exact sequence of homologies we get that $\operatorname{width}_R N = \operatorname{width}_R H = d$. Since M and H are Ext-finite modules, then so is N. Now from [89, (1.6)] it follows that $\operatorname{id}_R N = 0$. This yields that $g = 0$, which is a contradiction, hence $\operatorname{width}_R M < \infty$. On the other hand, since $\operatorname{width}_R H = d$, we get $\operatorname{depth}_R H = 0$ by [54, (14.18)], and this yields that $\operatorname{depth}_R N = 0$. So we obtain that $\operatorname{width}_R N < \infty$ by [54, (14.18)] again. Since R is a Cohen-Macaulay ring, from Lemma 2.1.1 we get that $\operatorname{Ext}^g_R(T, N) \neq 0$ for some $T \in \mathcal{P}_0(R)$ of finite length. From the long exact sequence induced by $(*)$ and [38, Proposition (2.4)] we find that there is an exact sequence as follows

$$0 = \operatorname{Ext}^g_R(T, H) \to \operatorname{Ext}^g_R(T, N) \to \operatorname{Ext}^g_R(T, M) \to 0.$$

Therefore

$$\operatorname{Gid}_R M = \sup\{i \mid \operatorname{Ext}^i_R(T, M) \neq 0 \text{ for some } T \in \mathcal{P}_0 \text{ with } \ell_R(T) < \infty\},$$

and so we find that

$$\begin{aligned}\operatorname{Gid}_R M &= \sup\{i \mid \operatorname{Hom}_R(\operatorname{Ext}^i_R(T, M), E(k)) \neq 0 \text{ for some } T \in \mathcal{P}_0 \text{ with } \ell_R(T) < \infty\} \\ &= \sup\{i \mid \operatorname{Tor}^R_i(T, \operatorname{Hom}(M, E(k))) \neq 0 \text{ for some } T \in \mathcal{P}_0 \text{with } \ell_R(T) < \infty\}.\end{aligned}$$

On the other hand, since $\operatorname{depth}_R T = 0$ and $\operatorname{depth}_R \operatorname{Hom}_R(M, E(k)) = \operatorname{width}_R M$, from [90, (2.3)], it follows that the right side of the second equality is $\operatorname{depth} R - \operatorname{width}_R M$ as desired. □

In the rest of this section we introduce a class of modules called *Grothendieck modules*. We find a dual result for the Auslander-Bridger formula for this kind of modules of maximal dimension.

Definition 2.1.5. *Let $(R, \mathfrak{m})$ be local ring. An R-module M of Krull dimension n, is said to be a* Grothendieck module *if* $\mathrm{H}^n_{\mathfrak{m}}(M) \neq 0$.

The following result is analogous to a classical result due to H. Bass [14]. In [91] Takahashi proved the following theorem for finite modules, under the additional assumption that the base ring admits a dualizing complex. In [98] Yassemi, proved Takahashi's result, without assuming that the ring admits a dualizing complex.

Theorem 2.1.6. *Let $(R, \mathfrak{m})$ be a local ring and let M be a Grothendieck module with $\mathrm{Gid}_R\, M < \infty$. If $\dim M = \dim R$, then R is a Cohen-Macaulay ring and $\mathrm{Gid}_R\, M = \operatorname{depth} R$.*

Proof. Let $g = \mathrm{Gid}_R\, M$ and let $n = \dim M$. By a similar argument to that of [87, (3.1)] it is easy to see that $\mathrm{H}^i_{\mathfrak{m}}(H) = 0$ when $\mathrm{Gid}_R\, H = 0$ and $i > 0$. Now from Lemma 2.1.1 it is clear that for $i > 0$, $\mathrm{H}^i_{\mathfrak{m}}(M) = \mathrm{H}^i_{\mathfrak{m}}(N)$ when N is an $\mathcal{I}_0$-precover of M such that $\mathrm{id}_R\, N = g$. It is easy to see that in this case $\mathrm{H}^i_{\mathfrak{m}}(N) = 0$ for $i > g$. On the other hand, since $\dim N = \dim M$, N is a Grothendieck module too. Therefore $n \leq \mathrm{id}_R\, N$. Now we have the following (in)equalities:

$$n = \dim R \leq \mathrm{id}_R\, N = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \leq \operatorname{ht} \mathfrak{p}$$

in which the second equality holds by [23, (3.1)], so we obtain $\mathfrak{p} = \mathfrak{m}$. This ends our proof. □

Recall from [17] that an R-module M is said to be have rank r, if $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of rank r, for all prime ideals $\mathfrak{p} \in \mathrm{Ass}(R)$. It is clear that, finite modules with positive rank are of maximal Krull dimension.

Corollary 2.1.7. *If M is a finite module of positive rank with finite Gorenstein injective dimension, then R is a Cohen-Macaulay ring.*

It is interesting to know that there is a large class of non-finite modules satisfying both conditions of Theorems 2.1.4 and 2.1.6.

Example 2.1.8. *Let $(R, \mathfrak{m}, k)$ be an n-Gorenstein local ring which is not regular, and let L be an R-module with $\ell_R(L) < \infty$ and $\mathrm{id}_R L = \infty$. Let $E^{\cdot}$ be the minimal injective resolution of L. Therefore, each term of $E^{\cdot}$, is direct sum of finitely many copies of $E(k)$, the injective envelope of k. So all terms of $E^{\cdot}$ are $\mathfrak{m}$-torsion, in the sense of [16]. Let H be the r-th cosyzygy of this resolution for $r \geqslant n$. By [42, (4.2) and (4.7)] H is Gorenstein injective and* Ext*-finite. So, by [38, (6.5)] H is a mock finite module. Viewing [38, (6.6)] we get that, the first GI-syzygy of H is mock finite too. Set $M = H \oplus R$, now it is clear that M is an* Ext*-finite, Grothendieck R-module with* $\dim M = \dim R$.

2.2 The GF-Perfect modules

Let M be a finite R-module, the notion $\mathrm{grade}_R M$ was defined by Rees as the least integer $\ell \geq 0$ such that $\mathrm{Ext}^{\ell}_R(M, R) \neq 0$. In [83] Rees proved that the $\mathrm{grade}_R M$ is the maximum lengths of R-regular elements in $\mathrm{Ann}_R(M)$. It is easy to see that $\mathrm{grade}_R M$ is the least integer $\ell \geqslant 0$ such that $\mathrm{Ext}^{\ell}_R(M, P) \neq 0$ for some projective R-module P. When M is a non-finite R-module there is not any extension of this important invariant, however in any extension of grade, a homological view is useful. In this section for an arbitrary R-module M we introduce a new invariant denoted by $\text{F-grade}_R M$, such that when M is finite then $\text{F-grade}_R M = \text{grade}_R M$. One important concept closely related to the grade of modules is *perfectness*. A finite R-module M with $\text{pd}_R M < \infty$ is said to be *perfect* if $\text{grade}_R M = \text{pd}_R M$. This concept was generalized by Foxby in [52] where he defined *quasi-perfect* modules. A finite R-module M with $\text{G-dim}_R M < \infty$ is said to be quasi-perfect if $\text{grade}_R M = \text{G-dim}_R M$ The perfect and quasi-perfect modules over Cohen-Macaulay local rings are Cohen-Macaulay modules, see [17] and [52], respectively.

Definition 2.2.1. *Let M be an R-module. The* Flat grade *of M is denoted by* $\text{F-grade}_R M$*, and it is defined by the following formula*

$$\text{F-grade}_R M = \inf\{i | \operatorname{Ext}^i_R(M, F) \neq 0 \text{ for some flat } R\text{-module } F\}.$$

By definition it is clear that $\text{F-grade}_R M \leq \text{grade}_R M$.

Remark 2.2.2. *Let M be a finite R-module and suppose that* $\text{F-grade}_R M = \ell$. *Then there is a flat R-module F such that* $\operatorname{Ext}^\ell_R(M, F) \neq 0$. *Since M is finite,* $\operatorname{Ext}^\ell_R(M, F) \simeq \operatorname{Ext}^\ell_R(M, R) \otimes_R F$ *and so* $\operatorname{Ext}^\ell_R(M, R) \neq 0$. *Therefore,* $\text{grade}_R M \leqslant \ell$. *Now we have* $\text{F-grade}_R M = \text{grade}_R M$.

It is not difficult to see that $\text{F-grade}_R M \leq \text{Gfd}_R M$, it is also a trivial consequence of the following proposition. Recall from [95, Definition (3.1.1)] that an R-module C is called *cotorsion*, if for all flat R-modules F, $\operatorname{Ext}^1_R(F, C) = 0$. The following proposition shows that F-grade and Gfd can be computed via cotorsion flat modules.

Proposition 2.2.3. *Let R be a ring and let M be an R-module. Then*

$$\text{F-grade}_R M = \inf\{i | \operatorname{Ext}^i_R(M, F) \neq 0 \text{ for some cotorsion flat } R\text{-module } F\},$$

and if $\operatorname{Gfd}_R M < \infty$*, then*

$$\operatorname{Gfd}_R M = \sup\{i | \operatorname{Ext}_R^i(M, F) \neq 0 \text{ for some cotorsion flat } R\text{-module } F\}.$$

Proof. If $\text{F-grade}_R M = \infty$, then $\operatorname{Ext}_R^i(M, F) = 0$ for all flat R-modules and for all $i \geq 0$, thus the right side is infinity too. Let $\text{F-grade}_R M = n$, therefore $\operatorname{Ext}_R^n(M, F) \neq 0$ for some flat R-module F. Now let Q be the pure injective envelope of F cf. [95]. By [95, (3.1.6)] $Q/F = H$ is flat, thus $\operatorname{Ext}_R^i(M, H) = 0$ for $i < n$. On the other hand the exact sequence $0 \longrightarrow F \longrightarrow Q \longrightarrow H \longrightarrow 0$ gives rise to an injection

$$0 \longrightarrow \operatorname{Ext}_R^n(M, F) \longrightarrow \operatorname{Ext}_R^n(M, Q) \longrightarrow \cdots .$$

Therefore $\operatorname{Ext}_R^n(M, Q) \neq 0$. Keep in mind that pure injective modules are cotorsion. Now let $g = \operatorname{Gfd}_R M < \infty$. From [61] we can find an injective module J such that $\operatorname{Tor}_g^R(M, J) \neq 0$. Therefore $\operatorname{Hom}_R(\operatorname{Tor}_g^R(M, J), Q) \neq 0$ for a faithfully injective R-module Q. This yields that $\operatorname{Ext}_R^g(M, \operatorname{Hom}_R(J, Q)) \neq 0$. By setting $F = \operatorname{Hom}_R(J, Q)$ and considering the simple fact that F is a flat cotorsion R-module, we see that the right side is greater than or equal to g. On the other hand, let F be a cotorsion flat R-module. Therefore, from [43, (2.3)] it follows that there is a flat R-module H, and injective R-modules J_1 and J_2 such that $F \oplus H = \operatorname{Hom}_R(J_1, J_2)$. Let for some $i > g$, $\operatorname{Ext}_R^i(M, F) \neq 0$. Hence $\operatorname{Ext}_R^i(M, \operatorname{Hom}_R(J_1, J_2)) = \operatorname{Hom}_R(\operatorname{Tor}_i^R(M, J_1), J_2) \neq 0$ and so $\operatorname{Tor}_i^R(M, J_1) \neq 0$. From this and [61] we have $\operatorname{Gfd}_R M > g$, which is a contradiction. □

Definition 2.2.4. *Let M be an R-module with* $\operatorname{Gfd}_R M < \infty$*. We call M Gorenstein flat perfect (GF-perfect for short) if* $\text{F-grade}_R M = \operatorname{Gfd}_R M$.

Note that a finite R-module M is quasi-perfect if and only if it is GF-perfect, because $\mathrm{G}\text{-}\dim_R M = \mathrm{Gfd}_R M$ by [39] and $\mathrm{F}\text{-}\mathrm{grade}_R M = \mathrm{grade}_R M$ by Remark 2.2.2.

Lemma 2.2.5. *Let R be a ring and let M be a GF-perfect R-module, then for $\mathfrak{p} \in \mathrm{Supp}_R M$, $M_\mathfrak{p}$ is a GF-perfect $R_\mathfrak{p}$-module.*

Proof. Since $\mathrm{Gfd}_R M < \infty$ it is easy to see that $\mathrm{Gfd}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$. Thus we have $\mathrm{F}\text{-}\mathrm{grade}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \mathrm{Gfd}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$. Set $\mathrm{F}\text{-}\mathrm{grade}_{R_\mathfrak{p}} M_\mathfrak{p} = n$. So there is a flat $R_\mathfrak{p}$-module Q such that $\mathrm{Ext}^n_{R_\mathfrak{p}}(M_\mathfrak{p}, Q) \neq 0$. Consider an R-projective resolution $P_. \longrightarrow M$, hence $P_{.\mathfrak{p}}$ is a projective resolution for $M_\mathfrak{p}$. Now we have the following equalities

$$\begin{aligned} 0 \neq \mathrm{Ext}^n_{R_\mathfrak{p}}(M_\mathfrak{p}, Q) &= \mathrm{H}^n(\mathrm{Hom}_{R_\mathfrak{p}}(P_{.\mathfrak{p}}, Q)) \\ &= \mathrm{H}^n(\mathrm{Hom}_R(P_., Q)) = \mathrm{Ext}^n_R(M, Q) \end{aligned}$$

Since Q is also flat as an R-module, then $\mathrm{F}\text{-}\mathrm{grade}_R M \leqslant n$. Hence we have the following chain of inequalities

$$\mathrm{F}\text{-}\mathrm{grade}_R M \leq \mathrm{F}\text{-}\mathrm{grade}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \mathrm{Gfd}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \mathrm{Gfd}_R M.$$

Since M is GF-perfect so $\mathrm{Gfd}_R M = \mathrm{F}\text{-}\mathrm{grade}_R M$, therefore $M_\mathfrak{p}$ is also GF-perfect $R_\mathfrak{p}$-module. □

The following lemma is well known [53]. We include a proof here for completeness.

Lemma 2.2.6. *Let $(R, \mathfrak{m})$ be a local ring with* $\mathrm{cmd}\, R = \dim R - \mathrm{depth}\, R \leq 1$. *If $\mathfrak{p}$, $\mathfrak{q} \in \mathrm{Spec}(R)$ and $\mathfrak{p} \subseteq \mathfrak{q}$, then* $\mathrm{depth}\, R_\mathfrak{p} \leq \mathrm{depth}\, R_\mathfrak{q}$.

Proof. If $\mathrm{cmd}\, R = 0$ there is nothing to prove. Now we can assume $\mathrm{cmd}\, R = 1$. We will induct on $d = \dim R$. If $d = 0$ it is trivial. Assume $d > 0$, and let $\mathfrak{p} \subseteq \mathfrak{q}$. If

$\mathfrak{q} = \mathfrak{m}$, viewing [6], we get that $\operatorname{depth} R_{\mathfrak{p}} \leqslant \dim R - 1 = \operatorname{depth} R_{\mathfrak{q}}$. Now let $\mathfrak{q} \neq \mathfrak{m}$, since $\dim R_{\mathfrak{q}} \lneqq d$ and $\operatorname{cmd} R_{\mathfrak{q}} \leqslant \operatorname{cmd} R \leqslant 1$, by the induction hypothesis we have $\operatorname{depth} R_{\mathfrak{p}} = \operatorname{depth}(R_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}} \leqslant \operatorname{depth} R_{\mathfrak{q}}$. □

In the following theorem we generalized, results of Rees and Foxby in [83] and [52].

Theorem 2.2.7. *Let* $(R, \mathfrak{m})$ *be a local ring such that* $\operatorname{cmd} R \leqslant 1$. *Then for any* GF*-perfect* R*-module* M *we have:*

$$\operatorname{depth} R - \operatorname{depth}_R M \leq \text{F-grade}_R M \leq \dim R - \dim M.$$

In particular, if R *is a Cohen-Macaulay ring and* M *is of finite* depth *then* $\operatorname{depth}_R M = \dim M$.

Proof. First of all we show that for each $\mathfrak{p} \in \operatorname{Ass}_R(M)$, $\text{F-grade}_R M = \operatorname{depth} R_{\mathfrak{p}}$. Choose $\mathfrak{p} \in \operatorname{Ass}_R(M)$. Thus $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = 0$. Since $\operatorname{Gfd}_R M < \infty$ therefore by [24] and [61, (3.19)] we have

$$\operatorname{depth} R_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \operatorname{Gfd}_R M.$$

Since $\operatorname{Gfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty$, there is a prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$ such that

$$\operatorname{Gfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{q}}} M_{\mathfrak{q}}.$$

Hence by noting Lemma 2.2.6 we have:

$$\operatorname{Gfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} \leq \operatorname{Gfd}_R M - \operatorname{depth}_{R_{\mathfrak{q}}} M_{\mathfrak{q}}.$$

By Lemma 2.2.5 $M_{\mathfrak{p}}$ is GF-perfect as $R_{\mathfrak{p}}$-module and $\operatorname{Gfd}_R M = \operatorname{Gfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. Hence $\operatorname{depth}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} = 0$ and

$$\operatorname{depth} R_{\mathfrak{p}} \leq \operatorname{Gfd}_R M = \operatorname{Gfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \operatorname{depth} R_{\mathfrak{p}}.$$

Now our first claim is proved.

Choose $\mathfrak{p} \in \operatorname{Ass}_R(M)$ such that $\dim_R M = \dim R/\mathfrak{p}$. The following inequalities are clear

$$(*) \quad \dim_R M + \operatorname{grade}_R(\mathfrak{p}) \leq \dim R/\mathfrak{p} + \operatorname{ht}\mathfrak{p} \leq \dim R.$$

By [17, (1.2.10)] and the hypothesis that $\operatorname{cmd} R \leq 1$, we have:

$$\operatorname{grade}_R(\mathfrak{p}) = \inf\{\operatorname{depth} R_{\mathfrak{q}} | \mathfrak{p} \subseteq \mathfrak{q}\} = \operatorname{depth} R_{\mathfrak{p}}.$$

By our first claim since $\mathfrak{p} \in \operatorname{Ass}_R(M)$, $\text{F-grade}_R M = \operatorname{depth} R_{\mathfrak{p}} = \operatorname{grade}_R(\mathfrak{p})$. By $(*)$ we have

$$\dim_R M + \text{F-grade}_R M \leq \dim R.$$

Since $\text{F-grade}_R M = \operatorname{Gfd}_R M$ by [24] and [61, (3.19)] $\operatorname{depth} R - \operatorname{depth}_R M \leq \text{F-grade}_R M$.

□

The following Example shows that the hypothesis of finiteness of depth is necessary.

Example 2.2.8. *Let $(R, \mathfrak{m})$ be a local domain with $\dim R > 0$ and let K be its fraction field. It is clear that K is GF-perfect but $\operatorname{depth}_R K = \infty$ and $\dim_R K = \dim R$.*

The following result is also analogous to a classical result of Auslander and Bridger for the Gorenstein dimension [4]. We remark that, when R is a Cohen-Macaulay local ring and M is a finite R-module, it is clear that, $\operatorname{grade}_R M + \dim_R M = \dim R$. Viewing this, the following result is also an extension of this fact in the non-finite case.

Corollary 2.2.9. *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring and let M be a GF-perfect module of finite* depth*. Then*

$$\operatorname{Gfd}_R M + \operatorname{depth}_R M = \operatorname{depth} R.$$

Lemma 2.2.10. *Let R be a ring and let x be an R and M-regular element. Set $S = R/xR$. Then*

$$\operatorname{Rfd}_R M = \operatorname{Rfd}_S(M/xM).$$

Proof. Set $\bar{X} = X \otimes_R S$ for a module X. It is a simple computation that in the exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0,$$

when F is a flat R-module we have $\operatorname{Rfd}_R K = 0$, if $\operatorname{Rfd}_R M = 0$ and if $\operatorname{Rfd}_R M > 0$ then $\operatorname{Rfd}_R K = \operatorname{Rfd}_R M - 1$. We will induct on $n = \operatorname{Rfd}_R M$. Let L be a module such that $\operatorname{fd}_S L < \infty$, thus $\operatorname{fd}_R L < \infty$ by [10, Corollary 4.2 (b) (F)]. By [73, page 140], we have $\operatorname{Tor}_i^S(L, \overline{M}) = \operatorname{Tor}_i^R(L, M)$. So if $\operatorname{Rfd}_R M = 0$ thus $\operatorname{Rfd}_S \overline{M} = 0$. Now let $n > 0$. Consider the exact sequence $0 \to K \to F \to M \to 0$, where F is a free module. Since x is both F and M-regular, by [17, (1.1.5)] the following sequence is again exact

$$0 \longrightarrow \overline{K} \longrightarrow \overline{F} \longrightarrow \overline{M} \longrightarrow 0.$$

By the induction hypothesis $\operatorname{Rfd}_S \overline{K} = n - 1$ thus $\operatorname{Rfd}_S \overline{M} = n$. □

The following theorem is a generalization of a theorem due to Auslander and Bridger in [4] on the behavior of the Gorenstein dimension under base change.

Theorem 2.2.11. *Let R be a ring and let M be an R-module with $\operatorname{Gfd}_R M < \infty$. Let x be an R and M-regular element, and let $S = R/xR$. Then*

(1) $\operatorname{Gfd}_R M = \operatorname{Gfd}_S(M/xM)$.

(2) $\operatorname{Gfd}_R(M/xM) = \operatorname{Gfd}_R M + 1$.

Proof. Set $\bar{X} = X \otimes_R S$ for a module X. For part (1) we argue by induction on $g = \operatorname{Gfd}_R M$. Let $g = 0$, so M is a Gorenstein flat R-module. Consider a complete resolution of flat modules as the following

$$\cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} \xrightarrow{\partial_{-1}} F_{-2} \longrightarrow \cdots$$

such that $M \simeq \ker \partial_0$; and for all $i \in \mathrm{Z}$ each $\ker \partial_i = M_i$ is a Gorenstein flat R-module and has the flat resolution as

$$\cdots \longrightarrow F_{i+2} \longrightarrow F_{i+1} \longrightarrow M_i \longrightarrow 0.$$

Since x is M_i-regular and F_i-regular for all i, by [17, (1.1.5)] the following sequence is again exact

$$\cdots \longrightarrow \overline{F}_{i+2} \longrightarrow \overline{F}_{i+1} \longrightarrow \overline{M}_i \longrightarrow 0.$$

If we splice these sequences to each other we get a long exact sequence of flat S-modules

$$(*) \qquad \cdots \longrightarrow \overline{F}_1 \xrightarrow{\overline{\partial}_1} \overline{F}_0 \xrightarrow{\overline{\partial}_0} \overline{F}_{-1} \xrightarrow{\overline{\partial}_{-1}} \overline{F}_{-2} \longrightarrow \cdots .$$

Let J be an injective S-module, hence by [73, page 140] we have

$$\operatorname{Tor}_\ell^S(J, \overline{M}_i) = \operatorname{Tor}_\ell^R(J, M_i) \quad \text{for all } \ell \geqslant 0.$$

Since $\operatorname{id}_R J < \infty$ by [10, Corollary 4.2 (a) (F)], we have $\operatorname{Tor}_\ell^R(J, M_i) = 0$ for $\ell > 0$. So $(*)$ is a complete flat resolution of S-modules such that $\overline{M} = \ker \overline{\partial}_0$ and hence $\overline{M}$ is a Gorenstein flat S-module.

Now let $g > 0$, so there is a Gorenstein flat R-module G and an R-module L with $\operatorname{Gfd}_R L = g - 1$, such that the following sequence is exact

$$0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0.$$

Since x is G-regular, by [17, (1.1.5)] the following sequence of S-modules is exact

$$0 \longrightarrow \overline{L} \longrightarrow \overline{G} \longrightarrow \overline{M} \longrightarrow 0.$$

By the induction hypothesis, $\operatorname{Gfd}_S \overline{L} = g-1$ and $\operatorname{Gfd}_S \overline{G} = 0$, therefore $\operatorname{Gfd}_S \overline{M} < \infty$. Now [61, (3.19)] and Lemma 2.2.10 give the desired equality.

(2) Since x is M-regular we have

$$0 \longrightarrow M \xrightarrow{.x} M \longrightarrow \overline{M} \longrightarrow 0.$$

From [61] we have $\operatorname{Gfd}_R \overline{M} < \infty$, and by [61, (3.19)] and [90, (3.6)] we get $\operatorname{Gfd}_R \overline{M} = \operatorname{Rfd}_R \overline{M} = \operatorname{Rfd}_S \overline{M} + 1$. But from Lemma 2.2.10 we find that $\operatorname{Gfd}_S \overline{M} = \operatorname{Rfd}_S \overline{M} = \operatorname{Gfd}_R M$ and this completes the proof of part (2). $\square$

Corollary 2.2.12. *Let R be a ring and let M be a GF-perfect R-module. If x is an R and M-regular element and $S = R/xR$ and $\overline{M} = M/xM$, then $\overline{M}$ is a GF-perfect S-module.*

Proof. It follows from Theorem 2.2.11 that $\operatorname{Gfd}_S \overline{M} < \infty$, and so $\operatorname{F-grade}_S \overline{M} < \infty$. Let $n = \operatorname{Gfd}_R M = \operatorname{F-grade}_R M$ and $m = \operatorname{F-grade}_S \overline{M}$. Thus there is a flat S-module F, such that $\operatorname{Ext}_S^m(\overline{M}, F) \neq 0$. Since $\operatorname{fd}_R F < \infty$, using [73, Page 140] it is not difficult to see that $n \leq m$. On the other hand, by Theorem 2.2.11, $n = \operatorname{Gfd}_R M = \operatorname{Gfd}_S \overline{M} \geq \operatorname{F-grade}_S \overline{M} = m$, and so $n = m$. Now it follows that $\overline{M}$ is a GF-perfect S-module. $\square$

Chapter 3

Homological flat dimensions

3.1 Complete intersection flat dimension

The homological dimensions $\text{CI-dim}_R M$, $\text{G-dim}_R M$, and $\text{CM-dim}_R M$ were introduced by, respectively Avramov, Gasharov and Peeva in [13], by Auslander and Bridger [4], and by Gerko in [58], whenever M is finitely generated R-module. Furthermore, in this setup there is the next chain of inequalities:

$$(*) \qquad \operatorname{depth} R - \operatorname{depth}_R M \leq \text{CM-dim}_R M \leq \text{G-dim}_R M \leq \text{CI-dim}_R M \leq \operatorname{pd}_R M.$$

In the sequence $(*)$ any of the four homological dimensions might be infinite, but if one of them is finite then there are equalities in $(*)$ to the left of the dimensions.

The local ring $(R, \mathfrak{m}, k)$ is, respectively, regular, a complete intersection, Gorenstein, Cohen-Macaulay, if and only if, for each finitely generated non-zero module the corresponding dimension (respectively, pd, CI-dim, G-dim, CM-dim) is finite, and if and only if, respectively, $\operatorname{pd}_R k$, $\text{CI-dim}_R k$, $\text{G-dim}_R k$, $\text{CM-dim}_R k$ are finite.

For any, not necessarily finitely generated R-module M, the flat dimension of M over R is denoted by $\operatorname{fd}_R M$. There is always an inequality $\operatorname{fd}_R M \leq \operatorname{pd}_R M$, and equality holds if M is *finite*, that is finitely generated. In [24] Christensen, Foxby, and Frankild introduced the *large restricted flat dimension* which is denoted by Rfd. They showed that for all R-module M, there is an inequality

$$\operatorname{Rfd}_R M \leq \operatorname{fd}_R M$$

with equality if $\operatorname{fd}_R M < \infty$.

In [40] and [39] Enochs and Jenda have introduced the Gorenstein flat dimension $\operatorname{Gfd}_R M$ of any R-module M and proved that for a finite R-module M we have $\operatorname{Gfd}_R M = \operatorname{G-dim}_R M$. Holm has studied this concept further in [61] and proved that $\operatorname{Gfd}_R M$ is a refinement of $\operatorname{fd}_R M$ and that $\operatorname{Rfd}_R M$ is a refinement of $\operatorname{Gfd}_R M$. In other words, for any R-module M there is a chain of inequalities

$$\operatorname{Rfd}_R M \leq \operatorname{Gfd}_R M \leq \operatorname{fd}_R M,$$

and if one of these quantities is finite then there is equality everywhere to its left.

The main goal of this chapter is to introduce and study notions of complete intersection flat dimension (CIfd) and Cohen-Macaulay flat dimension (CMfd) as refinements of flat dimensions for every module M over a noetherian ring R. We plane to show that there is the following sequence of inequalities:

$$\operatorname{Rfd}_R M \leq \operatorname{CMfd}_R M \leq \operatorname{Gfd}_R M \leq \operatorname{CIfd}_R M \leq \operatorname{fd}_R M.$$

We say that R has a CM*-deformation if there exist a local ring Q and a perfect ideal J in Q such that $R = Q/J$. A CM*-quasi-deformation of R is a diagram of local homomorphisms $R \to R' \leftarrow Q$ with $R \to R'$ a flat extension and $R' \leftarrow Q$ a CM^*-deformation. We set $M' = M \otimes_R R'$.

Definition 3.1.1. *Let $M \neq 0$ be an R-module. The complete intersection flat dimension,*[1] *upper Gorenstein flat dimension, and upper Cohen-Macaulay flat dimension of M, are defined as:*

$$\mathrm{CI}\,\mathrm{fd}_R\, M := \inf\{\mathrm{fd}_Q\, M' - \mathrm{fd}_Q\, R' \mid R \to R' \leftarrow Q \text{ is a } \mathit{CI}\text{-quasi-deformation}\}$$

$$\mathrm{G}^*\,\mathrm{fd}_R\, M := \inf\{\mathrm{fd}_Q\, M' - \mathrm{fd}_Q\, R' \mid R \to R' \leftarrow Q \text{ is a } \mathit{G}^*\text{-quasi-deformation}\}$$

$$\mathrm{CM}^*\,\mathrm{fd}_R\, M := \inf\{\mathrm{fd}_Q\, M' - \mathrm{fd}_Q\, R' \mid R \to R' \leftarrow Q \text{ is a } \mathrm{CM}^*\text{-quasi-deformation}\},$$

respectively. We complement this by $\mathrm{H}\,\mathrm{fd}_R\, 0 = -\infty$ *for* $\mathrm{H} = \mathrm{CI}$, G^*, *and* CM^*.

Our first result says that the large restricted flat dimension is a refinement of the above H-flat dimensions.

Proposition 3.1.2. *Let $R \to S \leftarrow Q$ be a* CM*-quasi-deformation, and let M be an R-module. Then*

$$\mathrm{Rfd}_Q(M \otimes_R S) - \mathrm{Rfd}_Q\, S = \mathrm{Rfd}_R\, M.$$

Proof. First we prove the equality

$$\mathrm{Rfd}_S\, N + \text{G-dim}_Q\, S = \mathrm{Rfd}_Q\, N,$$

for an S-module N. To this end, choose by [24, (2.4)(b)] a prime ideal $\mathfrak{p}$ of S such that the first equality below holds. Let $\mathfrak{q}$ be the inverse image of $\mathfrak{p}$ in Q. Therefore

[1] The notion of complete intersection flat dimension was introduced by T. Sharif in [88].

there is an isomorphism $N_{\mathfrak{p}} \cong N_{\mathfrak{q}}$ of $Q_{\mathfrak{q}}$-modules and a CM-deformation $Q_{\mathfrak{q}} \to S_{\mathfrak{p}}$. Hence

$$\begin{aligned}
\operatorname{Rfd}_S N &= \operatorname{depth} S_{\mathfrak{p}} - \operatorname{depth}_{S_{\mathfrak{p}}} N_{\mathfrak{p}} \\
&= \operatorname{depth}_{Q_{\mathfrak{q}}} S_{\mathfrak{p}} - \operatorname{depth}_{Q_{\mathfrak{q}}} N_{\mathfrak{p}} \\
&= \operatorname{depth} Q_{\mathfrak{q}} - \operatorname{G-dim}_{Q_{\mathfrak{q}}} S_{\mathfrak{p}} - \operatorname{depth}_{Q_{\mathfrak{q}}} N_{\mathfrak{p}} \\
&\leq \operatorname{Rfd}_Q N - \operatorname{G-dim}_{Q_{\mathfrak{q}}} S_{\mathfrak{p}} \\
&= \operatorname{Rfd}_Q N - \operatorname{G-dim}_Q S.
\end{aligned}$$

The second equality holds since $Q_{\mathfrak{q}} \to S_{\mathfrak{p}}$ is surjective; the third equality holds by Auslander-Bridger formula [4]; the fourth equality is due to the G-perfectness assumption of S over Q; while the inequality follows from [24, (2.4)(b)]. Now by [90, (3.5)] we have

$$\operatorname{Rfd}_Q N \leq \operatorname{Rfd}_S N + \operatorname{Rfd}_Q S \leq \operatorname{Rfd}_Q N - \operatorname{G-dim}_Q S + \operatorname{Rfd}_Q S = \operatorname{Rfd}_Q N,$$

which is the desired equality.

Now we have

$$\begin{aligned}
\operatorname{Rfd}_Q(M \otimes_R S) &\leq \operatorname{Rfd}_S(M \otimes_R S) + \operatorname{Rfd}_Q S \\
&= \operatorname{Rfd}_S(M \otimes_R S) + \operatorname{G-dim}_Q S \\
&= \operatorname{Rfd}_Q(M \otimes_R S),
\end{aligned}$$

where the inequality was proven in [90, (3.5)], the first equality follows from the hypotheses, and the last follows from the above observation. Hence

$$\operatorname{Rfd}_Q(M \otimes_R S) - \operatorname{Rfd}_Q S = \operatorname{Rfd}_S(M \otimes_R S) = \operatorname{Rfd}_R M$$

where the second equality holds by [65, (8.5)]. □

Theorem 3.1.3. *Let M be an R-module. Then we have* $\mathrm{Rfd}_R\, M \leq \mathrm{H}\,\mathrm{fd}_R\, M$, *for* $\mathrm{H} = CI$, G^*, *and* CM^*, *with equality if* $\mathrm{H}\,\mathrm{fd}_R\, M$ *is finite. In this case we have*

$$\mathrm{H}\,\mathrm{fd}_R\, M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Proof. The inequality follows easily from the definitions of various H-flat dimensions introduced above, and Proposition 3.1.2. The last equality follows from [24, (2.4)(b)]. □

The following corollary is an immediate consequence of Theorem 3.1.3:

Corollary 3.1.4. *There is the following chain of inequalities:*

$$\mathrm{Rfd}_R\, M \leq \mathrm{CM}^*\,\mathrm{fd}_R\, M \leq G^*\,\mathrm{fd}_R\, M \leq \mathrm{CIfd}_R\, M \leq \mathrm{fd}_R\, M,$$

with equality to the left of any finite number.

In [52, (19.7)] Foxby proved an Intersection Theorem for flat dimension. More precisely, he showed that for M an R-module of finite flat dimension and of finite depth, and R admitting of a Hochster module (as is the case where R is equicharacteristic), one has:

$$\dim R \leq \dim_R M + \mathrm{fd}_R\, M.$$

Recall that the local ring $(R, \mathfrak{m}, k)$ is equicharacteristic if $\mathrm{char} R = \mathrm{char} k$, where $\mathrm{char} R$ denotes to the characteristic of the ring R. Now we extend Foxby's result to the homological flat dimensions in the following theorem:

Theorem 3.1.5. *Let M be an R-module of finite depth such that $\mathrm{H}\,\mathrm{fd}_R M < \infty$. Suppose that R is an equicharacteristic zero ring, then:*

$$\dim R \leq \dim_R M + \mathrm{H}\,\mathrm{fd}_R M,$$

for $\mathrm{H} = CI$, G^*, *and* CM^*.

The proof of this theorem makes use of Lemma 3.1.6 below and the notion of the *Cohen-Macauley defect* ($\operatorname{cmd} R$) of a ring R which is defined as:

$$\operatorname{cmd} R := \dim R - \operatorname{depth} R.$$

Lemma 3.1.6. *Let $Q \to R'$ be any CM^*-deformation. Then $\operatorname{cmd} R' \leq \operatorname{cmd} Q$.*

Proof. Suppose that $J = \ker(Q \to R')$. Since J is a perfect ideal of Q, we have $\mathrm{pd}_Q R' = \mathrm{grade}_Q J$. The proof of the Lemma is easily completed by noting that $\operatorname{depth} Q - \operatorname{depth}_Q R' = \mathrm{pd}_Q R' = \mathrm{grade}_Q J \leq \operatorname{ht} J \leq \dim Q - \dim R'$, in which the first equality follows the Auslander-Buchsbaum formula. □

Proof of Theorem 3.1.5. It is sufficient to prove the Theorem for $\mathrm{H} = \mathrm{CM}^*$. Choose a CM^*-quasi-deformation $R \to R' \leftarrow Q$ such that $\mathrm{fd}_Q(M \otimes_R R') < \infty$ and $\mathrm{CM}^*\,\mathrm{fd}_R M = \mathrm{fd}_Q(M \otimes_R R') - \mathrm{fd}_Q R'$. It can be seen that Q is an equicharacteristic zero ring. Since $R \to R'$ is a flat extension and $\operatorname{depth}_R M < \infty$, it follows from [64, (2.6)] that $\operatorname{depth}_{R'}(M \otimes_R R') < \infty$. Therefore we obtain $\operatorname{depth}_Q(M \otimes_R R') < \infty$ since $Q \to R'$ is surjective. By Lemma 3.1.6 there is an inequality $\operatorname{cmd} R' \leq \operatorname{cmd} Q$. Then

$\operatorname{cmd} R + \operatorname{cmd} R'/\mathfrak{m}R' \leq \operatorname{cmd} Q$. So we have:

$$\begin{aligned}
\dim R &\leq \operatorname{cmd} Q - \operatorname{cmd} R'/\mathfrak{m}R' + \operatorname{depth} R \\
&= \dim Q - \dim R'/\mathfrak{m}R' - \operatorname{depth} Q + \operatorname{depth} R + \operatorname{depth} R'/\mathfrak{m}R' \\
&= \dim Q - \dim R'/\mathfrak{m}R' - \operatorname{depth} Q + \operatorname{depth} R' \\
&= \dim Q - \dim R'/\mathfrak{m}R' - \operatorname{pd}_Q R' \\
&\leq \dim_Q(M \otimes_R R') + \operatorname{fd}_Q(M \otimes_R R') - \operatorname{fd}_Q R' - \dim R'/\mathfrak{m}R' \\
&= \dim_{R'}(M \otimes_R R') + \operatorname{CM}^* \operatorname{fd}_R M - \dim R'/\mathfrak{m}R' \\
&= \dim_R M + \operatorname{CM}^* \operatorname{fd}_R M,
\end{aligned}$$

where the third equality holds by the Auslander-Buchsbaum formula; and the second inequality holds from Foxby's Theorem [52, (19.7)]. To prove the fifth equality assume that M is the direct union of finite submodules M_i of M (for i in a directed set I). Then

$$\dim_R M = \sup\{\dim_R M_i | i \in I\}.$$

So we get that $M \otimes_R R'$ is the direct union of $M_i \otimes_R R'$. Consequently by the above observation we have:

$$\begin{aligned}
\dim_{R'}(M \otimes_R R') &= \sup\{\dim_{R'}(M_i \otimes_R R') | i \in I\} \\
&= \sup\{\dim_R M_i + \dim R'/\mathfrak{m}R' | i \in I\} \\
&= \sup\{\dim_R M_i | i \in I\} + \dim R'/\mathfrak{m}R' \\
&= \dim_R M + \dim R'/\mathfrak{m}R',
\end{aligned}$$

where the second equality follows from [17, (A.11)]. □

In the rest of this section let M is a finite R-module. Since for a finite R-module M, there is the equality $\mathrm{fd}_R\, M = \mathrm{pd}_R\, M$ and $\mathrm{Gfd}_R\, M = \text{G-dim}_R\, M$ by [39], we have

$$\mathrm{CM}^*\,\mathrm{fd}_R\, M = \inf\{\mathrm{pd}_Q\, M' - \mathrm{pd}_Q\, R' |\, R \to R' \leftarrow Q \textit{ is a } \mathrm{CM}^*\textit{-quasi-deformation}\},$$

$\mathrm{CMfd}_R\, M = \text{CM-dim}_R\, M$, $\mathrm{G}^*\,\mathrm{fd}_R\, M = \mathrm{G}^*\text{-}\dim_R M$, and $\mathrm{CI}\,\mathrm{fd}_R\, M = \text{CI-dim}_R\, M$.

Remark 3.1.7. *It can be seen that if* $\mathrm{CM}^*\,\mathrm{fd}_R\, M < \infty$*, then there is the equality*

$$\mathrm{CM}^*\,\mathrm{fd}_R\, M + \mathrm{depth}_R\, M = \mathrm{depth}\, R.$$

Let $\mathrm{Syz}_n^R(M)$ to denote the n-th syzygy module of M. Then by an argument similar to that of [94, (2.5)] we have the following proposition:

Proposition 3.1.8. *For each $n \geq 0$ there is the equality*

$$\mathrm{CM}^*\,\mathrm{fd}_R\, Syz_n^R(M) = \max\{\mathrm{CM}^*\,\mathrm{fd}_R\, M - n, 0\}.$$

Theorem 3.1.9. *The following conditions are equivalent:*

(i) *The ring R is Cohen-Macaulay.*

(ii) $\mathrm{CM}^*\,\mathrm{fd}_R\, M < \infty$ *for every not necessarily finite R-module M.*

(iii) $\mathrm{CM}^*\,\mathrm{fd}_R\, M < \infty$ *for every finite R-module M.*

(iv) $\mathrm{CM}^*\,\mathrm{fd}_R\, M = 0$ *for every finite R-module M with* $\mathrm{depth}_R\, M \geqslant \mathrm{depth}\, R$.

(v) $\mathrm{CM}^*\,\mathrm{fd}_R\, k = \mathrm{depth}\, R$.

(vi) $\mathrm{CM}^*\,\mathrm{fd}_R\, k < \infty$.

Proof. $(i) \Rightarrow (ii)$ Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Since R is Cohen-Macaulay, so is $\widehat{R}$. Therefore by Cohen's structure theorem, $\widehat{R}$ is isomorphic to Q/J, where Q is a regular local ring. By Cohen-Macaulay-ness of $\widehat{R}$ and regularity of Q, the ideal J is perfect. Thus $R \to \widehat{R} \leftarrow Q$ is a CM*-quasi-deformation. Since Q is regular $\mathrm{fd}_Q(M \otimes_R \widehat{R})$ is finite. Thus $\mathrm{CM}^* \,\mathrm{fd}_R\, M$ is finite.

$(ii) \Rightarrow (iii)$ is trivial.

$(iii) \Rightarrow (iv)$ follows by applying Remark 3.1.7 to the R-module M.

$(iv) \Rightarrow (v)$ by [17, (1.3.7)] we have $\operatorname{depth}_R \operatorname{Syz}_n^R(k) \geq \min(n, \operatorname{depth} R)$. In particular, if we choose $n \geq \operatorname{depth} R$ we get $\mathrm{CM}^* \,\mathrm{fd}_R\, M = 0$. Thus by Remark 3.1.7 $\mathrm{CM}^* \,\mathrm{fd}_R\, k = \operatorname{depth} R$.

$(v) \Rightarrow (vi)$ is trivial.

$(vi) \Rightarrow (i)$ follows from $\mathrm{CM}\text{-}\dim_R k \leq \mathrm{CM}^* \,\mathrm{fd}_R\, k$ and [57, Theorem (3.9)]. □

One can actually state similar theorems for upper Gorenstein flat and complete intersection flat dimensions.

As a consequence of the New Intersection Theorem of Peskine and Szpiro [78], Hochster [60] and P. Roberts [84] and [85] we have:

$$(*) \qquad \operatorname{cmd} R \leq \operatorname{cmd}_R M.$$

The New Intersection Theorem is not true for CI-dimension, G*-dimension, G-dimension, and CM-dimension, see Examples [89, (3.2)] and [98, (2.20)]. But the inequality $(*)$ holds for G*-dimension, see [89, (2.1)]. For G-dimension and CM-dimension we do not know whether the inequality $(*)$ holds. However it holds for upper Cohen-Macaulay flat dimension as shown by the following theorem:

Theorem 3.1.10. *Let M be a finite R-module with finite upper Cohen-Macaulay dimension. Then* $\operatorname{cmd} R \leq \operatorname{cmd}_R M$.

Proof. Since $\mathrm{CM}^* \operatorname{fd}_R M < \infty$, there exists a CM^*-quasi-deformation $R \to R' \leftarrow Q$ such that $\operatorname{pd}_Q M' < \infty$, where $M' = M \otimes_R R'$. From the surjectivity of $Q \to R'$ we have $\operatorname{cmd}_Q M' = \operatorname{cmd}_{R'} M'$. Because $R \to R'$ is a flat extension, the following (in)equalities hold:

$$\begin{aligned}\operatorname{cmd} R + \operatorname{cmd}_R R'/\mathfrak{m}R' &= \operatorname{cmd} R' \leq \operatorname{cmd} Q \leq \operatorname{cmd}_Q M' \\ &= \operatorname{cmd}_R M + \operatorname{cmd} R'/\mathfrak{m}R',\end{aligned}$$

where the first inequality holds by Lemma 3.1.6, and the second one is by the New Intersection Theorem. This gives us the desired inequality. □

Corollary 3.1.11. *If M is a Cohen-Macaulay module with* $\mathrm{CM}^* \operatorname{fd}_R M < \infty$, *then the base ring R is Cohen-Macaulay.*

3.2 Cohen-Macaulay flat dimension

In this section we introduce the notion of *Cohen-Macaulay flat dimension* denoted by CMfd. For a finite R-module M it coincides with the Cohen-Macaulay dimension $\operatorname{CM-dim}_R M$ of Gerko. And we show that, for an R-module M we have the following sequence of inequalities

$$\operatorname{Rfd}_R M \leq \operatorname{CMfd}_R M \leq \operatorname{Gfd}_R M \leq \operatorname{CI} \operatorname{fd}_R M \leq \operatorname{fd}_R M,$$

with equality to the left of any finite number.

Definition 3.2.1. *Let $M \neq 0$ be an R-module. The Cohen-Macaulay flat dimension of M, is defined as:*

$$\operatorname{CMfd}_R M := \inf\{\operatorname{Gfd}_Q M' - \operatorname{Gfd}_Q R' \mid R \to R' \leftarrow Q \text{ is a } \operatorname{CM}\text{-quasi-deformation}\}.$$

We complement this by $\operatorname{CMfd}_R 0 = -\infty$.

Remark 3.2.2. *By taking the trivial* CM-*quasi-deformation $R \to R \leftarrow R$, one has* $\operatorname{CMfd}_R M \leq \operatorname{Gfd}_R M$, *and using Proposition 3.1.2 we have, when* $\operatorname{CMfd}_R M < \infty$, *then* $\operatorname{CMfd}_R M = \operatorname{Rfd}_R M$.

Notice that there is a notion of Cohen-Macaulay flat dimension in [62] which is different with ours.

Theorem 3.2.3. [2] *The following conditions are equivalent:*

(i) The ring R is Cohen-Macaulay.

(ii) $\operatorname{CMfd}_R M < \infty$ *for every R-module M.*

(iii) $\operatorname{CMfd}_R M < \infty$ *for every finite R-module M.*

(iv) $\operatorname{CMfd}_R k < \infty$.

[2] In fall 2003 I had taken the Homological algebra course with Professor Yassemi. In one session he historically reviewed the homological dimensions.... He spoke about Gerko's Cohen-Macaulay dimension and stated the equivalencies of the followings:

(i) The ring R is Cohen-Macaulay.

(ii) $\text{CM-dim}_R M < \infty$ for every finite R-module M.

(iii) $\text{CM-dim}_R k < \infty$.

And said that my new PH.D. student have to define a Cohen-Macaulay dimension for arbitrary modules, and complete this theorem as:

(vi) $\text{CM-dim}_R M < \infty$ for every not necessarily finite R-module M.

Before proceeding any further it is necessary to investigate the effect of change of ring on various notions of homological flat dimensions.

Proposition 3.2.4. *Let M be an R-module. Let $R \to R'$ be a local flat extension, and $M' = M \otimes_R R'$. Then*

$$\mathrm{H}\,\mathrm{fd}_R M \leq \mathrm{H}\,\mathrm{fd}_{R'} M'$$

with equality when $\mathrm{H}\,\mathrm{fd}_{R'} M'$ is finite, for $\mathrm{H} = CI$, G^*, CM^*, and CM.

Proof. We prove the result for Cohen-Macaulay flat dimension and the proof of the other cases are similar to this one, so we omit them. Suppose that $\mathrm{CMfd}_{R'} M' < \infty$, and let $R' \to R'' \leftarrow Q$ be a CM-quasi-deformation with $\mathrm{Gfd}_Q M'' < \infty$, where $M'' = M' \otimes_{R'} R''$. Since $R \to R'$ and $R' \to R''$ are flat extensions, the local homomorphism $R \to R''$ is also flat. Hence $R \to R'' \leftarrow Q$ is a CM-quasi-deformation with $\mathrm{Gfd}_Q(M \otimes_R R'') < \infty$. It follows that $\mathrm{CMfd}_R M$ is finite. Now by Theorem 3.1.3 and [65, (8.5)], we have $\mathrm{CMfd}_R M = \mathrm{Rfd}_R M = \mathrm{Rfd}_{R'} M' = \mathrm{CMfd}_{R'} M'$. $\square$

Proposition 3.2.5. *Let $\widehat{R}$ be the completion of R relative to the $\mathfrak{m}$-adic topology. Then*

$$\mathrm{H}\,\mathrm{fd}_R M = \mathrm{H}\,\mathrm{fd}_{\widehat{R}}(M \otimes_R \widehat{R}),$$

for $\mathrm{H} = CI$, G^*, and CM^*.

Proof. We prove the result for $\mathrm{CM}^*\,\mathrm{fd}$ and the proof of the other cases are similar to this one. If $\mathrm{CM}^*\,\mathrm{fd}_R M = \infty$, then we obtain that $\mathrm{CM}^*\,\mathrm{fd}_{\widehat{R}}(M \otimes_R \widehat{R}) = \infty$ by 3.2.4. Now assume that $\mathrm{CM}^*\,\mathrm{fd}_R M < \infty$. It is sufficient to prove that $\mathrm{CM}^*\,\mathrm{fd}_{\widehat{R}}(M \otimes_R \widehat{R})$ is finite. Because in this case we have

$$\mathrm{CM}^*\,\mathrm{fd}_R M = \mathrm{Rfd}_R M = \mathrm{Rfd}_{\widehat{R}}(M \otimes_R \widehat{R}) = \mathrm{CM}^*\,\mathrm{fd}_{\widehat{R}}(M \otimes_R \widehat{R}),$$

in which the first and the last equalities follow from Theorem 3.1.3, and the middle one follows from [65, (8.5)].

For a CM*-quasi-deformation $R \to R' \leftarrow Q$ of R, we have $\widehat{R} \to \widehat{R'} \leftarrow \widehat{Q}$ is a CM*-quasi-deformation of $\widehat{R}$ with respect to their maximal ideal-adic completions. Now the equalities

$$\begin{aligned}\mathrm{fd}_Q(M \otimes_R R') &= \mathrm{fd}_{\widehat{Q}}(M \otimes_R R' \otimes_Q \widehat{Q}) = \mathrm{fd}_{\widehat{Q}}(M \otimes_R \widehat{R'}) \\ &= \mathrm{fd}_{\widehat{Q}}(M \otimes_R (\widehat{R} \otimes_{\widehat{R}} \widehat{R'})) = \mathrm{fd}_{\widehat{Q}}((M \otimes_R \widehat{R}) \otimes_{\widehat{R}} \widehat{R'}),\end{aligned}$$

show that $\mathrm{fd}_{\widehat{Q}}((M \otimes_R \widehat{R}) \otimes_{\widehat{R}} \widehat{R'})$ is finite which imply that $\mathrm{CM}^* \,\mathrm{fd}_{\widehat{R}}(M \otimes_R \widehat{R})$ is finite. □

The main result of this chapter is Theorem 3.2.6 below the proof of which strongly makes use results of Sather-Wagstaff [86, Theorem F] and Esmkhani and Tousi [44, Corollary 2.6].

Theorem 3.2.6. *Let M be an R-module. Then there is the inequality*

$$\mathrm{Gfd}_R\, M \leq \mathrm{CIfd}_R\, M$$

with equality if $\mathrm{CIfd}_R\, M$ *is a finite number.*

Proof. **Step 1.** Assume that R admits of a dualizing complex D. We can actually assume that $\mathrm{CI}\,\mathrm{fd}_R\, M$ is finite. So that by [86, Theorem F], there exists a CI-quasi-deformation $R \to R' \leftarrow Q$ such that Q is complete, the closed fibre $R'/\mathfrak{m}R'$ is artinian and Gorenstein, and $\mathrm{fd}_Q(M \otimes_R R')$ is finite. Therefore by Remark 1.1.5(a), $M \otimes_R R'$ belongs to the the Auslander class $\mathbf{A}(Q)$. On the other hand since the kernel of $Q \to R'$ is generated by Q-regular elements, using [9, Proposition 4.3] we deduce that it is a Gorenstein local homomorphism. Thus thanks to [8, Corollary (7.9)] we see that $M \otimes_R R'$ belongs to the the Auslander class $\mathbf{A}(R')$. Note that R' is a complete local ring, so it admits of a dualizing complex. Hence using Remark 1.1.5(a), we obtain that $\mathrm{Gfd}_{R'}(M \otimes_R R')$ is finite. Since $R'/\mathfrak{m}R'$ is a Gorenatein local ring, by [9, Proposition 4.2], we have $R \to R'$ is a Gorenstein local homomorphism. Therefore by [9, Theorem 5.1], the complex $D \otimes^{\mathbf{L}}_R R'$ is a dualizing complex of R'. Consequently by [25, Theorem 5.3], $\mathrm{Gfd}_R M$ is finite. Hence the equalities $\mathrm{Gfd}_R M = \mathrm{Rfd}_R M = \mathrm{CI\,fd}_R M$ hold.

Step 2. Now let R be any ring. Note that by Proposition 3.2.5 we have

$$\mathrm{CIfd}_R M = \mathrm{CIfd}_{\widehat{R}}(M \otimes_R \widehat{R}).$$

Since $\widehat{R}$ admits of a dualizing complex by Step 1 we have

$$\mathrm{Gfd}_{\widehat{R}}(M \otimes_R \widehat{R}) \leq \mathrm{CIfd}_{\widehat{R}}(M \otimes_R \widehat{R})$$

with equality if $\mathrm{CIfd}_{\widehat{R}}(M \otimes_R \widehat{R})$ is finite. Now assume that $\mathrm{CIfd}_R M$ is finite. Therefore $\mathrm{Gfd}_{\widehat{R}}(M \otimes_R \widehat{R})$ is finite. Consequently by [44, Corollary 2.6], $\mathrm{Gfd}_R M$ is finite and $\mathrm{Gfd}_R M = \mathrm{Gfd}_{\widehat{R}}(M \otimes_R \widehat{R})$. Hence

$$\mathrm{Gfd}_R M = \mathrm{Gfd}_{\widehat{R}}(M \otimes_R \widehat{R}) = \mathrm{CIfd}_{\widehat{R}}(M \otimes_R \widehat{R}) = \mathrm{CIfd}_R M.$$

This completes the proof. $\square$

Corollary 3.2.7. *Let M be an R-module. Then there is the following sequence of inequalities*

$$\mathrm{Rfd}_R M \leq \mathrm{CMfd}_R M \leq \mathrm{Gfd}_R M \leq \mathrm{CIfd}_R M \leq \mathrm{fd}_R M,$$

with equality to the left of any finite number.

Proposition 3.2.8. *Let M be an R-module. For each prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ there is an inequality*

$$\operatorname{H}\operatorname{fd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \operatorname{H}\operatorname{fd}_R M,$$

for $\mathrm{H} = CI$, G^*, CM^*, *and* CM.

Proof. We prove the result for $\operatorname{CMfd}_R M$, and the proof of the other cases are similar to this one. Choose a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$. Assume that $\operatorname{CMfd}_R M < \infty$ and fix a CM-quasi-deformation $R \to R' \leftarrow Q$ such that $\operatorname{Gfd}_Q M' < \infty$, where $M' = M \otimes_R R'$. Since $R \to R'$ is faithfully flat extension of rings, there is a prime ideal $\mathfrak{p}'$ in R' lying over $\mathfrak{p}$. Let $\mathfrak{q}$ be the inverse image of $\mathfrak{p}'$ in Q. The map $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'}$ is flat, and $R'_{\mathfrak{p}'} \leftarrow Q_{\mathfrak{q}}$ is a CM-deformation. Therefore the diagram $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'} \leftarrow Q_{\mathfrak{q}}$ is a CM-quasi-deformation with $\operatorname{Gfd}_{Q_{\mathfrak{q}}}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'}) = \operatorname{Gfd}_{Q_{\mathfrak{q}}} M'_{\mathfrak{q}'} \leq \operatorname{Gfd}_Q M' < \infty$. Hence $\operatorname{CM}\operatorname{fd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty$. So we obtain

$$\begin{aligned} \operatorname{CMfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} &= \operatorname{Rfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\leq \operatorname{Rfd}_R M \\ &= \operatorname{CMfd}_R M, \end{aligned}$$

in which the inequality holds by [24, (2.3)]. Thus the desired inequality follows. $\square$

Lemma 3.2.9. *Let Q be a local ring, and let $J \subseteq I$ be ideals of Q. Set $R = Q/J$. If J and I/J are perfect ideals of Q and R respectively, then I is a perfect ideal in Q.*

Proof. Since $\operatorname{pd}_Q R < \infty$ and $\operatorname{pd}_R Q/I < \infty$, by [5, (3.8)] there is an equality $\operatorname{pd}_Q Q/I = \operatorname{pd}_R Q/I + \operatorname{pd}_Q R$. By our assumption J is a perfect ideal of Q hence by [7, (2.7)] we have $\operatorname{grade}_Q Q/I = \operatorname{grade}_R Q/I + \operatorname{grade}_Q R$. Using the perfectness of J in Q and I/J in R, we see that I is a perfect ideal in Q. $\square$

Proposition 3.2.10. *Let* $\mathbf{x} = x_1, \dots, x_n$ *be a sequence of elements of* $\mathfrak{m}$*, constituting* R*- and* M*-regular elements. Set* $\overline{R} = R/(\mathbf{x})$ *and* $\overline{M} = M/(\mathbf{x})M$*. Then there are inequalities*

$$\mathrm{H}\,\mathrm{fd}_{\overline{R}}\,\overline{M} \leq \mathrm{H}\,\mathrm{fd}_R\,M, \text{ and } \mathrm{H}\,\mathrm{fd}_R\,\overline{M} \leq \mathrm{H}\,\mathrm{fd}_R\,M + n,$$

with equality when, $\mathrm{H}\,\mathrm{fd}_R\,M$ *is finite, for* $\mathrm{H} =$ *CI,* G^**, and* CM^**.*

Proof. Since the proof for $\mathrm{H} = \mathrm{CI}$ and G^* is analogous to CM^*, we only prove the proposition for $\mathrm{H} = \mathrm{CM}^*$. It is sufficient to prove the proposition for $\mathbf{x} = x$ with x an R-regular and M-regular element. We may assume that $\mathrm{CM}^*\,\mathrm{fd}_R\,M < \infty$ and choose a CM^*-quasi-deformation $R \to R' \leftarrow Q$ with $\mathrm{fd}_R\,M' < \infty$, where $M' = M \otimes_R R'$. Thus $R' = Q/J$, where J is a perfect ideal of Q. We construct a CM^*-quasi-deformation of $\overline{R}$. Choose $y \in Q$ mapping to $x \in R'$. Since x is R-regular, it is also R'-regular due to flatness of R' as an R-module. Set $I = (y) + J$ and note that $I/J = xR'$ is a perfect ideal of R'. Therefore by lemma 3.2.9, I is a perfect ideal in Q (for the case $\mathrm{H} = \mathrm{G}^*$ use [94, (2.11)]). Set $\overline{R'} = Q/I$, and note that $\overline{R} \to \overline{R'}$ is flat because $R \to R'$ is flat. Thus $\overline{R} \to \overline{R'} \leftarrow Q$ is a CM^*-quasi-deformation of $\overline{R}$.

Now we show that $\mathrm{fd}_Q(\overline{M} \otimes_{\overline{R}} \overline{R'})$ and $\mathrm{fd}_Q(\overline{M} \otimes_R R')$ are finite. We have the following isomorphisms

$$\overline{M} \otimes_{\overline{R}} \overline{R'} \cong \overline{M} \otimes_R \overline{R} \otimes_R R' \cong \overline{M} \otimes_R R'.$$

Since x is M-regular and $R \to R'$ is flat, the exact sequence $0 \to M \xrightarrow{x} M \to \overline{M} \to 0$ induces an exact sequence $0 \to M' \xrightarrow{x} M' \to \overline{M} \otimes_R R' \to 0$. So we obtain $\overline{M} \otimes_R R' \cong M'/xM'$ and we have $\mathrm{fd}_Q(M'/xM') = \mathrm{fd}_Q\,M' + 1$. Hence we get $\mathrm{CM}^*\,\mathrm{fd}_{\overline{R}}\,\overline{M}$ and

$\mathrm{CM}^* \operatorname{fd}_R \overline{M}$ are finite. Now the equalities

$$\mathrm{CM}^* \operatorname{fd}_{\overline{R}} \overline{M} = \operatorname{Rfd}_{\overline{R}} \overline{M} = \operatorname{Rfd}_R M = \mathrm{CM}^* \operatorname{fd}_R M,$$

where the second equality follows from 2.2.11, complete the proof of the first inequality in the assertion of the Theorem. The equalities

$$\mathrm{CM}^* \operatorname{fd}_R \overline{M} = \operatorname{Rfd}_R \overline{M} = \operatorname{Rfd}_{\overline{R}} \overline{M} + 1 = \mathrm{CM}^* \operatorname{fd}_R M + 1,$$

where the second equality follows from [90, (3.6)] and the third one holds by Lemma 2.2.10 complete the proof. $\square$

Proposition 3.2.11. *Let $\mathbf{x} = x_1, \ldots, x_n$ be a R-regular elements. Set $\overline{R} = R/(\mathbf{x})$. For an $\overline{R}$-module M, then there is the inequality*

$$n + \operatorname{CIfd}_{\overline{R}} M \leq \operatorname{CIfd}_R M,$$

with equality when $\operatorname{CIfd}_R M$ is finite.

Proof. As usual we may assume that $\operatorname{CIfd}_R M < \infty$ and choose a CI-quasi-deformation $R \to R' \leftarrow Q$ with $\operatorname{fd}_Q M' < \infty$, where $M' = M \otimes_R R'$. Consider $\overline{R} \to \overline{R} \leftarrow R$ as a CI-quasi-deformation. One checks readily that $\overline{R} \to R'' = \overline{R} \otimes_R R' \leftarrow Q$ is a CI-quasi-deformation of $\overline{R}$. From the equalities $M \otimes_R R' = (M \otimes_{\overline{R}} \overline{R}) \otimes_R R' = M \otimes_{\overline{R}} (\overline{R} \otimes_R R') = M \otimes_R R''$, we obtain that $\operatorname{fd}_Q(M \otimes_R R'')$ and so $\operatorname{CIfd}_{\overline{R}} M$ are finite. Now the equalities

$$n + \operatorname{CIfd}_{\overline{R}} M = n + \operatorname{Rfd}_{\overline{R}} M = \operatorname{Rfd}_R M = \operatorname{CIfd}_R M,$$

where the second one holds by [90, (3.6)] complete the proof. $\square$

Let $\varphi : R \to S$ be a local homomorphism of complete local rings. Let N be a finite S-module, and let $R \to R' \to S$ be a Cohen factorization of φ (cf. [12]). The following inequalities hold:

$$\operatorname{fd}_R N \leq \operatorname{pd}_{R'} N \leq \operatorname{fd}_R N + \operatorname{edim}(R'/\mathfrak{m}R')$$

$$\operatorname{Gfd}_R N \leq \text{G-dim}_{R'} N \leq \operatorname{Gfd}_R N + \operatorname{edim}(R'/\mathfrak{m}R'),$$

where $\operatorname{edim}(R'/\mathfrak{m}R')$ is the minimal number of generators of the maximal ideal of $R'/\mathfrak{m}R'$. The first inequality is by [11] and the latter uses the recent characterization by Christensen, Frankild, and Holm of certain Auslander categories in terms of finiteness of G-dimensions (cf. [25], and also [65, Theorem 8.2]).

Question 3.2.12. *Let $\varphi : R \to S$ be a local homomorphism of complete local rings. Let N be a finite S-module and let $R \to R' \to S$ be a Cohen factorization of φ. The question is whether the following inequalities hold:*

$$\operatorname{CIfd}_R N \leq \text{CI-dim}_{R'} N \leq \operatorname{CIfd}_R N + \mathit{edim}(R'/\mathfrak{m}R').$$

3.3 The depth formula

The point of this section is to prove the *depth formula* and note its immediate consequences.

Notation 3.3.1. *For R-modules M and N set*

$$\operatorname{fd}_R(M,N) = \sup\{i \mid \operatorname{Tor}_i^R(M,N) \neq 0\}.$$

In particular, if $\operatorname{Tor}_n^R(M,N) = 0$ *for all* n*, then* $\operatorname{fd}_R(M,N) = -\infty$*, else* $0 \leq \operatorname{fd}_R(M,N) \leq \infty$*. For a finite* R*-module* M*,* $\operatorname{fd}_R(M,k)$ *is the usual flat dimension of* M *which is also equal to its projective dimension* $\operatorname{pd}_R M$*. Moreover for such an* M*,* $\operatorname{fd}_R(M,N)$ *is finite for every finitely generated* N*.*

Theorem 3.3.2. *Let* M *and* N *be* R*-modules such that* $\operatorname{CIfd}_R M < \infty$*. If* $\operatorname{fd}_R(M,N) < \infty$*, then*

$$\operatorname{fd}_R(M,N) \geq \operatorname{depth} R - \operatorname{depth}_R M - \operatorname{depth}_R N$$

with equality if and only if $\operatorname{depth}_R \operatorname{Tor}_s^R(M,N) = 0$*, for* $s = \operatorname{fd}_R(M,N)$*.*

Proof. Since $\operatorname{CIfd}_R M < \infty$ there is, say a codimension c CI-quasi-deformation $R \to R' \leftarrow Q$, such that $\operatorname{fd}_Q M' < \infty$, where $M' = M \otimes_R R'$. By codimension c we mean that the kernel of the homomorphism $Q \to R'$ is generated by regular elements of length c. Choose $\mathfrak{p} \in \operatorname{Spec}(R')$ such that it is a minimal prime ideal containing $\mathfrak{m}R'$. Thus $\mathfrak{m} = \mathfrak{p} \cap R$ and $\mathfrak{p} = \mathfrak{q}/(\mathbf{x})$ for some $\mathfrak{q} \in \operatorname{Spec}(Q)$, where $(\mathbf{x}) = \ker(Q \to R')$. Now the diagram $R \to R'_{\mathfrak{p}} \leftarrow Q_{\mathfrak{q}}$ is a CI-quasi-deformation of the same codimension as $R \to R' \leftarrow Q$. It is clear that $\operatorname{pd}_Q R' = \operatorname{pd}_{Q_{\mathfrak{q}}} R'_{\mathfrak{p}}$. Also we have

$$\operatorname{fd}_{Q_{\mathfrak{q}}}(M \otimes_R R'_{\mathfrak{p}}) = \operatorname{fd}_{Q_{\mathfrak{q}}}(M \otimes_R (R' \otimes_Q Q_{\mathfrak{q}})) = \operatorname{fd}_{Q_{\mathfrak{q}}}((M \otimes_R R') \otimes_Q Q_{\mathfrak{q}}) \leq \operatorname{fd}_Q M' < \infty.$$

Hence $\operatorname{CIfd}_R M \leq \operatorname{fd}_{Q_{\mathfrak{q}}}(M \otimes_R R'_{\mathfrak{p}}) - \operatorname{fd}_{Q_{\mathfrak{q}}} R'_{\mathfrak{p}}$. Therefore we showed that complete intersection flat dimension can be computed from CI-quasi-deformations $R \to R' \leftarrow Q$ such that the closed fiber $R'/\mathfrak{m}R'$ is artinian.

Due to faithful flatness of R' we have the following equalities in which $N' = N \otimes_R R'$

$$s = \operatorname{fd}_R(M,N) = \operatorname{fd}_{R'}(M',N').$$

Assume that $c = 1$. Consider the change of rings spectral sequence

$$\operatorname{Tor}_p^{R'}(M', \operatorname{Tor}_q^Q(R', N')) \Rightarrow \operatorname{Tor}_{p+q}^Q(M', N').$$

If $q > 1$, then $\operatorname{Tor}_q^Q(R', N') = 0$ and for $q \leq 1$ $\operatorname{Tor}_q^Q(R', N') = N'$. Now the above spectral sequence generates the following long exact sequence

$$\cdots \to \operatorname{Tor}_{i+1}^{R'}(M', N') \to \operatorname{Tor}_{i-1}^{R'}(M', N') \to \operatorname{Tor}_i^Q(M', N') \to \operatorname{Tor}_i^{R'}(M', N') \to \cdots.$$

Therefore $\operatorname{Tor}_{s+1}^Q(M', N') = \operatorname{Tor}_s^{R'}(M', N')$. Iterating in the same manner we have

$$\operatorname{Tor}_s^{R'}(M', N') = \operatorname{Tor}_{s+c}^Q(M', N').$$

So $\sup\{i | \operatorname{Tor}_i^Q(M', N') \neq 0\} = s + c$. Since $\operatorname{depth}(R'/\mathfrak{m}R') = 0$ and $Q \to R'$ is surjective, the following equalities hold:

$$\operatorname{depth}_Q \operatorname{Tor}_s^{R'}(M', N') = \operatorname{depth}_{R'} \operatorname{Tor}_s^{R'}(M', N') = \operatorname{depth}_R \operatorname{Tor}_s^R(M, N),$$

and they are equal to $\operatorname{depth}_Q \operatorname{Tor}_{s+c}^Q(M', N')$. Since $\operatorname{fd}_Q M' < \infty$ it follows from [90, (2.3)] that

$$\begin{aligned} s + c \geq& \operatorname{depth} Q - \operatorname{depth}_Q M' - \operatorname{depth}_Q N' \\ =& \operatorname{depth} R + c - \operatorname{depth}_R M - \operatorname{depth}_R N, \end{aligned}$$

and equality holds if and only if $\operatorname{depth}_Q \operatorname{Tor}_{s+c}^Q(M', N') = 0$. Thus

$$s = \operatorname{fd}_R(M, N) \geq \operatorname{depth} R - \operatorname{depth}_R M - \operatorname{depth}_R N,$$

with equality if and only if $\operatorname{depth}_R \operatorname{Tor}_s^R(M, N) = 0$. □

Definition 3.3.3. *We say that M and N satisfy the* dependency formula *over R, if*

$$\operatorname{fd}_R(M, N) = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Supp}(M) \cap \operatorname{Supp}(N)\}.$$

Corollary 3.3.4. *Let M and N be R-modules such that* $\mathrm{CIfd}_R M < \infty$. *If* $\mathrm{fd}_R(M,N) < \infty$, *then M and N satisfy the dependency formula.*

Proof. It is easy to see that:

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \leq \mathrm{fd}_R(M,N).$$

Using Theorem 3.3.2 we have $\mathrm{fd}_R(M,N) = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ if and only if

$$\operatorname{depth}_{R_{\mathfrak{p}}} \operatorname{Tor}_s^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0,$$

or equivalently if and only if $\mathfrak{p} \in \operatorname{Ass}(\operatorname{Tor}_s^R(M,N))$ for $s = \mathrm{fd}_R(M,N) < \infty$. $\square$

In Theorem 3.2.6 we proved that $\mathrm{Gfd}_R M \leq \mathrm{CIfd}_R M$ for any R-module M. So it is natural to look for a dependency formula for Gorenstein flat dimension. In the following proposition we prove a dependency formula for Gorenstein flat dimension:

Proposition 3.3.5. *Let M and N be R-modules, such that* $\mathrm{Gfd}_R M < \infty$ *and* $\mathrm{id}_R N < \infty$. *Then then M and N satisfy the dependency formula.*

Proof. It is clear that $\mathrm{Gfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ and $\mathrm{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ are finite numbers, and we have $\mathrm{fd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq \mathrm{fd}_R(M,N)$ for $\mathfrak{p} \in \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$. Now from [52, (12.26)] and [26, (4.4)(a)] we get:

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \leq \mathrm{fd}_R(M,N),$$

with equality when $\mathfrak{p} \in \operatorname{Ass}(\operatorname{Tor}_s^R(M,N))$, for $s = \mathrm{fd}_R(M,N)$. $\square$

The following corollary due to Iyengar and Sather-Wagstaff [65, Theorem (8.7)] is an immediate consequence of the above proposition.

Corollary 3.3.6. *Let M be an R-module such that $\operatorname{Gfd}_R M < \infty$. Then*

$$\sup\{i \mid \operatorname{Tor}_i^R(M, E_R(k)) \neq 0\} = \operatorname{depth} R - \operatorname{depth}_R M,$$

where $E_R(k)$ denotes to the injective envelope of k.

Corollary 3.3.7. *Let $(R, \mathfrak{m})$ be a complete local ring and N and M be R-modules with M finite.*

(a) If $\operatorname{Gfd}_R N < \infty$, and $\operatorname{fd}_R M < \infty$, then

$$\sup\{i \mid \operatorname{Ext}_R^i(N, M) \neq 0\} = \operatorname{depth} R - \operatorname{depth}_R N.$$

(b) If $\operatorname{CIfd}_R N < \infty$, then

$$\sup\{i \mid \operatorname{Ext}_R^i(N, M) \neq 0\} = \operatorname{depth} R - \operatorname{depth}_R N,$$

provided that the left hand side is a finite number.

Proof. (a) Set $E = E_R(k)$, the injective envelope of k. Since R is complete we have $R = \operatorname{Hom}_R(E, E)$. Therefore we have

$$\begin{aligned}
\operatorname{Ext}_R^i(N, M) &\cong \operatorname{Ext}_R^i(N, M \otimes_R R) \cong \operatorname{Ext}_R^i(N, M \otimes_R \operatorname{Hom}_R(E, E)) \\
&\cong \operatorname{Ext}_R^i(N, \operatorname{Hom}_R(\operatorname{Hom}_R(M, E), E)) \\
&\cong \operatorname{Hom}_R(\operatorname{Tor}_i^R(N, \operatorname{Hom}_R(M, E), E).
\end{aligned}$$

Consequently we have

$$\sup\{i \mid \operatorname{Ext}_R^i(N, M) \neq 0\} = \operatorname{fd}_R(N, \operatorname{Hom}_R(M, E)).$$

Since $\mathrm{fd}_R M < \infty$ we have $\mathrm{id}_R \operatorname{Hom}_R(M, E) < \infty$. Now Proposition 3.3.5, gives the result.

(b) Similarly to that of part (a) one has

$$\sup\{i| \operatorname{Ext}^i_R(N, M) \neq 0\} = \mathrm{fd}_R(N, \operatorname{Hom}_R(M, E)),$$

which is equal to $\operatorname{depth} R - \operatorname{depth}_R N$ by Corollary 3.3.4 and [97, Lemma 2.2]. □

The following example shows that the completeness assumption of R is crucial in part (a) of the above corollary.

Example 3.3.8. *Let* $(R, \mathfrak{m})$ *be a local domain which is not complete with respect to the* $\mathfrak{m}$*-adic topology. In [2, (3.3)] it is shown that* $\operatorname{Hom}_R(\widehat{R}, R) = 0$*. Therefore, when* $N = \widehat{R}$ *and* $M = R$*. It is clear that the right hand side of the first equality equal to zero which is not equal to the left hand side.*

Corollary 3.3.9. *Let* M *and* N *be* R*-modules;*

(a) If $\mathrm{CIfd}_R M < \infty$ *then the following are equivalent:*

(i) $\operatorname{Tor}^R_n(N, M) = 0$ $n \gg 0$.

(ii) $\operatorname{Tor}^R_n(N, M) = 0$ $n > \mathrm{CIfd}_R M$.

(b) If R *is a complete local ring and* $\mathrm{CIfd}_R N < \infty$*, and* M *a finite* R*-module, then the following are equivalent:*

(i) $\operatorname{Ext}^n_R(N, M) = 0$ $n \gg 0$.

(ii) $\operatorname{Ext}^n_R(N, M) = 0$ $n > \operatorname{depth} R - \operatorname{depth}_R M$.

Proof. (a) If for all integer n, $\operatorname{Tor}_n^R(M,N)=0$, then the assertion holds. So assume for some integer ℓ, $\operatorname{Tor}_\ell^R(M,N)\neq 0$. Therefore $s=\operatorname{fd}_R(M,N)<\infty$. Now by Theorem 3.3.2, $s=\operatorname{depth} R_{\mathfrak{p}}-\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}-\operatorname{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ for some $\mathfrak{p}\in\operatorname{Supp}(M)\cap\operatorname{Supp}(N)$. Now choose an integer $n>\operatorname{CIfd}_R M=\operatorname{Rfd}_R M\geq \operatorname{depth} R_{\mathfrak{p}}-\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}\geq s$. Therefore $\operatorname{Tor}_n^R(N,M)=0$.

(b) It follows easily from Corollary 3.3.7(b). □

3.4 Homological Injective Dimensions

It is well known that flat dimension and injective dimension are dual of each other. In particular there are the following equalities:

$$\operatorname{fd}_R M^\vee=\operatorname{id}_R M \textit{ and } \operatorname{id}_R M^\vee=\operatorname{fd}_R M,$$

where $M^\vee=\operatorname{Hom}_R(M,E_R(k))$ and $E_R(k)$ is the injective envelope of k over R. In this section we introduce dual of the complete intersection flat dimension and the Cohen-Macaulay flat dimension.

Definition 3.4.1. *Let $M\neq 0$ be an R-module. The complete intersection injective dimension, upper Gorenstein injective dimension, and upper Cohen-Macaulay injective dimension of M, are defined as:*

$$\operatorname{CI}\operatorname{id}_R M:=\inf\{\operatorname{id}_Q M'-\operatorname{fd}_Q R'|\ R\to R'\leftarrow Q \text{ is a } \textit{CI}\text{-quasi-deformation}\}$$

$$\operatorname{G}^*\operatorname{id}_R M:=\inf\{\operatorname{id}_Q M'-\operatorname{fd}_Q R'|\ R\to R'\leftarrow Q \text{ is a } \textit{G}^*\text{-quasi-deformation}\}$$

$$\operatorname{CM}^*\operatorname{id}_R M:=\inf\{\operatorname{id}_Q M'-\operatorname{fd}_Q R'|\ R\to R'\leftarrow Q \text{ is a } \operatorname{CM}^*\text{-quasi-deformation}\},$$

respectively. We complement this by $\operatorname{H}\operatorname{fd}_R 0=-\infty$ *for* $\operatorname{H}=\operatorname{CI}$, G^*, *and* CM^*.

In [86, Definition 2.6] Sather-Wagstaff introduced the upper complete intersection injective dimension of an R-module M as

$$\mathrm{CI}^*\,\mathrm{id}_R\, M := \inf \left\{ \mathrm{id}_Q\, M' - \mathrm{fd}_Q\, R' \,\middle|\, \begin{array}{l} R \to R' \leftarrow Q \textit{ is a } \mathrm{CI}\textit{-quasi-deformation} \\ \textit{such that } R' \textit{ has Gorenstein formal} \\ \textit{fibre and } R'/\mathfrak{m}R' \textit{ is Gorenstein} \end{array} \right\}.$$

We use the upper complete intersection injective dimension for a dual version of Theorem 3.2.6. The following theorem shows that the upper Cohen-Macaulay injective dimension characterizes Cohen-Macaulay local rings.

Theorem 3.4.2. *The following conditions are equivalent.*

(i) *The ring R is Cohen-Macaulay.*

(ii) $\mathrm{CM}^*\,\mathrm{id}_R\, M < \infty$ *for every R-module M.*

(iii) $\mathrm{CM}^*\,\mathrm{id}_R\, M < \infty$ *for every finite R-module M.*

(iv) $\mathrm{CM}^*\,\mathrm{id}_R\, k < \infty$.

Proof. $(i) \Rightarrow (ii)$ Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Since R is Cohen-Macaulay, so is $\widehat{R}$. Therefore by Cohen's structure theorem, $\widehat{R}$ is isomorphic to Q/J, where Q is a local regular ring. Hence due to Cohen-Macaulay-ness of $\widehat{R}$ and regularity of Q, the ideal J is perfect. Thus $R \to \widehat{R} \leftarrow Q$ is a CM^*-quasi-deformation. Since Q is regular $\mathrm{id}_Q(M \otimes_R \widehat{R})$ is finite, so $\mathrm{CM}^*\,\mathrm{id}_R\, M$ is finite.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (i)$ Suppose $\mathrm{CM}^*\,\mathrm{id}_R\, k < \infty$. So that there exists a CM^*-quasi-deformation $R \to R' \leftarrow Q$, such that $\mathrm{id}_Q(k \otimes_R R')$ is finite. It is clear that $k \otimes_R R'$ is a finite

Q-module. Consequently Q is a Cohen-Macaulay ring by the Bass Theorem. We plan to show that R' is a Cohen-Macaulay ring. Let $I = \ker(Q \to R')$ which is perfect by definition. We have

$$\begin{aligned} \operatorname{ht} I &= \operatorname{grade}(I, Q) \\ &= \operatorname{pd}_Q R' \\ &= \operatorname{depth} Q - \operatorname{depth}_Q R' \\ &= \operatorname{depth} Q - \operatorname{depth} R' \\ &= \dim Q - \operatorname{depth} R' \\ &= \operatorname{ht} I + \dim R' - \operatorname{depth} R', \end{aligned}$$

in which the equalities follow from Cohen-Macaulay-ness of Q; perfectness of I; Auslander-Buchsbaum formula; [17, (1.2.26)]; Cohen-Macaulay-ness of Q; and [73, Page 250] respectively. Therefore we obtain that $\dim R' - \operatorname{depth} R' = 0$, that is R' is Cohen-Macaulay. Now [17, Theorem (2.1.7)] gives us the desired result. □

In the same way one can show that the upper Gorenstein injective dimension detects the Gorenstein property and the complete intersection injective dimension detects the complete intersection propoerty of local rings.

The proof of the above theorem says some thing more, viz., a local ring R is Cohen-Macaulay if and only if there exists a finite R-module of finite upper Cohen-Macaulay injective dimension. In other words every finite R-module is a test module for the Cohen-Macaulay property of a local ring. So we state the following corollary, which is analogous to the definition of a Gorenstein ring:

Corollary 3.4.3. *A local ring R is Cohen-Macaulay if and only if* $\mathrm{CM}^*\operatorname{id}_R R < \infty$*.*

In [55, Theorem (4.5)] Foxby and Frankild proved that if a ring admits a cyclic module of finite Gorenstein injective dimension, then the base ring is Gorenstein. Parallel to their result we have the the following proposition.

Proposition 3.4.4. *If $G^*id_R C < \infty$ for a cyclic R-module C, then R is a Gorenstein local ring.*

Proof. There is a G^*-quasi-deformation $R \to R' \leftarrow Q$ such that $\operatorname{id}_Q(C \otimes_R R')$ is finite. Since $C \otimes_R R'$ is a cyclic R'-module and R' is a cyclic module over Q, we see that $C \otimes_R R'$ is a cyclic module over Q. So that Q is Gorenstein by [79]. Hence R' is a Gorenstein ring because the kernel of $Q \to R'$ is a Gorenstein ideal. Consequently R is Gorenstein. □

Lemma 3.4.5. *There is an equality*

$$\mathrm{H}\operatorname{id}_R M = \inf\left\{ \operatorname{id}_Q M' - \operatorname{fd}_Q R' \;\middle|\; \begin{array}{c} R \to R' \leftarrow Q \text{ is an H-quasi-deformation such} \\ \text{that the closed fibre of } R \to R' \text{ is artinian} \end{array} \right\},$$

for $\mathrm{H} = \mathrm{CI}^*$, CI, G^*, *and* CM^*.

Proof. We prove the lemma for $\mathrm{H} = \mathrm{CM}^*$ only since the other cases are similar. Let $R \to R' \leftarrow Q$ be an CM^*-quasi-deformation. Choose $\mathfrak{p} \in \operatorname{Spec}(R')$ such that it is a minimal prime ideal containing $\mathfrak{m}R'$; thus $\mathfrak{m} = \mathfrak{p} \cap R$ and $\mathfrak{p} = \mathfrak{q}/J$ for some $\mathfrak{q} \in \operatorname{Spec}(Q)$, where $J = \ker(Q \to R')$. Now the diagram $R \to R'_{\mathfrak{p}} \leftarrow Q_{\mathfrak{q}}$ is a CM^*-quasi-deformation. It is clear that $\operatorname{pd}_Q R' = \operatorname{pd}_{Q_{\mathfrak{q}}} R'_{\mathfrak{p}}$. Also we have

$$\operatorname{id}_{Q_{\mathfrak{q}}}(M \otimes_R R'_{\mathfrak{p}}) = \operatorname{id}_{Q_{\mathfrak{q}}}(M \otimes_R (R' \otimes_Q Q_{\mathfrak{q}})) = \operatorname{id}_{Q_{\mathfrak{q}}}(M' \otimes_Q Q_{\mathfrak{q}}) \leq \operatorname{id}_Q M' < \infty.$$

Hence $\mathrm{CM}^*\operatorname{id}_R M \leq \operatorname{id}_{Q_{\mathfrak{q}}}(M \otimes_R R'_{\mathfrak{p}}) - \operatorname{pd}_{Q_{\mathfrak{q}}} R'_{\mathfrak{p}}$. So the proof in complete. □

Recall that $\operatorname{width}_R M = \inf\{i| \operatorname{Tor}_i^R(M,k) \neq 0\}$. It is the dual notion for $\operatorname{depth}_R M$. In particular by [24, (4.8)] we have $\operatorname{width}_R M = \operatorname{depth}_R \operatorname{Hom}_R(M, E_R(k))$, where $E_R(k)$ denote for injective envelope of k over R.

The *Chouinard injective dimension* is denoted by $\operatorname{Ch}\operatorname{id}_R M$ and is defined as,

$$\operatorname{Ch}\operatorname{id}_R M := \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Remark 3.4.6. *We do not introduce Chouinard flat dimension because it coincides with* Rfd *since by [24, (2.4)(b)], we have*

$$\operatorname{Rfd}_R M := \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

It is proved in [23] that for an R-module M, $\operatorname{Ch}\operatorname{id}_R M$ is a refinement of $\operatorname{id}_R M$, that is

$$\operatorname{Ch}\operatorname{id}_R M \leq \operatorname{id}_R M,$$

with equality if $\operatorname{id}_R M$ is finite.

In the Theorem 3.4.11 we partly extend this relation for our homological injective dimensions. As of the writing of this result, the author do not know if the equality holds in general. Before stating the theorem we need some lemmas.

Lemma 3.4.7. *Suppose that $Q \to S$ is a surjective local homomorphism and N is an S-module. Then we have*

$$\operatorname{width}_S N = \operatorname{width}_Q N.$$

Proof. We have the following equalities:

$$
\begin{aligned}
\operatorname{width}_S N &= \operatorname{depth}_S \operatorname{Hom}_S(N, E_S(k)) \\
&= \operatorname{depth}_S \operatorname{Hom}_S(N, \operatorname{Hom}_Q(S, E_Q(k))) \\
&= \operatorname{depth}_S \operatorname{Hom}_Q(N, E_Q(k)) \\
&= \operatorname{depth}_Q \operatorname{Hom}_Q(N, E_Q(k)) \\
&= \operatorname{width}_Q N,
\end{aligned}
$$

where the first one is by [24, (4.8)]; the second one is by [16, (10.1.15)]; the third one is by adjointness of Hom and tensor; the fourth one is true since $Q \to S$ is surjective; while the last one is again by [24, (4.8)]. Here we used k for the residue fields of Q and S, and $E_Q(k)$ and $E_S(k)$ for the injective envelopes of k over respectively Q and S. □

Dualizing the proof of Proposition 3.1.2 and using the above lemma one easily shows

Proposition 3.4.8. *Let $Q \to S$ be a* CM*-deformation, and N be an S-module. Then there is the equality:*

$$\operatorname{Ch id}_S N + \text{G-dim}_Q S = \operatorname{Ch id}_Q N.$$

Lemma 3.4.9. *Suppose that $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is a local ring homomorphism, and M is an R-module. Then we have*

$$\operatorname{width}_S(M \otimes_R S) = \operatorname{width}_R M.$$

Proof. Let F_M be a flat resolution of M over R. Therefore $F_M \otimes_R S$ is a flat resolution of $M \otimes_R S$ over S. So we have

$$\begin{aligned}
\operatorname{width}_S(M \otimes_R S) &= \inf\{i | \operatorname{Tor}_i^S(M \otimes_R S, l) \neq 0\} \\
&= \inf\{i | \operatorname{H}_i((F_M \otimes_R S) \otimes_S l) \neq 0\} \\
&= \inf\{i | \operatorname{H}_i(F_M \otimes_R l) \neq 0\} \\
&= \inf\{i | \operatorname{Tor}_i^R(M, l) \neq 0\} \\
&= \inf\{i | \operatorname{Tor}_i^R(M, k) \neq 0\} \\
&= \operatorname{width}_R M.
\end{aligned}$$

□

Lemma 3.4.10. *Let $R \to S$ be a flat local homomorphism and let M be an R-module. Then*

$$\operatorname{Ch}\operatorname{id}_R M \leq \operatorname{Ch}\operatorname{id}_S(M \otimes_R S).$$

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\operatorname{Ch}\operatorname{id}_R M = \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p}$. Let $\mathfrak{q} \in \operatorname{Spec}(S)$ contain $\mathfrak{p}S$ minimally. Since $R \to S$ is a flat local homomorphism we have $\mathfrak{p} = \mathfrak{q} \cap R$ and $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q}$. Hence:

$$\begin{aligned}
\operatorname{Ch}\operatorname{id}_R M &= \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} \\
&= \operatorname{depth} S_\mathfrak{q} - \operatorname{width}_{S_\mathfrak{q}}(M_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}) \\
&= \operatorname{depth} S_\mathfrak{q} - \operatorname{width}_{S_\mathfrak{q}}(M \otimes_R S)_\mathfrak{q} \\
&\leq \operatorname{Ch}\operatorname{id}_S(M \otimes_R S),
\end{aligned}$$

in which the second equality holds by Lemma 3.4.9 and the fact that $R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ has artinian closed fibre. $\square$

Theorem 3.4.11. *Suppose that M is an R-module such that $\mathrm{H}\,\mathrm{id}_R\, M < \infty$ for* $\mathrm{H} = CI^*$, CI, G^*, *or* CM^*. *Then there is the inequality*

$$\mathrm{Ch}\,\mathrm{id}_R\, M \leq \mathrm{H}\,\mathrm{id}_R\, M,$$

and if M is a finite module we have

$$\mathrm{Ch}\,\mathrm{id}_R\, M = \mathrm{H}\,\mathrm{id}_R\, M = \operatorname{depth} R.$$

Proof. We prove the theorem for $\mathrm{H} = \mathrm{CM}^*$ and the other cases are similar. Choose by Lemma 3.4.5 a CM^*-quasi-deformation $R \to R' \leftarrow Q$, such that $\mathrm{CM}^*\,\mathrm{id}_R\, M = \mathrm{id}_Q\, M' - \mathrm{fd}_Q\, R'$, where $M' = M \otimes_R R'$, and the closed fibre of $R \to R'$ is artinian. Hence we have

$$\begin{aligned}\mathrm{CM}^*\,\mathrm{id}_R\, M =& \,\mathrm{id}_Q\, M' - \mathrm{fd}_Q\, R' \\ =& \,\mathrm{Ch}\,\mathrm{id}_Q\, M' - \mathrm{fd}_Q\, R' \\ =& \,\mathrm{Ch}\,\mathrm{id}_{R'}\, M' \geq \mathrm{Ch}\,\mathrm{id}_R\, M,\end{aligned}$$

in which the second equality comes by [23], and the third one by Proposition 3.4.8; while the inequality is by Lemma 3.4.10.

Now let M be a finite R-module, therefore M' is a finite Q-module. So by the Bass Theorem [73, (18.9)], and the Auslander-Buchsbaum formula, and the fact that the closed fibre of $R \to R'$ is artinian we have:

$$\mathrm{CM}^*\,\mathrm{id}_R\, M = \mathrm{id}_Q\, M' - \mathrm{fd}_Q\, R' = \operatorname{depth} Q - \operatorname{depth} Q + \operatorname{depth} R' = \operatorname{depth} R' = \operatorname{depth} R.$$

On the other hand, since $\mathrm{CM}^* \operatorname{id}_R M < \infty$ and M is finite, R is a Cohen-Macaulay ring. Thus using Lemma 2.2.6 we see that $\mathrm{Ch} \operatorname{id}_R M = \operatorname{depth} R$. Hence $\mathrm{CM}^* \operatorname{id}_R M = \mathrm{Ch} \operatorname{id}_R M = \operatorname{depth} R$. □

Corollary 3.4.12. *Let M be an R-module. Then we have the following chain of inequalities:*

$$\mathrm{Ch} \operatorname{id}_R M \leq \mathrm{CM}^* \operatorname{id}_R M \leq \mathrm{G}^* id_R M \leq \mathrm{CIid}_R M \leq \mathrm{CI}^* \operatorname{id}_R M \leq \operatorname{id}_R M,$$

with equality to the left of any finite number for finite modules or, if $\operatorname{id}_R M < \infty$ for arbitrary module M.

Now we prove the dual result of Theorem 3.2.6.

Theorem 3.4.13. *Suppose that R has a dualizing complex D, and M is an R-module. Then there is an inequality*

$$\mathrm{Gid}_R M \leq \mathrm{CI}^* \operatorname{id}_R M.$$

Proof. We can actually assume that $\mathrm{CI}^* \operatorname{id}_R M$ is finite. So that by [86, Proposition 3.5], there exists a CI-quasi-deformation $R \to R' \leftarrow Q$ such that Q is complete, the closed fibre $R'/\mathfrak{m}R'$ is artinian and Gorenstein and $\operatorname{id}_Q(M \otimes_R R')$ is finite. Therefore by Remark 1.1.5(b), $M \otimes_R R'$ belongs to the the Bass class $\mathbf{B}(Q)$. By the same argument as in the proof of Theorem 3.2.6 one can show that $M \otimes_R R'$ belongs to the Bass class $\mathbf{B}(R')$. Consequently Remark 1.1.5(b), gives us that $\mathrm{Gid}_{R'}(M \otimes_R R') < \infty$. Since $R'/\mathfrak{m}R'$ is a Gorenatein local ring, by [9, Proposition 4.2], we have $R \to R'$ is a Gorenstein local homomorphism. Therefore by [9, Theorem 5.1], the complex $D \otimes^{\mathbf{L}}_R R'$ is a dualizing complex of R'. Consequently [25, Theorem 5.3] gives finiteness of $\operatorname{Gid}_R M$. Hence using [25, Theorem 6.8] we have $\operatorname{Gid}_R M = \operatorname{Ch}\operatorname{id}_R M \leq \operatorname{CI}^* \operatorname{id}_R M$ as desired. □

By the above theorem we have

$$\operatorname{Ch}\operatorname{id}_R M \leq \operatorname{Gid}_R M \leq \operatorname{CI}^* \operatorname{id}_R M \leq \operatorname{id}_R M.$$

Now we define a Cohen-Macaulay injective dimension to complete this sequence of inequalities. Notice that there is a notion of Cohen-Macaulay injective dimension in [62] which is different with ours.

Definition 3.4.14. *Let $M \neq 0$ be an R-module. The Cohen-Macaulay injective dimension of M, is defined by ,*

$$\operatorname{CMid}_R M := \inf\{\operatorname{Gid}_Q M' - \operatorname{Gfd}_Q R' \mid R \to R' \leftarrow Q \text{ is a } \operatorname{CM}\text{-quasi-deformation}\}.$$

We complement this by $\operatorname{CMid}_R 0 = -\infty$.

Therefore by taking the trivial CM-quasi-deformation $R \to R \leftarrow R$, one has $\operatorname{CMid}_R M \leq \operatorname{Gid}_R M$. Suppose that R has a dualizing complex. Then by [25, Proposion 5.5] one has $\operatorname{Gid}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \operatorname{Gid}_R M$ for each prime ideal $\mathfrak{p}$ of R. So that by the proof of Lemma 3.4.5, one has

$$\operatorname{CMid}_R M = \inf \left\{ \operatorname{Gid}_Q M' - \operatorname{Gfd}_Q R' \left| \begin{array}{l} R \to R' \leftarrow Q \textit{ is a } \operatorname{CM}\textit{-quasi-deformation} \\ \textit{such that the closed fibre of} \\ R \to R' \textit{ is artinian} \end{array} \right. \right\}.$$

Therefore as in Theorem 3.4.11 one can show that $\operatorname{Ch}\operatorname{id}_R M \leq \operatorname{CM}\operatorname{id}_R M$. Hence when the ring R admits a dualizing complex, then there is the following sequence of inequalities

$$\operatorname{Ch}\operatorname{id}_R M \leq \operatorname{CMid}_R M \leq \operatorname{Gid}_R M \leq \operatorname{CI}^* \operatorname{id}_R M \leq \operatorname{id}_R M,$$

with equality to the left of any finite number for finite modules or, if $\operatorname{id}_R M$ or $\operatorname{Gid}_R M$ if finite, for arbitrary module M.

Theorem 3.4.15. *The following conditions are equivalent.*

(i) The ring R is Cohen-Macaulay.

(ii) $\operatorname{CMid}_R M < \infty$ *for every R-module M.*

(iii) $\operatorname{CMid}_R M < \infty$ *for every finite R-module M.*

(iv) $\operatorname{CMid}_R k < \infty$.

Proof. $(i) \Rightarrow (ii)$ It follows by the inequality $\operatorname{CMid}_R M \leq \operatorname{CM}^* \operatorname{id}_R M$ and Theorem 3.4.2.

$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are trivial.

$(iv) \Rightarrow (i)$ Suppose $\operatorname{CM}^* \operatorname{id}_R k < \infty$. So there exists a CM-quasi-deformation $R \to R' \leftarrow Q$, such that $\operatorname{Gid}_Q(k \otimes_R R')$ is finite. Since $k \otimes_R R'$ is a cyclic module over Q, we see that Q is a Gorenstein ring by [55, Theorem 4.5]. The rest of proof is the same as that of Theorem 3.4.2. $\square$

3.5 The Auslander-Buchsbaum formula

In this section we give a necessary and sufficient condition for our homological flat dimensions to satisfy a formula of Auslander-Buchsbaum type. Our main result is Theorem 3.5.4 below. Recall that $\operatorname{grade}(\mathfrak{p}, M) = \inf\{i \mid \operatorname{Ext}^i_R(R/\mathfrak{p}, M) \neq 0\}$.

Lemma 3.5.1. *Let R be a local ring, and M an R-module of finite depth. Then* $\operatorname{depth}_R M \leq \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$ *if and only if* $\operatorname{depth}_R M \leq \operatorname{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof. The only if part is trivial. For the if part we show that for modules M and N with N finite we have $\operatorname{Ext}^i_R(N, M) = 0$ for $i < \operatorname{depth}_R M - \dim N$. We do this by induction on $\dim N$. If $\dim N = 0$, then N has finite length, and in this case an easy induction proves the result. Now let $\dim N = t$. By a method similar to that of [73, (17.1)] it is sufficient to take $N = R/\mathfrak{p}$ such that $\dim R/\mathfrak{p} = t$. Let $s < \operatorname{depth}_R M - t \leq \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$. We have to show that $E = \operatorname{Ext}^s_R(R/\mathfrak{p}, M) = 0$. If $E \neq 0$, there is a non-zero element $e \in E$. Since $E_\mathfrak{p} = 0$ there is an element $u \in R \backslash \mathfrak{p}$ such that $ue = 0$. Now the exact sequence $0 \to R/\mathfrak{p} \xrightarrow{.u} R/\mathfrak{p} \to N' \to 0$, gives rise the exact sequence

$$\operatorname{Ext}^s_R(N', M) \to \operatorname{Ext}^s_R(R/\mathfrak{p}, M) \xrightarrow{.u} \operatorname{Ext}^s_R(R/\mathfrak{p}, M),$$

in which the left most module equal to zero by the induction hypothesis. So u is injective and therefore $e = 0$, which is a contradiction. □

Proposition 3.5.2. *Let M be an R-module such that* $\operatorname{Rfd}_R M + \operatorname{depth}_R M = \operatorname{depth} R$. *Then*

$$\operatorname{depth}_R M \leq \operatorname{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$. *The converse is true over Cohen-Macaulay rings.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be an arbitrary prime ideal. Therefore we have

$$\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \operatorname{Rfd}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \operatorname{Rfd}_R M = \operatorname{depth} R - \operatorname{depth}_R M.$$

So that

$$\begin{aligned}\operatorname{depth}_R M &\leq \operatorname{depth} R - \operatorname{depth} R_{\mathfrak{p}} + \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\leq \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p}.\end{aligned}$$

Now from Lemma 3.5.1 we obtain that $\operatorname{depth}_R M \leq \operatorname{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Next suppose that R is a Cohen-Macaulay ring. Choose a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\operatorname{Rfd}_R M = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. Then from the hypothesis and [73, Page 250] we have:

$$\begin{aligned}\operatorname{Rfd}_R M &= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &= \dim R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\leq \dim R - \dim R/\mathfrak{p} - \operatorname{grade}(\mathfrak{p}, M) \\ &\leq \dim R - \operatorname{depth}_R M \\ &= \operatorname{depth} R - \operatorname{depth}_R M \\ &\leq \operatorname{Rfd}_R M,\end{aligned}$$

which completes the proof. □

Combining Proposition 3.5.2, and [24, Theorem (3.4)] we have

Corollary 3.5.3. *Let R be a ring and M be an R-module. The following then are equivalent:*

(i) $\operatorname{Rfd}_R M + \operatorname{depth}_R M = \operatorname{depth} R$ *for every R-module M of finite depth.*

(ii) R is a Cohen-Macaulay ring and $\operatorname{depth}_R M \leq \operatorname{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$ *for every R-module M of finite depth, and for all* $\mathfrak{p} \in \operatorname{Spec}(R)$*.*

This is an extension of [24, Theorem (3.4)]. Now we state the main result of this section.

Theorem 3.5.4. *Let R be a Cohen-Macaulay local ring and let M be an R-module of finite* $\mathrm{H}\operatorname{fd}_R M$ *for* $\mathrm{H} = CI$, G^*, CM^*, *and* CM. *Then* $\mathrm{H}\operatorname{fd}_R M + \operatorname{depth}_R M = \operatorname{depth} R$ *if and only if* $\operatorname{depth}_R M \leq \operatorname{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Supp}(M)$*.*

Dual to the Proposition 3.5.2 one can prove the following.

Proposition 3.5.5. *Let R be a local ring, and M an R-module such that* $Ch\operatorname{id}_R M + \operatorname{depth}_R M = \operatorname{depth} R$*. Then*

$$\operatorname{width}_R M \leq \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim R/\mathfrak{p}$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$*. The converse is true over Cohen-Macaulay rings.*

Chapter 4

Filter ring under flat base change

4.1 Filter rings

Throughout this chapter, all modules are finite (i.e. finitely generated). In [27] Cuong, Schenzel, and Trung introduced the notion of filter regular sequence as an extension of the more known concept of regular sequence. By using this notion they defined the filter modules. An R-module is called *filter module* if every system of parameters is a filter regular sequence. On the other hand they studied modules M such that the length of the i-th local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(M)$ is finite for all $i < \dim M$. These kind of modules are called *generalized Cohen-Macaulay*. In general, every generalized Cohen-Macaulay module is a filter module, and the converse is true when R is a quotient of a Cohen-Macaulay ring. Consider the following standard fact, cf. [17, Theorem 2.1.7]: Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings. Then S is a Cohen-Macaulay ring if and only if R and $S/\mathfrak{m}S$ are Cohen–Macaulay.

It seems natural to ask: what parts of the above results hold when we replace "a Cohen-Macaulay ring" with "a filter ring". In this section we answer this.

Let $(R, \mathfrak{m})$ be a local ring. Recall from [27] that a sequence $x_1, \cdots, x_n$ of elements in $\mathfrak{m}$ is said to be an M-filter regular sequence if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(M/(x_1, \cdots, x_{i-1})M)\backslash\{\mathfrak{m}\}$ for all $i = 1, ..., n$.

Definition 4.1.1. *An R-module M is called filter module if every system of parameters of M is an M-filter regular sequence. A ring is called filter ring if it is a filter module over itself.*

Proposition 4.1.2. *([27, Satz 2.5]) Let $d = \dim M > 0$. Then the following are equivalent:*

(1) M is a filter module.

(2) $\mathrm{depth}\, M_{\mathfrak{p}} = d - \dim R/\mathfrak{p}$*, for all* $\mathfrak{p} \in \mathrm{Supp}(M)\backslash\{\mathfrak{m}\}$.

(3) $M_{\mathfrak{p}}$ is Cohen-Macaulay, for all $\mathfrak{p} \in \mathrm{Supp}(M)\backslash\{\mathfrak{m}\}$*, and* $\mathrm{Supp}(M)$ *is catenary and equidimensional.*

(4) $M_{\mathfrak{p}}$ is Cohen-Macaulay, of dimension $\dim M_{\mathfrak{p}} = \dim M - \dim R/\mathfrak{p}$*, for all* $\mathfrak{p} \in \mathrm{Supp}(M)\backslash\{\mathfrak{m}\}$.

Now we are ready to prove the main theorem of this section.

Theorem 4.1.3. *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension of local rings. Then the following hold:*

(a) *If* $\dim S/\mathfrak{m}S > 0$*, then* S *is a filter ring if and only if* R *and* $k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ *are Cohen-Macaulay for all* $\mathfrak{q} \in \operatorname{Spec}(S)\backslash\{\mathfrak{n}\}$ *and* $\mathfrak{p} = \mathfrak{q} \cap R$*, and* $S/\mathfrak{p}S$ *is catenary and equidimensional for all minimal prime ideal* $\mathfrak{p}$ *of* R*.*

(b) *If* $\dim S/\mathfrak{m}S = 0$*, then* S *is a filter ring if and only if* R *is a filter ring and* $k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ *is Cohen-Macaulay for all* $\mathfrak{q} \in \operatorname{Spec}(S)\backslash\{\mathfrak{n}\}$ *and* $\mathfrak{p} = \mathfrak{q} \cap R$*, and* $S/\mathfrak{p}S$ *is catenary and equidimensional for all minimal prime ideal* $\mathfrak{p}$ *of* R*.*

Proof. To prove (a) suppose S is a filter ring. Choose $\mathfrak{q} \in \operatorname{Spec}(S)\backslash\{\mathfrak{n}\}$ such that $\mathfrak{m} = \mathfrak{q} \cap R$. Consider the flat local homomorphism $R \to S_\mathfrak{q}$. By Proposition 4.1.2 $S_\mathfrak{q}$ is Cohen-Macaulay and so R is Cohen-Macaulay. Let $\mathfrak{q} \in \operatorname{Spec}(S)\backslash\{\mathfrak{n}\}$ and set $\mathfrak{p} = \mathfrak{q} \cap R$. Since $R_\mathfrak{p} \to S_\mathfrak{q}$ is flat homomorphism of local rings, and $S_\mathfrak{q}$ is Cohen-Macaulay, hence $k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ is Cohen-Macaulay too. In this case $S/\mathfrak{p}S$ is catenery and equidimensional for all $\mathfrak{p} \in \operatorname{Min}(R)$, cf. [93, Theorem 1.4]. Conversely, suppose that R and $k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ are Cohen-Macaulay for all $\mathfrak{q} \in \operatorname{Spec}(S)\backslash\{\mathfrak{n}\}$ and $\mathfrak{p} = \mathfrak{q} \cap R$ and $S/\mathfrak{p}S$ is catenary and equidimensional for all $\mathfrak{p} \in \operatorname{Min}(R)$. Let $\mathfrak{q} \in \operatorname{Spec}(S)\backslash\{\mathfrak{n}\}$, and $\mathfrak{p} = \mathfrak{q} \cap R$. From the flat local extension $R_\mathfrak{p} \to S_\mathfrak{q}$, $S_\mathfrak{q}$ is Cohen-Macaulay. By catenarity and equidimensionality assumption on $S/\mathfrak{p}S$ for all $\mathfrak{p} \in \operatorname{Min}(R)$, S is catenary and equidimensional, cf. [93, Theorem 1.4]. Therefore $\dim S_\mathfrak{q} = \dim S - \dim S/\mathfrak{q}$, cf. [73, Page 250]. This ends the proof of (a).

To prove (b) suppose that S is a filter ring. Then in particular R and $S/\mathfrak{p}S$ are catenary and equidimensional for all $\mathfrak{p} \in \operatorname{Min}(R)$. Since $\dim S/\mathfrak{m}S = 0$, then for all $\mathfrak{p} \in \operatorname{Spec}(R)\backslash\{\mathfrak{m}\}$ there is a prime ideal $\mathfrak{q} \in \operatorname{Spec}(S)\backslash\{\mathfrak{n}\}$ such that $\mathfrak{p} = \mathfrak{q} \cap R$. Consider the flat local homomorphism $R_\mathfrak{p} \to S_\mathfrak{q}$. Since $S_\mathfrak{q}$ is Cohen-Macaulay by assumption, then $R_\mathfrak{p}$ is Cohen-Macaulay of dimension $\dim R_\mathfrak{p} = \dim R - \dim R/\mathfrak{p}$, and $k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ is Cohen-Macaulay. Now by Proposition 4.1.2 R is a filter ring. Conversely, assume that R is a filter ring and $k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ is Cohen-Macaulay for all $\mathfrak{q} \in \operatorname{Spec}(S)\backslash\{\mathfrak{n}\}$ and $\mathfrak{p} = \mathfrak{q} \cap R$, and for all $\mathfrak{p} \in \operatorname{Min}(R)$, $S/\mathfrak{p}S$ is catenary and equidimensional. By the same way as above S is catenary and equidimensional. So that, by [73, Page 250], $\dim S_\mathfrak{q} = \dim S - \dim S/\mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Spec}(S)\backslash\{\mathfrak{n}\}$. Now for $\mathfrak{q} \in \operatorname{Spec}(S)\backslash\{\mathfrak{n}\}$, let $\mathfrak{p} = \mathfrak{q} \cap R$. From the flat local homomorphism $R_\mathfrak{p} \to S_\mathfrak{q}$ and Cohen-Macaulayness of $k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$, $S_\mathfrak{q}$ is a Cohen-Macaulay ring. Therefore by Proposition 4.1.2, S is a filter ring. □

In the following corollary a ring R is said to be a locally filter ring, if $R_\mathfrak{p}$ is a filter ring for each prime ideal $\mathfrak{p}$.

Corollary 4.1.4. *Let k be a field. Let R and S be two k-algebras, such that $R \otimes_k S$ is a locally filter ring, then R and S are locally filter rings.*

Proof. Consider the natural flat homomorphism $R \to R \otimes_k S$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Since LO (lying over) happens between R and $R \otimes_k S$, there is a prime ideal $\mathfrak{q} \in \operatorname{Spec}(R \otimes_k S)$, such that $\mathfrak{q} \cap R = \mathfrak{p}$. Then $R_\mathfrak{p} \to (R \otimes_k S)_\mathfrak{q}$ is a flat local homomorphism of local rings. Hence $R_\mathfrak{p}$ is a filter ring, that means R is a locally filter ring. □

Corollary 4.1.5. *Let k be a field and let R be a k-algebra and K an algebraic field extension of k, such that $K \otimes_k R$ is locally equidimensional. Then R is a locally filter ring if and only if $K \otimes_k R$ is a locally filter ring.*

Proof. "Only if" Let $\mathfrak{n}$ be a maximal ideal of $K \otimes_k R$, since the natural homomorphism $R \to K \otimes_k R$ is integral (it means that $K \otimes_k R$ is integral over R), then $\mathfrak{n} \cap R = \mathfrak{m}$ is a maximal ideal of R. Consider the flat local homomorphism $R_\mathfrak{m} \to (K \otimes_k R)_\mathfrak{n}$, and note that $\dim(K \otimes_k R)_{\mathfrak{n}}/(\mathfrak{m}R_{\mathfrak{m}})(K \otimes_k R)_{\mathfrak{n}} = 0$. For $\mathfrak{Q} \in \operatorname{Spec}(K \otimes_k R)$ with $\mathfrak{Q} \subsetneqq \mathfrak{n}$, set $\mathfrak{p}R_{\mathfrak{m}} = \mathfrak{Q}(K \otimes_k R)_{\mathfrak{n}} \cap R_{\mathfrak{m}}$. We have the following isomorphism:

$$(R_{\mathfrak{m}})_{\mathfrak{p}R_{\mathfrak{m}}}/(\mathfrak{p}R_{\mathfrak{m}})(R_{\mathfrak{m}})_{\mathfrak{p}R_{\mathfrak{m}}} \otimes_{(R_{\mathfrak{m}})_{\mathfrak{p}R_{\mathfrak{m}}}} [(K \otimes_k R)_{\mathfrak{n}}]_{\mathfrak{Q}(K\otimes_k R)_{\mathfrak{n}}} \simeq R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (K \otimes_k R)_{\mathfrak{Q}}.$$

Since by [92, Theorem 6] $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \otimes_R (K \otimes_k R) \simeq R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \otimes_k K$ is Cohen-Macaulay we get that $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (K \otimes_k R)_{\mathfrak{Q}}$ is Cohen-Macaulay too. Since R is catenary, $K \otimes_k R$ is catenary too, cf. [17, (2.1.11)]. Hence by keeping in mind the locally equidimensionality assumption on $K \otimes_k R$ we get the result using Theorem 4.1.3(b).

□

Theorem 4.1.6. *Suppose $(R, \mathfrak{m})$ is a filter ring. Let I be an ideal of R generated by an R-filter regular sequence. Then R/I is a filter ring.*

Proof. If $\dim R/I = 0$, then R/I is already a filter ring. So suppose that $\dim R/I > 0$. Let $\mathfrak{p} \in \operatorname{Min}(I)$ such that $\dim R/\mathfrak{p} = \dim R/I$. Assume I is generated by R-filter regular sequence $x_1, \ldots, x_s$. Then by [70, (3.5)] we have

$$s \leqslant \text{f-depth}(I, R) \leqslant \operatorname{ht} I \leqslant \operatorname{ht} \mathfrak{p} \leqslant s,$$

where f-depth(I, R) is the maximum length of an R-filter regular sequence in I. Therefore $s = \operatorname{ht} I = \operatorname{ht} \mathfrak{p}$. In order to prove that R/I is filter ring, using the Proposition 2.2, it is enough to prove that for all $\mathfrak{q}/I \in \operatorname{Spec}(R/I)\backslash\{\mathfrak{m}/I\}$, we have

$$\operatorname{depth}(R/I)_{\mathfrak{q}/I} = \dim R/I - \dim R/\mathfrak{q}.$$

But since R is a filter ring and $\mathfrak{q} \neq \mathfrak{m}$, $R_{\mathfrak{q}}$ is Cohen-Macaulay, and $IR_{\mathfrak{q}}$ is generated by a regular sequence, we have the following equalities:

$$\begin{aligned}\operatorname{depth}(R/I)_{\mathfrak{q}/I} &= \operatorname{depth}(R_{\mathfrak{q}}/IR_{\mathfrak{q}}) = \dim R_{\mathfrak{q}} - s\\ &= \operatorname{ht}\mathfrak{q} - \operatorname{ht}\mathfrak{p} = \dim R/\mathfrak{p} - \dim R/\mathfrak{q}\\ &= \dim R/I - \dim R/\mathfrak{q}.\end{aligned}$$

Now the assertion holds. □

An ideal I of R is called *unmixed up to* $\mathfrak{m}$*-primary* if I has no any embedded prime divisors except probably the maximal ideal $\mathfrak{m}$. Note that any unmixed ideal is unmixed up to $\mathfrak{m}$-primary. This is well-known that a ring R is Cohen-Macaulay if and only if every ideal I of R generated by $\operatorname{ht} I$ elements is unmixed. The following result is a general case of this result for filter rings.

Theorem 4.1.7. *Let $(R, \mathfrak{m})$ be a local ring. Then the following hold:*

(a) If R is a filter ring then every ideal I of R generated by $\operatorname{ht} I$ *elements is unmixed up to $\mathfrak{m}$-primary.*

(b) The converse of (a) holds if R is catenary and equidimensional.

Proof. Let I be an ideal of R and $\sqrt{I} \neq \mathfrak{m}$. Let $n = \operatorname{ht} I$ and $I = (x_1, \cdots, x_n)$. Suppose that $\mathfrak{p}, \mathfrak{q} \in \operatorname{Ass} R/I \setminus \{\mathfrak{m}\}$ and $\mathfrak{p} \subset \mathfrak{q}$. Then $\mathfrak{p}R_{\mathfrak{q}}, \mathfrak{q}R_{\mathfrak{q}} \in \operatorname{Ass} R_{\mathfrak{q}}/IR_{\mathfrak{q}}$. Since $R_{\mathfrak{q}}$ is Cohen-Macaulay, $\dim R_{\mathfrak{q}}/IR_{\mathfrak{q}} = \dim R_{\mathfrak{q}} - \operatorname{ht} IR_{\mathfrak{q}} = \dim R_{\mathfrak{q}} - n$. So $x_1/1, \cdots, x_n/1$ is $R_{\mathfrak{q}}$-sequence. Thus $R_{\mathfrak{q}}/IR_{\mathfrak{q}}$ is Cohen-Macaulay and $\mathfrak{p}R_{\mathfrak{q}} = \mathfrak{q}R_{\mathfrak{q}}$ and hence $\mathfrak{p} = \mathfrak{q}$.

For part (b) let $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ with $\operatorname{ht}\mathfrak{p} = n$. Then there exist $x_1, \cdots, x_n \in \mathfrak{p}$ such that $\operatorname{ht}(x_1, \cdots, x_i) = i$ for all $i = 1, \cdots, n$. By assumption $(x_1, \cdots, x_i)$ is unmixed up to $\mathfrak{m}$-primary for all $i = 1, \cdots, n$, so x_{i+1} does not belong to any $\mathfrak{q} \in \operatorname{Ass} R/(x_1, \cdots, x_i) \setminus \{\mathfrak{m}\}$. This means that $x_1, \cdots, x_n$ is R-filter regular sequence. Therefore $x_1/1, \cdots, x_n/1$ is $R_{\mathfrak{p}}$-regular sequence. So that $n \leq \operatorname{depth} R_{\mathfrak{p}} \leq \operatorname{ht} \mathfrak{p} = n$ and hence $R_{\mathfrak{p}}$ is Cohen-Macaulay. Since R is catenary and equidimensional, using Proposition 2.2, R is a filter ring . $\square$

Example 4.1.8. *Let $R = k[[X, Y, Z]]/(X) \cap (Y, Z)$, where k is any field. Let x, y and z be images of X, Y, and Z in R. Then R is a 2-dimensional local ring, having $\{(x), (y, z)\}$ as associated primes. Hence R is not equidimensional, while all localization of R with respect to a non-maximal prime ideal is a Cohen-Macaulay ring. This shows that in Theorem 4.1.7(b), the assumption "catenary and equidimensional" is crucial.*

4.2 Filter modules

Recall from [75] that a sequence $x_1, \cdots, x_n$ of elements in $\mathfrak{m}$ is said to be a generalized M-filter regular sequence of M if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M/(x_1, \cdots, x_{i-1}))$ satisfying $\dim R/\mathfrak{p} > 1$ for all $i = 1, \ldots, n$. It is clear that $x_1, \cdots, x_n$ is a generalized regular sequence if and only if the modules $(x_1, \ldots, x_{i-1})M :_M x_i/(x_1, \ldots, x_{i-1})M$ is of dimension at most 1 for all $i = 1, \ldots, n$. Note that these modules are of dimension at most 0 if and only if $x_1, \cdots, x_n$ is a filter regular sequence and that they are isomorphic to zero module (or of dimension $-\infty$) if and only if $x_1, \cdots, x_n$ are regular sequence.

Definition 4.2.1. *(see [76]) An R-module M is called a generalized filter module if* *any system of parameter of M is a generalized M-filter regular sequence. A ring is called a generalized filter ring if it is a generalized filter module over itself.*

Theorem 4.2.2. *Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension of local rings, and let M be an R-module. If $M \otimes_R S$ is a filter (resp. generalized filter) S-module, then*

(1) If $\dim S/\mathfrak{m}S = 0$, then M is a filter (resp. generalized filter) R-module.

(2) If $\dim S/\mathfrak{m}S = 1$, then M is a Cohen-Macaulay (resp. filter module) R-module.

(3) If $\dim S/\mathfrak{m}S > 1$, then M is a Cohen-Macaulay R-module.

Proof. First of all suppose that $x_1, \cdots, x_n$ is a system of parameters of M. For simplicity set $\mathfrak{a}_i = (x_1, \cdots, x_i)$ for all $i = 1, \cdots, n$. Due to flatness of S over R, we have $M \otimes_R S/\mathfrak{a}_n(M \otimes_R S) \simeq (M/\mathfrak{a}_n M) \otimes_R S$. So by [17, Theorem (A.11)]

$$\begin{aligned}
\dim_S M \otimes_R S/\mathfrak{a}_n(M \otimes_R S) &= \dim_S (M/\mathfrak{a}_n M) \otimes_R S \\
&= \dim_R M/\mathfrak{a}_n M + \dim S/\mathfrak{m}S \\
&= \dim M - n + \dim S/\mathfrak{m}S \\
&= \dim_S (M \otimes_R S) - n.
\end{aligned}$$

This means that $\varphi(x_1), \ldots, \varphi(x_n)$ is a subset of a system of parameters of $M \otimes_R S$. Again since S is flat we have:

$$\mathfrak{a}_{i-1}(M \otimes_R S) :_{M \otimes_R S} x_i S/\mathfrak{a}_{i-1}(M \otimes_R S) \simeq (\mathfrak{a}_{i-1}M :_M x_i/\mathfrak{a}_{i-1}M) \otimes_R S$$

for all $i = 1, \ldots, n$.

Case 1. If $M \otimes_R S$ is a filter S-module, then for all $1 \leqslant i \leqslant n$ we have:

$$\begin{aligned}\dim_R(\mathfrak{a}_{i-1}M :_M x_i/\mathfrak{a}_{i-1}M) + \dim S/\mathfrak{m}S &= \dim_S((\mathfrak{a}_{i-1}M :_M x_i/\mathfrak{a}_{i-1}M) \otimes_R S)\\ &= \dim_S(\mathfrak{a}_{i-1}(M \otimes_R S) :_{M\otimes_R S} x_iS/\mathfrak{a}_{i-1}(M \otimes_R S))\\ &= 0,\end{aligned}$$

where the third equality follows from the fact that $M \otimes_R S$ is a filter S-module.

Case 2. If $M \otimes_R S$ is a generalized filter S-module, then as above for all $1 \leqslant i \leqslant n$ we have:

$$\begin{aligned}\dim_R(\mathfrak{a}_{i-1}M :_M x_i/\mathfrak{a}_{i-1}M) + \dim S/\mathfrak{m}S &= \dim_S((\mathfrak{a}_{i-1}M :_M x_i/\mathfrak{a}_{i-1}M) \otimes_R S)\\ &= \dim_S(\mathfrak{a}_{i-1}(M \otimes_R S) :_{M\otimes_R S} x_iS/\mathfrak{a}_{i-1}(M \otimes_R S))\\ &\leqslant 1,\end{aligned}$$

where the inequality follows from the fact that $M \otimes_R S$ is a generalized filter S-module. □

Corollary 4.2.3. *Let k be a field. Let R and S be two k-algebras, such that $R \otimes_k S$ is locally a generalized filter ring, then R and S are locally generalized filter rings.*

Proof. Exactly the same proof as Corollary 4.1.4. □

4.3 Filter dimension

In this section we define the filter dimension of finite modules, designed on the upper Cohen-Macaulay dimension.

Definition 4.3.1. *Let*

$$\mathrm{f}(M) = \min\{\operatorname{depth} M_{\mathfrak{p}} + \dim R/\mathfrak{p} | \mathfrak{p} \in \operatorname{Supp} M \setminus \{\mathfrak{m}\}\}.$$

It is easy to see that $\mathrm{f}(M) \leq \dim M$ *and equality holds if and only if* M *is a filter module.*

Let $\mathfrak{a} = \operatorname{Ann}_R M$. Then $\operatorname{grade} M = \operatorname{grade}(\mathfrak{a}, R) \leq \operatorname{ht} \mathfrak{a}$ and so $\operatorname{grade} M \leq \dim R - \mathrm{f}(R/\mathfrak{a})$.

Definition 4.3.2. *An* R*-module* M *with finite projective dimension is called f-perfect, if*

$$\operatorname{grade} M = \min\{\operatorname{pd}_R M, \dim R - \mathrm{f}(R/\operatorname{Ann}_R(M))\}.$$

An ideal $\mathfrak{a}$ is called f-perfect if $R/\mathfrak{a}$ is an f-perfect module. We say that R has an f-deformation, if there exists a local ring Q and an f-perfect ideal $\mathfrak{a}$ in Q such that $R = Q/\mathfrak{a}$. A filter quasi-deformation of R is a diagram of local homomorphisms $R \to R' \leftarrow Q$ with $R \to R'$ is flat and $Q \to R'$ is a filter deformation. Set $M' = M \otimes_R R'$. With these notations we can define filter dimension of an R-module M, $\operatorname{f-dim}_R M$, as the formula

$$\operatorname{f-dim}_R M = \inf\{\operatorname{pd}_Q M' - \operatorname{grade}(\mathfrak{a}, Q) |\, R \to R' \leftarrow Q \text{ is a filter quasi–deformation}\},$$

where $\mathfrak{a} = \ker(Q \to R')$.

Theorem 4.3.3. *Let* $(R, \mathfrak{m}, k)$ *be a local ring. Then the following statements are true:*

(a) If $\operatorname{f-dim}_R k < \infty$ *then* R *is a filter ring.*

(b) If $\hat{R}$ is a filter ring then $\text{f-dim}_R\, k < \infty$.

Proof. (a) Let $\text{f-dim}_R\, k < \infty$. Let $R \to R' \leftarrow Q$ be the corresponding diagram and $R' = Q/\mathfrak{a}$. If $\operatorname{grade}(\mathfrak{a}, Q) = \operatorname{pd}_Q Q/\mathfrak{a}$, then $\text{CM-dim}_R\, k < \infty$. Thus using [58] R is Cohen-Macaulay and hence a filter ring. Assume that $\operatorname{grade}(\mathfrak{a}, Q) = \dim Q - \operatorname{f}(Q/\mathfrak{a})$. Since $\operatorname{grade}(\mathfrak{a}, Q) \leq \operatorname{ht} \mathfrak{a}$ and $\operatorname{f}(Q/\mathfrak{a}) \leq \dim Q/\mathfrak{a}$, we have that $\operatorname{grade}(\mathfrak{a}, Q) = \operatorname{ht} \mathfrak{a}$ and $\operatorname{f}(Q/\mathfrak{a}) = \dim Q/\mathfrak{a}$. Therefore $Q/\mathfrak{a} = R'$ is a filter ring and so R is a filter ring.

(b) Assume $\hat{R}$ is a filter ring. Then by Cohen's structure theorem there exists a regular ring Q with $\hat{R} = Q/\mathfrak{a}$. Thus $\operatorname{grade}(\mathfrak{a}, Q) = \dim Q - \dim Q/\mathfrak{a} = \dim Q - \operatorname{f}(Q/\mathfrak{a})$ and $\operatorname{pd}_Q Q/\mathfrak{a} < \infty$. Therefore $R \to \hat{R} \leftarrow Q$ is a filter quasi deformation with $\operatorname{pd}_Q(M \otimes_R \hat{R}) < \infty$ for all finite R-module M. Hence $\text{f-dim}_R\, M < \infty$. $\square$

Remark 4.3.4. *There exists filter ring which its completion is not filter. for example, Ferrand and Raynaud [56] have constructed a two-dimensional local integral domain R such that the $\mathfrak{m}$-adic completion $\hat{R}$ has a one-dimensional associated prime ideal. Thus R is a filter ring but it is not a generalized Cohen-Macaulay ring and so $\hat{R}$ is not a filter ring.*

Part II

Multiplicative ideal theory

Chapter 5

Going-down and semistar operations

Throughout this chapter, D denotes a (commutative integral) domain with identity and K denotes the quotient field of D. For several decades, star operations, as described in [58, Section 32], have proven to be an essential tool in multiplicative ideal theory, allowing one to study various classes of domains. Nearly 15 years ago, Okabe and Matsuda [77] introduced the concept of a semistar operation to extend the notion of a star operation. Since then, semistar operations have been much studied and, thanks to a greater flexibility than for star operations, have permitted a finer study and classification of domains. For instance, semistar-theoretic analogues of the classical notions of Noetherian and Prüfer domains have been introduced: see [37] and [36] for the basics on $\star$-Noetherian domains and $P{\star}MD$s, respectively. Among semistar-theoretic analogues of classical properties, we note that [21, Lemma 2.15] has recently found an instance where $\widetilde{\star}$-INC, $\widetilde{\star}$-LO and $\widetilde{\star}$-GU arise naturally. Given that INC, LO and GU are usually introduced together with the classical going-down property GD (cf. [69, page 28]), it seems reasonable to develop a semistar-theoretic analogue of GD and then, using it, to introduce a semistar-theoretic analogue of the going-down domains that were introduced in [29], [35]. Both of these goals are accomplished in this thesis.

Indeed, if $D \subseteq T$ is an extension of domains and $\star$ (resp., $\star'$) is a semistar operation on D (resp., T), we define in Section 5.1 what it means for $D \subseteq T$ to satisfy the $(\star, \star')$-GD property. Sufficient conditions are given for $(\star, \star')$-GD, generalizing classical sufficient conditions for GD such as flatness, openness of the contraction map of spectra and (to some extent) the hypotheses of the classical going-down theorem. Moreover, if $\star$ is a semistar operation on a domain D, we define in Section 5.2 what it means for D to be a $\star$-GD domain. Let $\widetilde{\star}$ be the canonical *spectral* semistar operation *of finite type* associated to $\star$ (whose definition is recalled below). In determining whether a domain D is a $\widetilde{\star}$-GD domain, the domain extensions T of D for which suitable $(\widetilde{\star}, \star')$-GD is tested can be the $\widetilde{\star}$-valuation overrings of D, the simple overrings of D, or all T (see Theorem 5.2.14), thus generalizing [35, Theorem 1]. In Theorem 5.2.11, $P\star MD$s are characterized as the $\widetilde{\star}$-treed (resp., $\widetilde{\star}$-GD) domains D which are $\widetilde{\star}$-finite conductor domains such that $D^{\widetilde{\star}}$ is integrally closed, thus generalizing [71, Theorem 1]. The final two results of this section give several characterizations of the $\widetilde{\star}$-Noetherian domains D of $\widetilde{\star}$-dimension 1 in terms of the behavior of the $(\star, \star')$-linked overrings of D and the $\star$-Nagata rings $Na(D, \star)$. In fact, Corollary 5.2.20 can be viewed as a semistar-theoretic generalization of the Krull-Akizuki Theorem, while Theorem 5.2.21 generalizes the fact [29, Corollary 2.3] that Noetherian going-down domains must have (Krull) dimension at most 1. In Section 5.3 we give some new characterization of $\star$-quasi-Prüfer domain introduced recently by Chang and Fontana in [21].

For the remainder of the Introduction, we recall some background related to semistar operations. Let $\overline{\mathcal{F}}(D)$ denote the set of all nonzero D-submodules of K. Let $\mathcal{F}(D)$ be the set of all nonzero *fractional* ideals of D; i.e., $E \in \mathcal{F}(D)$ if $E \in \overline{\mathcal{F}}(D)$ and there exists a nonzero element $r \in D$ with $rE \subseteq D$. Let $f(D)$ be the set of all nonzero finitely generated fractional ideals of D. Obviously, $f(D) \subseteq \mathcal{F}(D) \subseteq \overline{\mathcal{F}}(D)$. As in [77], a *semistar operation on* D is a map $\star : \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathcal{F}}(D)$, the following three properties hold:

$\star_1$: $(xE)^\star = xE^\star$;

$\star_2$: $E \subseteq F$ implies that $E^\star \subseteq F^\star$;

$\star_3$: $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Recall from [77, Proposition 5] that if $\star$ is a semistar operation on D, then, for all $E, F \in \overline{\mathcal{F}}(D)$, the following basic formulas follow easily from the above axioms:

(1) $(EF)^\star = (E^\star F)^\star = (EF^\star)^\star = (E^\star F^\star)^\star$;

(2) $(E + F)^\star = (E^\star + F)^\star = (E + F^\star)^\star = (E^\star + F^\star)^\star$;

(3) $(E : F)^\star \subseteq (E^\star : F^\star) = (E^\star : F) = (E^\star : F)^\star$, if $(E : F) \neq (0)$;

(4) $(E \cap F)^\star \subseteq E^\star \cap F^\star = (E^\star \cap F^\star)^\star$ if $(E \cap F) \neq (0)$.

It is convenient to say that a *(semi)star operation on* D is a semistar operation which, when restricted to $\mathcal{F}(D)$, is a star operation (in the sense of [58, Section 32]). It is easy to see that a semistar operation $\star$ on D is a (semi)star operation on D if and only if $D^\star = D$.

Let $\star$ be a semistar operation on the domain D. For every $E \in \overline{\mathcal{F}}(D)$, put $E^{\star_f} := \cup F^\star$, where the union is taken over all finitely generated $F \subseteq E$. It is easy to see that $\star_f$ is a semistar operation on D, and $\star_f$ is called *the semistar operation of finite type associated to* $\star$. Note that $(\star_f)_f = \star_f$. A semistar operation $\star$ is said to be of *finite type* if $\star = \star_f$; thus, $\star_f$ is of finite type. We say that a nonzero ideal I of D is a *quasi-$\star$-ideal of* D if $I^\star \cap D = I$; a *quasi-$\star$-prime (ideal of* D) if I is a prime quasi-$\star$-ideal of D; and a *quasi-$\star$-maximal (ideal of* D) if I is maximal in the set of all proper quasi-$\star$-ideals of D. Each quasi-$\star$-maximal ideal is a prime ideal. It was shown in [46, Lemma 4.20] that if $D^\star \neq K$, then each proper quasi-$\star_f$-ideal of D is contained in a quasi-$\star_f$-maximal ideal of D. We denote by $\mathrm{QMax}^\star(D)$ (resp., $\mathrm{QSpec}^\star(D)$) the set of all quasi-$\star$-maximal ideals (resp., quasi-$\star$-prime ideals) of D. When $\star$ is a (semi)star operation, the notion of quasi-$\star$-ideal is equivalent to the classical notion of $\star$-ideal (i.e., a nonzero ideal I of D such that $I^\star = I$).

If $\star_1$ and $\star_2$ are semistar operations on D, one says that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$ for each $E \in \overline{\mathcal{F}}(D)$. (cf. [77, page 6].) This is equivalent to saying that $(E^{\star_1})^{\star_2} = E^{\star_2} = (E^{\star_2})^{\star_1}$ for each $E \in \overline{\mathcal{F}}(D)$ (cf. [77, Lemma 16]). Obviously, for each semistar operation $\star$ defined on D, we have $\star_f \leq \star$. Let d_D (or, simply, d) denote the identity (semi)star operation on D. Clearly, $d_D \leq \star$ for all semistar operations $\star$ on D.

If Δ is a set of prime ideals of a domain D, then there is an associated semistar operation on D, denoted by $\star_\Delta$, defined as follows:

$$E^{\star_\Delta} := \cap\{ED_P | P \in \Delta\}, \text{ for each } E \in \overline{\mathcal{F}}(D).$$

One calls $\star_\Delta$ the *spectral semistar operation associated to* Δ. A semistar operation $\star$ on a domain D is called a *spectral semistar operation* if there exists a subset Δ of the prime spectrum of D, $\mathrm{Spec}(D)$, such that $\star = \star_\Delta$. When $\Delta := \mathrm{QMax}^{\star_f}(D)$, we set $\widetilde{\star} := \star_\Delta$; i.e.,

$$E^{\widetilde{\star}} := \cap\{ED_P | P \in \mathrm{QMax}^{\star_f}(D)\}, \text{ for each } E \in \overline{\mathcal{F}}(D).$$

It has become standard to say that a semistar operation $\star$ is *stable* if $(E \cap F)^\star = E^\star \cap F^\star$ for all E, $F \in \overline{\mathcal{F}}(D)$. ("Stable" has replaced the earlier usage, "quotient", in [77, Definition 21].) All spectral semistar operations are stable [46, Lemma 4.1(3)]. In particular, for any semistar operation $\star$, we have that $\widetilde{\star}$ is a stable semistar operation of finite type [46, Corollary 3.9].

Let $\star$ be a semistar operation on a domain D, T an overring of D, and $\iota : D \hookrightarrow T$ the corresponding inclusion map. In a canonical way, one can define an associated semistar operation $\star_\iota$ on T by $E \mapsto E^{\star_\iota} := E^\star$, for each $E \in \overline{\mathcal{F}}(T) (\subseteq \overline{\mathcal{F}}(D))$.

The most widely studied (semi)star operations on D have been the d, v, $t := v_f$, and $w := \widetilde{v}$ operations, where $E^v := (E^{-1})^{-1}$, in which $E^{-1} := (D : E) := \{x \in K | xE \subseteq D\}$.

Let D be a domain with quotient field K, and let X be an indeterminate over K. For each $f \in K[X]$, we let $c_D(f)$ denote the content of the polynomial f, i.e., the (fractional, if $f \neq 0$) ideal of D generated by the coefficients of f. Let $\star$ be a semistar operation on D. If $N_\star := \{g \in D[X] | g \neq 0 \text{ and } c_D(g)^\star = D^\star\}$, then

$N_\star = D[X] \setminus \bigcup\{P[X] | P \in \mathrm{QMax}^\star(D)\}$ is a saturated multiplicative subset of $D[X]$. The ring of fractions

$$Na(D, \star) := D[X]_{N_\star}$$

is called the $\star$-*Nagata domain (of* D *with respect to the semistar operation* $\star$*)*. When $\star = d$, the identity (semi)star operation on D, then $Na(D, d)$ coincides with the classical Nagata domain $D(X)$ (as in, for instance [74, page 18], [58, Section 33] and [49]).

The ring $Na(D, \star)$ will figure in a number of results in Sections 2 and 3. In Remark 5.0.5, we record an important dimension-theoretic property of $Na(D, \star)$. First, we recall some relevant material.

In [37, Section 3], El Baghdadi, Fontana and Picozza defined and studied semistar Noetherian domains, i.e., domains having the ascending chain condition on quasi-semistar ideals.

Let $\star$ be a semistar operation on a domain D. Then the $\star$-*Krull dimension of* D is defined as

$$\star\text{-}\dim(D) := \sup\{n \mid P_1 \subset \cdots \subset P_n \text{ for some quasi-}\star\text{-prime ideals } P_i \text{ of } D\}.$$

It is known (see [37, Lemma 2.11]) that

$$\widetilde{\star}\text{-}\dim(D) = \sup\{\mathrm{ht}(P) \mid P \text{ is a quasi-}\widetilde{\star}\text{-prime ideal of } D\}.$$

Thus, if $\star = d$, then $\widetilde{\star}\text{-}\dim(D) = \star\text{-}\dim(D) = \dim(D)$, the usual (Krull) dimension of D. We close the Introduction by stating a result which shows that in the $\widetilde{\star}$-Noetherian case, $\widetilde{\star}\text{-}\dim(D)$ has another natural interpretation. Remark 5.0.5 can be proved by slightly modifying the argument given by Chang [19, Corollary 2.10] for the star operation case.

Remark 5.0.5. *Let $\star$ be a semistar operation on D. If D is a $\widetilde{\star}$-Noetherian domain, then $\widetilde{\star}\text{-}\dim(D) = \dim(Na(D, \star))$.*

5.1 The $(\star, \star')$-GD property

In this section, we define the semistar-theoretic version of the going-down property GD and then identify some sufficient conditions for this new property. The key definition is the following.

Definition 5.1.1. *Let $D \subseteq T$ be an extension of domains. Let $\star$ and $\star'$ be semistar operations on D and T, respectively. We say that $D \subseteq T$ satisfies $(\star, \star')$-GD if, whenever $P_0 \subset P$ are quasi-$\star$-prime ideals of D and Q is a quasi-$\star'$-prime ideal of T such that $Q \cap D = P$, there exists a quasi-$\star'$-prime ideal Q_0 of T such that $Q_0 \subseteq Q$ and $Q_0 \cap D = P_0$.*

The above definition generalizes the classical GD property in the following sense. If $D \subseteq T$ are domains, then $D \subseteq T$ satisfies (d_D, d_T)-GD if and only if $D \subseteq T$ satisfies GD.

Let D be a domain and $\star$ a semistar operation on D. Recall that the *$\star$-integral closure* $D^{[\star]}$ of D (or, the *semistar integral closure of D with respect to the semistar operation* $\star$) is the integrally closed overring of $D^\star$ defined by $D^{[\star]} := \{(F^\star : F^\star) | F \in f(D)\}$ ([77, page 12], [50, Definition 4.1]). The domain D is said to be *quasi-$\star$-integrally closed* if $D^\star = D^{[\star]}$. It was proved in [37, Lemma 4.13 (2)] that D is quasi-$\widetilde{\star}$-integrally closed if and only if $D^{\widetilde{\star}}$ is integrally closed. One can thus see (or prove directly) that if D is integrally closed, then D is quasi-$\widetilde{\star}$-integrally closed.

Theorem 5.1.3 will present our first sufficient condition for the $(\star, \star')$-GD property, in a result that has some of the flavor of the classical going-down theorem of Krull. First, we give a useful technical result.

Lemma 5.1.2. *Let $D \subseteq T$ be an extension of domains. Let $\star$ be a semistar operation on D. Suppose that D is quasi-$\widetilde{\star}$-integrally closed and that for all $u \in T$, there exists $I \in f(D)$ such that $uI^{\widetilde{\star}} \subseteq I^{\widetilde{\star}}$. Then $Na(D, \star) \hookrightarrow T[X]_{N_\star}$ is an integral extension and $Na(D, \star)$ is integrally closed.*

Proof. Consider any element $u \in T$. By hypothesis, $uI^{\widetilde{\star}} \subseteq I^{\widetilde{\star}}$ for some $I \in f(D)$. Hence, $uINa(D, \star) = uI^{\widetilde{\star}}Na(D, \star) \subseteq I^{\widetilde{\star}}Na(D, \star) = INa(D, \star)$. (See [19, Lemma 2.3] for the star operation case of the preceding reasoning.) As $INa(D, \star) \in f(Na(D, \star))$, it follows that u is integral over $Na(D, \star)$ (cf. [69, Theorem 12]).

Now, consider any element $v \in T[X]_{N_\star}$. We can write $v = f/g$, where $f \in T[X]$ and $g \in N_\star$. The result of the preceding paragraph ensures that each coefficient of f is integral over $Na(D, \star)$. Moreover, g and all powers of X are units of $Na(D, \star)$. Therefore, $v = f/g$ is integral over $Na(D, \star)$ (cf. [69, Theorem 13]). This proves the first assertion.

For the second assertion, note that by [49, Corollary 3.5], we have $Na(D, \star) = Na(D^{\widetilde{\star}}, (\widetilde{\star})_\iota)$, where $\iota : D \hookrightarrow D^{\widetilde{\star}}$ is the canonical inclusion map. So $Na(D, \star) = Na(D^{\widetilde{\star}}, (\widetilde{\star})_\iota) = D^{\widetilde{\star}}[X]_{N_{(\widetilde{\star})_\iota}}$ is integrally closed, since the class of integrally closed domains is stable under adjunction of indeterminates and the formation of rings of fractions. $\square$

Theorem 5.1.3. *Let $D \subseteq T$ be an extension of domains. Let $\star$ and $\star'$ be semistar operations on D and T, respectively. Suppose that D is quasi-$\widetilde{\star}$-integrally closed and that for all $u \in T$, there exists $I \in f(D)$ such that $uI^{\widetilde{\star}} \subseteq I^{\widetilde{\star}}$. Then $D \subseteq T$ satisfies* $(\star, \widetilde{\star'})$-GD.

Proof. By Lemma 5.1.2, $Na(D,\star) \hookrightarrow T[X]_{N_\star}$ is an integral extension and $Na(D,\star)$ is integrally closed. Let $P_0 \subset P$ be quasi-$\star$-prime ideals of D and Q a quasi-$\widetilde{\star'}$-prime ideal of T such that $Q \cap D = P$. Our task is to find a quasi-$\widetilde{\star'}$-prime ideal Q_0 of T such that $Q_0 \subseteq Q$ and $Q_0 \cap D = P_0$.

Note that $P_0Na(D,\star) \subseteq PNa(D,\star)$ are prime ideals of $Na(D,\star)$ and $QT[X]_{N_\star}$ is a prime ideal of $T[X]_{N_\star}$ such that $QT[X]_{N_\star} \cap Na(D,\star) = PNa(D,\star)$. Therefore, by the classical going-down theorem (as in, for example, [3, Theorem 5.16]), there exists a prime ideal L_0 of $T[X]$ such that $L_0T[X]_{N_\star} \subseteq QT[X]_{N_\star}$ and $L_0T[X]_{N_\star} \cap Na(D,\star) = P_0Na(D,\star)$. Consider the prime ideal $Q_0 := L_0 \cap T$ of T. By intersecting the preceding inclusion with $T[X]$, we find that $L_0 \subseteq QT[X]$. Hence, $Q_0 = L_0 \cap T \subseteq QT[X] \cap T = Q$. It remains to prove that $Q_0 \cap D = P_0$ and that Q_0 is a quasi-$\widetilde{\star'}$-prime ideal of T.

Recall that $L_0T[X]_{N_\star} \cap Na(D,\star) = P_0Na(D,\star)$. Intersecting with $D[X]$, we find that $L_0T[X] \cap D[X] = P_0D[X]$. If $a \in Q_0 \cap D$, then $a \in L_0T[X] \cap D[X] \cap D = P_0D[X] \cap D = P_0$, and so $Q_0 \cap D \subseteq P_0$. For the reverse inclusion, consider any element $b \in P_0$. Then $b \in P_0D[X] = L_0T[X] \cap D[X]$, and so $b \in L_0T[X] \cap T \cap D = (L_0 \cap T) \cap D = Q_0 \cap D$. Thus, $Q_0 \cap D = P_0$. Finally, since $Q_0 \subseteq Q$ and Q is a quasi-$\widetilde{\star'}$-prime ideal of T, we can conclude that Q_0 is a quasi-$\widetilde{\star'}$-prime ideal of T, by appealing to [46, Lemma 4.1, Remark 4.5]. □

Theorem 5.1.3 does not give a complete semistar-theoretic analogue of the classical going-down theorem. Such an analogue would not require T to be an overring of R, although the hypotheses that $I \in f(D)$ and $uI^{\widetilde{\star}} \subseteq I^{\widetilde{\star}}$ in Theorem 5.1.3 do imply that $u \in K$. A completely satisfactory analogue of the classical result would posit that D is quasi-$\star$-integrally closed and that for all $u \in T$, there exists a nonzero finitely generated D-submodule I of T such that $u(IT)^{\widetilde{\star'}} \subseteq (IT)^{\widetilde{\star'}}$. We do not know if these conditions imply that the extension of domains $D \subseteq T$ satisfies $(\star, \widetilde{\star'})$-GD.

Remark 5.1.4. *The condition of Lemma 5.1.2 that for all $u \in T$, there exists $I \in f(D)$ such that $uI^{\widetilde{\star}} \subseteq I^{\widetilde{\star}}$ is equivalent to the condition that $T \subseteq D^{\widetilde{\star}}$.*

Some satisfactory semistar-theoretic analogues of classical sufficient conditions for GD will be given in Corollary 5.1.6 and Propositions 5.1.8 and 5.1.9. First, we need the following definition. Let D be a domain and T an overring of D. Let $\star$ and $\star'$ be semistar operations on D and T, respectively. One says that T is $(\star, \star')$-*linked to* D (or that T is a $(\star, \star')$-*linked overring of* D) if

$$F^\star = D^\star \Rightarrow (FT)^{\star'} = T^{\star'}$$

for each nonzero finitely generated ideal F of D. (The preceding definition generalizes the notion of "t-linked overring" which was introduced in [33].) It was proved in [36, Theorem 3.8] that T is $(\star, \star')$-linked to D if and only if $Na(D, \star) \subseteq Na(T, \star')$. It will be useful to observe that these conditions are also equivalent to $N_\star \subseteq N_{\star'}$. For a proof, note that if $N_\star \subseteq N_{\star'}$, then

$$Na(D, \star) = D[X]_{N_\star} \subseteq T[X]_{N_\star} \subseteq (T[X]_{N_\star})_{N_{\star'}} = T[X]_{N_{\star'}} = Na(T, \star').$$

We show next that a $(\star, \star')$-linked overring extension satisfies $(\star, \widetilde{\star'})$-GD if the corresponding extension of semistar-theoretic Nagata rings satisfies GD.

Theorem 5.1.5. *Let D be a domain and T an overring of D. Let $\star$ and $\star'$ be semistar operations on D and T, respectively, such that T is a $(\star, \star')$-linked overring of D. If $Na(D, \star) \subseteq Na(T, \star')$ satisfies* GD*, then $D \subseteq T$ satisfies $(\star, \widetilde{\star'})$-*GD.

Proof. Let $P_0 \subseteq P$ be quasi-$\star$-prime ideals of D and Q a quasi-$\widetilde{\star'}$-prime ideal of T such that $Q \cap D = P$. Then $P_0 Na(D, \star) \subseteq PNa(D, \star)$ are prime ideals of $Na(D, \star)$ and $QNa(T, \star')$ is a prime ideal of $Na(T, \star')$. We claim that $QNa(T, \star') \cap Na(D, \star) = PNa(D, \star)$.

Suppose that $u \in QNa(T, \star') \cap Na(D, \star)$. Since $Na(T, \star') = Na(T, \widetilde{\star'})$, there exist $f \in D[X]$, $g \in N_\star$, $f' \in QT[X]$, and $g' \in N_{\widetilde{\star'}}$ such that $u = f/g = f'/g'$. Note that $fg' = gf'$, and so $c_T(fg') = c_T(gf') \subseteq Q$. Write $f = a_0 + a_1 X + \cdots + a_n X^n$ and $g' = t_0 + t_1 X + \cdots + t_m X^m$, where each $a_i \in D$ and each $t_j \in T$. As $g' \in N_{\widetilde{\star'}}$, we have $c_T(g')^{\widetilde{\star'}} = T^{\widetilde{\star'}}$. By the Dedekind-Mertens Lemma,

$$c_T(f)^m c_T(fg') = c_T(f)^{m+1} c_T(g').$$

Therefore,

$$(c_T(f)^{m+1})^{\widetilde{\star'}} = (c_T(f)^{m+1})^{\widetilde{\star'}} c_T(g')^{\widetilde{\star'}} \subseteq ((c_T(f)^{m+1})^{\widetilde{\star'}} (c_T(g'))^{\widetilde{\star'}})^{\widetilde{\star'}}$$

$$= (c_T(f)^{m+1} c_T(g'))^{\widetilde{\star'}} = (c_T(f)^m c_T(fg'))^{\widetilde{\star'}} \subseteq Q^{\widetilde{\star'}},$$

where we have used properties of semistar operations and, for the second inclusion, the fact that $c_T(fg') \subseteq Q$. Now, $c_T(f)^{m+1} \subseteq (c_T(f)^{m+1})^{\widetilde{\star'}} \cap T \subseteq Q^{\widetilde{\star'}} \cap T = Q$, and so $c_T(f) \subseteq Q$. This means that each coefficient a_i of f is in Q. It follows that $a_i \in Q \cap D = P$ for each i, and so $f \in PD[X]$. Thus, $u = f/g \in PNa(D, \star)$. This proves one inclusion for the above claim. For the reverse inclusion, the fact that T

is $(\star, \star')$-linked over D yields $Na(D, \star) \subseteq Na(T, \star')$, and so, since $P \subseteq Q$, we have $PNa(D, \star) \subseteq PNa(T, \star') \subseteq QNa(T, \star')$. This completes the proof of the claim.

Since $Na(D, \star) \subseteq Na(T, \star') = T[X]_{N_{\star'}}$ satisfies GD, we may now essentially repeat the proof of Theorem 5.1.3, thus showing that $D \subseteq T$ satisfies $(\star, \widetilde{\star'})$-GD. $\square$

We next state an immediate consequence of Theorem 5.1.5. Let $\star$ be a semistar operation on a domain D, and let $\iota : D \hookrightarrow D^{\widetilde{\star}}$ denote the canonical inclusion map. Then the extension $D \subseteq D^{\widetilde{\star}}$ satisfies $(\star, (\widetilde{\star})_\iota)$-GD. (Indeed, $Na(D, \star) \subseteq Na(D^{\widetilde{\star}}, (\widetilde{\star})_\iota)$ trivially satisfies GD because $Na(D, \star) = Na(D^{\widetilde{\star}}, (\widetilde{\star})_\iota)$.)

It is known that if a unital extension of commutative rings $A \subseteq B$ is flat (that is, B is a flat A-module), then $A \subseteq B$ satisfies GD. (For a particularly accessible proof of this, see [34]. Much more is true: see [59, Corollaire 3.9.4 (ii), page 254].) Before giving the semistar-theoretic version of this result in Corollary 5.1.6, we need to recall the following material from [36]. Let $D \subseteq T$ be domains with the same quotient field, and let $\star$ (resp., $\star'$) be a semistar operation on D (resp., T). Then T is said to be *$(\star, \star')$-flat over D* if T is $(\star, \star')$-linked over D and $D_{Q \cap D} = T_Q$ for each quasi-$\star_f$-prime ideal Q of T. It was proved in [36, Theorem 4.5] that (for $D, T, \star$, and $\star'$ as above) T is $(\star, \star')$-flat over D if and only if $Na(D, \star) \subseteq Na(T, \star')$ is a flat extension. Note that the "$(\star, \star')$-flat" terminology is appropriate, for a theorem of Richman [82, Theorem 2] shows that if D is a domain and T is an overring of D, then T is (d_D, d_T)-flat over D if and only if $D \subseteq T$ is a flat extension.

Corollary 5.1.6. *Let D be a domain and T an overring of D. Let $\star$ and $\star'$ be semistar operations on D and T, respectively, such that T is $(\star, \star')$-flat over D. Then $D \subseteq T$ satisfies $(\star, \widetilde{\star'})$-GD.*

Proof. Since T is a $(\star, \star')$-flat overring of D, then $Na(D, \star) \subseteq Na(T, \star')$ is a flat extension by [36, Theorem 4.5]. By the above comments, $Na(D, \star) \subseteq Na(T, \star')$ satisfies GD, and so the assertion follows from Theorem 5.1.5. $\square$

Recall that the classical Nagata ring associated with a domain D is defined to be $D(X) = Na(D, d_D)$.

Corollary 5.1.7. *Let T be an overring of a domain D. If the extension of classical Nagata rings $D(X) \subseteq T(X)$ satisfies* GD*, then $D \subseteq T$ satisfies* GD.

Proof. Apply Theorem 5.1.5, with $\star := d_D$ and $\star' := d_T$. $\square$

Let $A \subseteq B$ be a unital extension of commutative rings. It is known that if the canonical contraction map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is open (relative to the Zariski topology), then $A \subseteq B$ satisfies GD. (For a particularly accessible proof of this, see [34]. Much more is true: see [59, Corollaire 3.9.4 (i), page 254].) Before giving some semistar-theoretic versions of this result in the final two results of this section, we need to introduce the following material. Let $\star$ and $\star'$ be semistar operations on domains D and T, respectively. Suppose that T is a $(\star, \star')$-linked overring of D. We next show that the contraction map on prime spectra restricts to a well defined function

$$F : \mathrm{QSpec}^{\widetilde{\star'}}(T) \to \mathrm{QSpec}^{\widetilde{\star}}(D),\ Q \mapsto Q \cap D$$

of topological spaces which is continuous (with respect to the subspace topology induced by the Zariski topology).

If $Q \in \mathrm{QSpec}^{\widetilde{\star'}}(T)$, we will prove that $P := F(Q) = Q \cap D$ is a quasi-$\widetilde{\star}$-ideal of D. Note first that $P^{\widetilde{\star}} \neq D^{\widetilde{\star}}$. Indeed, if, on the contrary, $P^{\widetilde{\star}} = D^{\widetilde{\star}}$, then $(PT)^{\widetilde{\star'}} = T^{\widetilde{\star'}}$

by the linkedness hypothesis, whence $Q^{\widetilde{\star'}} = T^{\widetilde{\star'}}$ (since $PT \subseteq Q$), and so $Q = T$, the desired contradiction. It follows that $P^{\widetilde{\star}} \cap D$ is unequal to D, hence is a (proper) quasi-$\widetilde{\star}$-ideal, and hence is contained in some quasi-$\widetilde{\star}$-maximal ideal (and, *a fortiori*, some quasi-$\widetilde{\star}$-prime ideal), say W, of D. As $P \subseteq P^{\widetilde{\star}} \cap D \subseteq W$, we have that $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$, as asserted.

Proposition 5.1.8. *Let $\star$ and $\star'$ be semistar operations on D and T, respectively. Suppose that T is a $(\star, \star')$-linked overring of D. Let $F : \mathrm{QSpec}^{\widetilde{\star'}}(T) \to \mathrm{QSpec}^{\widetilde{\star}}(D)$ be the canonical continuous function (where the topologies are induced by the Zariski topology). If the map F is open, then $D \subseteq T$ satisfies $(\star, \widetilde{\star'})$-GD.*

Proof. We modify the proof given in [34] for the classical analogue. Suppose that the assertion fails. Then there exist quasi-$\star$-prime ideals $P \subset P_1$ of D and a quasi-$\widetilde{\star'}$-prime ideal Q_1 of T such that $Q_1 \cap D = P_1$ and no $Q \in \mathrm{QSpec}^{\widetilde{\star'}}(T)$ satisfies both $Q \subset Q_1$ and $Q \cap D = P$. Recall from [46, Lemma 4.1, and Remark 4.5] that any prime ideal of T which is contained in Q_1 must inherit from Q_1 the property of being a quasi-$\widetilde{\star'}$-prime ideal of T. Therefore, $D \subseteq T$ does not satisfy GD. Applying [69, Exercise 37, page 44], we find $Q \in \mathrm{Spec}(T)$ such that Q is minimal over PT, $Q \subseteq Q_1$, and

$$PT \cap (D \setminus P)(T \setminus Q) \neq \emptyset.$$

By the above-cited material from [46], $Q \in \mathrm{QSpec}^{\widetilde{\star'}}(T)$.

The above display leads to an equation $\sum_{i=1}^{n} p_i t_i = dt$, for some elements $p_i \in P$, $t_i \in T$, $d \in D \setminus P$ and $t \in T \setminus Q$. Let $X := \mathrm{Spec}(T)$, and let X_t denote the set of elements of X that do not contain t. Note that X_t is a basic Zariski-open subset of X. Then $X_t^{\widetilde{\star'}} := X_t \cap \mathrm{QSpec}^{\widetilde{\star'}}(T)$ is an open subset of $\mathrm{QSpec}^{\widetilde{\star'}}(T)$. By assumption, $F(X_t^{\widetilde{\star'}})$ is an open subset of $\mathrm{QSpec}^{\widetilde{\star}}(D)$. It follows that $P \in F(X_t^{\widetilde{\star'}})$ since $Q \in X_t^{\widetilde{\star'}}$ and $P \subseteq Q \cap D$. Hence, there exists $Q_0 \in X_t^{\widetilde{\star'}}$ such that $Q_0 \cap D = P$. Then $dt = \sum p_i t_i \in PT \subseteq Q_0$, although $t \notin Q_0$. As Q_0 is prime, $d \in Q_0$, whence $d \in Q_0 \cap D = P$, the desired contradiction, to complete the proof. $\square$

We pause to give an application of the preceding result. Let D be a domain, $\star$ a semistar operation on D, T an overring of D, $\iota : D \hookrightarrow T$ the canonical inclusion map, and $\star_\iota$ the induced semistar operation on T. Let $F : Q\,\mathrm{Spec}^{\widetilde{(\star_\iota)}}(T) \to Q\,\mathrm{Spec}^{\widetilde{\star}}(D)$ be the canonical continuous function (where the topologies are induced by the Zariski topology). If the map F is open, then $D \subseteq T$ satisfies $(\star, \widetilde{(\star_\iota)})$-$GD$. (Proof: it suffices to show that the extension $D \subseteq T$ is $(\star, \star_\iota)$-linked. We will prove that if $L \in f(D)$ is such that $L^\star = D^\star$, then $(LT)^{\star_\iota} = T^{\star_\iota}$. Indeed, since $1 \in L^{\star_\iota}$ and $1 \in T^{\star_\iota}$, it is enough to note that $1 \in (L^{\star_\iota} T^{\star_\iota})^{\star_\iota} = (LT)^{\star_\iota}$.)

For a more complete semistar-theoretic analogue of the classical result on open contraction maps [59, Corollaire 3.9.4 (i), page 254] than was provided in Proposition 5.1.8, one would consider an arbitrary extension $D \subseteq T$ of domains, not necessarily overrings, with semistar operations $\star$ and $\star'$ on D and T, respectively. The difficulty in handling this level of generality is that there does not seem to be a canonical way to define a continuous function $\mathrm{QSpec}^{\star'}(T) \to \mathrm{QSpec}^{\star}(D)$. Nevertheless, we can offer the following companion for Proposition 5.1.8 when the extension domain is not necessarily an overring.

Proposition 5.1.9. *Let $D \subseteq T$ be an extension of domains, with $\star$ and $\star'$ semistar operations on D and T, respectively. If the continuous function $G : \mathrm{QSpec}^{\widetilde{\star'}}(T) \to \mathrm{Spec}(D)$, $Q \mapsto Q \cap D$, is an open map, then $D \subseteq T$ satisfies $(\star, \widetilde{\star'})$-GD.*

Proof. One can modify the proof of Proposition 5.1.8. $\square$

5.2 The $\star$-GD domains

Going-down domains were introduced in [29] and [35]. The most natural examples of going-down domains are arbitrary Prüfer domains, arbitrary domains of (Krull) dimension at most 1, and certain pullbacks. In this section, we define the semistar-theoretic version of going-down domains and then study this new class of domains. The key definition is the following.

Definition 5.2.1. *Let D be a domain and $\star$ a semistar operation on D. Then D is said to be a $\star$-going-down domain (for short, a $\star$-*GD *domain) if, for every overring T of D and every semistar operation $\star'$ on T, the extension $D \subseteq T$ satisfies* $(\star, \widetilde{\star'})$-GD.

The first result of this section collects some important basic facts. In view of the above-cited works, Proposition 5.2.2 (a) implies that if D is a domain, then D is a going-down domain if and only if D is a d_D-GD domain.

Proposition 5.2.2. *(a) Let D be a domain and $\star$ a semistar operation on D. Then D is a $\star$-*GD *domain if and only if, for every overring T of D, the extension $D \subseteq T$ satisfies* $(\star, d_T)$-GD.

(b) Let D be a domain and $\star_1$, $\star_2$ semistar operations on D. If $\star_1 \leq \star_2$, then $\mathrm{QSpec}^{\star_2}(D) \subseteq \mathrm{QSpec}^{\star_1}(D)$.

*(c) Let D be a domain and $\star_1$, $\star_2$ semistar operations on D. If $\star_1 \leq \star_2$ and D is a $\star_1$-*GD *domain, then D is a $\star_2$-*GD *domain.*

(d) If D is a going-down domain and $\star$ is any semistar operation on D, then D is a $\star$-GD domain. In particular, if $\star$ is a semistar operation on D and D is either a Prüfer domain or a one-dimensional domain, then D is a $\star$-GD domain.

(e) Let D be a domain and $\star$ a semistar operation on D. If $\star$-$\dim(D) = 1$, then D is a $\star$-GD domain.

Proof. (a) The "only if" implication is trivial (by considering $\star' := d_T$). Conversely, suppose that T is an overring of D such that the extension $D \subseteq T$ satisfies $(\star, d_T)$-GD. We will show that if $\star'$ is a semistar operation on T, then $D \subseteq T$ satisfies $(\star, \widetilde{\star'})$-GD. Let $P_0 \subset P_1$ be quasi-$\star$-prime ideals of D and Q_1 a quasi-$\widetilde{\star'}$-prime ideal of T such that $Q_1 \cap D = P_1$. Since $D \subseteq T$ satisfies $(\star, d_T)$-GD, there exists $Q_0 \in \mathrm{Spec}(T)$ satisfying both $Q_0 \subseteq Q_1$ and $Q_0 \cap D = P_0$. To conclude, it suffices to show that Q_0 is a quasi-$\widetilde{\star'}$-prime ideal of T, and this, in turn, holds since Q_0 is contained in the quasi-$\widetilde{\star'}$-prime ideal Q_1 [46, Lemma 4.1 and Remark 4.5].

(b) The following argument is well known (cf. [77, Lemma 16]). Consider any $P \in \mathrm{QSpec}^{\star_2}(D)$. As $\star_1 \leq \star_2$, we have $(P^{\star_1})^{\star_2} = P^{\star_2}$, and so $P \subseteq P^{\star_1} \cap D \subseteq (P^{\star_1})^{\star_2} \cap D = P^{\star_2} \cap D = P$. Thus, $P = P^{\star_1} \cap D$, i.e., $P \in \mathrm{QSpec}^{\star_1}(D)$.

(c) In view of (b), one can easily prove the assertion by adapting the proof of the 'if' implication in (a).

(d) By (a), the hypothesis means that D is a d_D-GD domain. Since $d_D \leq \star$, the first assertion is a consequence of (c). The second assertion then follows from the above remarks.

(e) If $\star$-$\dim(D) = 1$, it follows from the material recalled in the Introduction that every quasi-$\star$-prime ideal of D is a quasi-$\star$-maximal ideal of D, so that D does not possess a chain of two distinct quasi-$\star$-prime ideals $P_0 \subset P_1$, whence $D \subseteq T$ satisfies $(\star, \widetilde{\star'})$-GD vacuously whenever $\star'$ is a star operation on an overring T of D. $\square$

Proposition 5.2.2 (d) established that every going-down domain D is a $\star$-GD domain. The following example shows that the converse is false.

Example 5.2.3. Let K be a field; X and Y algebraically independent indeterminates over K; $D := K[X, Y]$; and Δ the set of prime ideals P of D with $\mathrm{ht}(P) = 1$. Let $\star := \star_\Delta$, the spectral semistar operation associated to Δ. Then $\mathrm{QMax}^{\widetilde{\star}}(D) = \Delta$ (use [46, Lemma 4.1(4)]), and hence $\widetilde{\star}$-$\dim(D) = 1$. Therefore, while D is a $\widetilde{\star}$-GD domain (by Proposition 5.2.2 (e)), it cannot be a going-down domain, by [28, Proposition 7], since D is a Noetherian domain of (Krull) dimension 2.

Let $\star$ be a semistar operation on a domain D. As in [48] and [36] (cf. also [63] for the case of a star operation), D is called a *Prüfer $\star$-multiplication domain* (for short, a $P\star MD$) if each finitely generated ideal of D is $\star_f$-invertible; i.e., if $(II^{-1})^{\star_f} = D^\star$ for all $I \in f(D)$. When $\star = v$, we recover the classical notion of a $PvMD$; when $\star = d$, the identity (semi)star operation, we recover the notion of a Prüfer domain.

Remark 5.2.4. *Note that in the example 5.2.3, $\star$ is in fact a (semi)star operation. Indeed since D is a Krull domain then $D^\star = \bigcap_{P\in\Delta} D_P = D$. Therefore for all $P \in \Delta = \mathrm{QMax}^{\widetilde{\star}}(D)$, D_P is a valuation domain. So using [48, Theorem 3.1], D is a $P\widetilde{\star}MD$=$P\star MD$. Thus since $\star$ is a (semi)star operation then $\star_f = \widetilde{\star} = t = w$ by [48, Proposition 3.15].*

Recall (cf. [30, Remark 2.7 (b)]) that the class of going-down domains is stable under the formation of rings of fractions. Proposition 5.2.5 will establish the semistar-theoretic version of this fact. First, we need the following definition.

Let $\star$ be a semistar operation on a domain D. For each multiplicatively closed subset S of D, we consider the inclusion map $\iota : D \hookrightarrow D_S$ and the semistar operation $\star_S := \star_\iota$. If $P \in \operatorname{Spec}(D)$, we let $\star_P := \star_{D \setminus P}$.

Proposition 5.2.5. *Let D be a domain and $\star$ a semistar operation on D such that D is a $\star$-*GD *domain. Then D_S is a $\star_S$-*GD *domain for each multiplicatively closed subset S of D.*

Proof. By Proposition 5.2.2 (a), we need to show that if T is an overring of D_S, then $D_S \subseteq T$ satisfies $(\star_S, d_T)$-GD. Let $P_0D_S \subset PD_S$ be quasi-$\star_S$-prime ideals of D_S and Q a prime ideal of T such that $Q \cap D_S = PD_S$. Our task is to find $Q_0 \in \operatorname{Spec}(T)$ such that $Q_0 \subseteq Q$ and $Q_0 \cap D_S = P_0D_S$.

As PD_S is a quasi-$\star_S$-prime ideal of D_S, we have $(PD_S)^{\star_S} \cap D_S = PD_S$, and so $(PD_S)^\star \cap D = P$. Therefore,

$$P^\star \cap D \subseteq (PD_S)^\star \cap D = P \subseteq P^\star \cap D,$$

whence $P^\star \cap D = P$; i.e., P is a quasi-$\star$-prime ideal of D. The same is true of P_0. Since D is a $\star$-GD domain, the extension $D \subseteq T$ satisfies $(\star, d_T)$-GD. As $Q \cap D = P$, it follows that there exists $Q_0 \in \operatorname{Spec}(T)$ such that $Q_0 \subseteq Q$ and $Q_0 \cap D = P_0$. Since $Q_0 \cap D_S = P_0D_S$, the proof is complete. $\square$

Comparing parts (d) and (e) of Proposition 5.2.2, one may ask for a semistar-theoretic version of the fact that each Prüfer domain is a going-down domain. Proposition 5.2.6 presents such a generalization. First, we need to recall the following information.

Proposition 5.2.6. *Let D be a domain and $\star$ a semistar operation on D. If D is a $P\star MD$, then D is a $\star$-*GD *domain.*

Proof. By Proposition 5.2.2 (a), we need to show that if T is an overring of D, then $D \subseteq T$ satisfies $(\star, d_T)$-GD. Suppose that $P_0 \subset P$ are quasi-$\star$-prime ideals of D and Q is a prime ideal of T such that $Q \cap D = P$. Our task is to find $Q_0 \in \text{Spec}(T)$ such that $Q_0 \subseteq Q$ and $Q_0 \cap D = P_0$. Note that $P_0 D_P \subseteq PD_P$ and $QT_Q \cap D_P = PD_P$. Since D is a $P\star MD$, [48, Theorem 3.1] ensures that D_P is a valuation domain. Therefore, D_P is a going-down domain. In particular, $D_P \subseteq T_Q$ satisfies GD. Hence, there exists a prime ideal $Q_0 T_Q$ of T_Q such that $Q_0 T_Q \subseteq QT_Q$ and $Q_0 T_Q \cap D_P = P_0 D_P$. Then $Q_0 \subseteq Q$ and $Q_0 \cap D = P_0$, as desired. $\square$

Recall from [28, Corollary 4] that Prüfer domains can be characterized as the integrally closed finite conductor going-down domains. In light of Proposition 5.2.6, one may ask for a semistar-theoretic version of this fact. Corollary 5.2.12 will present such a result. First, we will need some definitions and preliminary results.

Recall from [29] that a commutative ring A is said to be *treed* in case $\text{Spec}(A)$, as a poset under inclusion, is a tree. As the semistar analogue, we next define the notion of a $\star$-*treed domain.* Let D be a domain and $\star$ a semistar operation on D. Then D is said to be a $\star$-*treed domain* if $\text{QSpec}^\star(D)$, as a poset under inclusion, is a tree; i.e., if no quasi-$\star$-prime ideal of D contains incomparable quasi-$\star$-prime ideals of D. The proof of the following theorem is a straightforward adaptation of the proof of [29, Theorem 2.2]; for the sake of completeness, we include the details.

Theorem 5.2.7. *Let D be a domain and $\star$ a semistar operation on D. If D is a* $\star$-GD *domain, then D is a $\star$-treed domain.*

Proof. Suppose the assertion fails. Then some quasi-$\star$-prime ideal M of D contains incomparable quasi-$\star$-prime ideals, P_1 and P_2, of D. Choose $b \in P_1 \backslash P_2$ and $c \in P_2 \backslash P_1$. Set $u := bc^{-1}$. By the (u, u^{-1})-Lemma [69, Theorem 55], MD_M survives in either $D_M[u]$ or $D_M[u^{-1}]$, say, in $D_M[u]$. Choose a maximal ideal N of $D_M[u]$ which contains MD_M. Necessarily, $N \cap D_M = MD_M$, and so $N \cap D = M$. Since D is a $\star$-GD domain, $D \subseteq D_M[u]$ satisfies $(\star, d_{D_M[u]})$-GD. Thus, there exists a prime ideal Q of $D_M[u]$ contained in N and lying over P_2. Then $c \in Q$ and $b = cu \in Q \cap D = P_2 D$, contradicting the choice of b. $\square$

Remark 5.2.8. Suppose that one's view of the meaning of "tree" is such that the above definition of a $\star$-*treed domain* is altered to mean a domain D with a semistar operation $\star$ on D such that no quasi-$\star$-maximal ideal of D contains incomparable quasi-$\star$-prime ideals of D. In that case, Theorem 5.2.7 should be restated as follows: if D is a domain and $\star$ is a semistar operation on D such that D is a $\star_f$-GD domain, then D is a $\star_f$-treed domain. (To see that the proof of Theorem 5.2.7 carries over to the present setting, one needs also to handle the possibility that $D^\star = K$; however, in this case, it is easy to see that $E^\star = K$ for all $E \in \overline{\mathcal{F}}(D)$ and so one vacuously has that any quasi-$\star_f$-prime ideal is contained in a quasi-$\star$-maximal ideal.) If one uses this variant definition of "$\star$-treed domain", then $\star$ should be replaced with $\star_f$ in Theorem 5.2.11 and its applications below.

Recall that a domain D is called a *finite conductor domain* if every intersection of two principal ideals of D is finitely generated. All Prüfer damains are finite conductor domains. Let D be a domain and $\star$ a semistar operation on D. We define D to be a $\star$-*finite conductor* domain if, for all a, $b \in D$, there exist finitely many elements $u_1, \cdots, u_n \in D$ such that $((a) \cap (b))^\star = (u_1, \cdots, u_n)^\star$.

We recall the following theorem of McAdam.

Theorem 5.2.9. *([71, Theorem 1]) Let D be an integrally closed finite conductor domain whose prime ideals are linearly ordered by inclusion. Then D is a valuation domain.*

Theorem 5.2.11 will give a semistar-theoretic analogue of Theorem 5.2.9. First, we isolate a useful fragment of the argument.

Lemma 5.2.10. *Let D be a domain and $\star$ a semistar operation on D. If D is a $P\star MD$, then D is a $\widetilde{\star}$-finite conductor domain.*

Proof. First, note that $Na(D, \star)$ is a Prüfer domain by [48, Theorem 3.1], and hence is a finite conductor domain. Let $a, b \in D$. Consider the following equations:

$$\begin{aligned} ((a) \cap (b))^{\widetilde{\star}} =& ((a) \cap (b))Na(D, \star) \cap K = (f_1, \cdots, f_n)Na(D, \star) \cap K \\ =& (u_{ij})Na(D, \star) \cap K = (u_{ij})^{\widetilde{\star}}, \end{aligned}$$

where the first and fourth equations are via [49, Proposition 3.4]; the second equation holds since $Na(D, \star)$ is a finite conductor domain and there are finitely many elements $f_1, \cdots, f_n \in D[X]$ such that $((a) \cap (b))Na(D, \star) = (f_1, \cdots, f_n)Na(D, \star)$; the third equation holds since

$$(f_1, \cdots, f_n)Na(D, \star) = (u_{ij})Na(D, \star),$$

where u_{ij} are the coefficients of $f_1, \cdots, f_n$ (the point being that since D is a $P\star MD$, $Na(D,\star)$ coincides with the Kronecker function ring $Kr(D,\star)$ and [50, Theorem 3.11 (2)] can be applied). The assertion follows. $\square$

Theorem 5.2.11. *Let D be a domain and let $\star$ be a semistar operation on D. Then D is a $P\star MD$ if and only if the following three conditions hold:*

(1) $D^{\widetilde{\star}}$ is an integrally closed domain,

(2) D is a $\widetilde{\star}$-finite conductor domain, and

(3) D is $\widetilde{\star}$-treed.

Proof. Suppose that D is a $P\star MD$. Then $Na(D,\star) = Na(D^{\widetilde{\star}}, (\widetilde{\star})_\iota)$ is a Prüfer domain by [49, Corollary 3.5] and [48, Theorem 3.1], where $\iota : D \hookrightarrow D^{\widetilde{\star}}$ denotes the canonical inclusion. Since $(\widetilde{\star})_\iota$ is a (semi)star operation on $D^{\widetilde{\star}}$ and $D^{\widetilde{\star}}$ is a $P(\widetilde{\star})_\iota MD$ by [48, Corollary 3.6], $D^{\widetilde{\star}}$ is integrally closed by [58, Theorem 34.6, Proposition 34.7, and Theorem 34.11]. Thus, (1) holds. Moreover, (2) holds by Lemma 5.2.10. By applying Proposition 5.2.6 and Theorem 5.2.7, (3) also holds.

Conversely suppose that (1), (2) and (3) hold. To show that D is a $P\star MD$, it suffices, by [48, Theorem 3.1], to prove that D_M is a valuation domain for each $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$. Since $M^{\widetilde{\star}} \neq D^{\widetilde{\star}}$, we have $D^{\widetilde{\star}}_{M^{\widetilde{\star}}} = D_M$ by [80, Proposition 1.4]. Hence, by (1), D_M is integrally closed. It also then follows from (3) that $\mathrm{Spec}(D_M)$ is linearly ordered under inclusion. Next, we show that D_M is a finite conductor domain. Let $a, b \in D$. Since D is a $\widetilde{\star}$-finite conductor domain, there exist $u_1, \ldots, u_n \in D$ such that $(Da \cap Db)^{\widetilde{\star}} = (\sum Du_i)^{\widetilde{\star}}$. We claim that $D_M a \cap D_M b = \sum D_M u_i$.

Consider the following equations:

$$\begin{aligned} aD_M \cap bD_M =& (aD \cap bD)D_M = (aD \cap bD)^{\widetilde{\star}} D_M \\ =& (u_1, \cdots, u_n)^{\widetilde{\star}} D_M = (u_1, \cdots, u_n)D_M, \end{aligned}$$

where the first equation holds by the flatness of D_M over D; the second and fourth equations hold by [46, Theorem 4.1 (2)]; and the third equation follows easily from the choice of $u_1, \ldots, u_n$. This proves the above claim, and so D_M is a finite conductor domain. Hence, by McAdam's result (Theorem 5.2.9), D_M is a valuation domain, as desired. □

By combining Theorem 5.2.11 with Theorem 5.2.7 and Proposition 5.2.6, we get the next result. Note that Corollary 5.2.12 is the semistar-theoretic analogue of the characterization of Prüfer domains in [28, Corollary 4].

Corollary 5.2.12. *Let D be a domain and $\star$ a semistar operation on D. Then D is a $P\star MD$ if and only if the following three conditions hold:*

(1) $D^{\widetilde{\star}}$ is an integrally closed domain,

(2) D is a $\widetilde{\star}$-finite conductor domain, and

(3) D is a $\widetilde{\star}$-GD domain.

In view of Corollary 5.2.12, it seems natural to seek to determine test overrings for the "$\widetilde{\star}$-GD domain" property. Theorem 5.2.14 will present the semistar-theoretic analogue of the test overring result for going-down domains [35, Theorem 1], adding additional conditions involving the w-operation. First, we give the analogue of [35, Proposition].

Proposition 5.2.13. *Let D be a domain, $\star$ a semistar operation on D, and T a quasilocal treed domain containing D. If $D \subseteq D[u]$ satisfies $(\widetilde{\star}, d_{D[u]})$-GD for each u in T, then $D \subseteq T$ satisfies $(\widetilde{\star}, d_T)$-GD.*

Proof. (Sketch) Suppose that the assertion fails. Then there exist quasi-$\widetilde{\star}$-prime ideals $P \subseteq P_1$ of D and $Q_1 \in \mathrm{Spec}(T)$ such that $Q_1 \cap D = P_1$ and no $Q \in \mathrm{Spec}(T)$ satisfies both $Q \subseteq Q_1$ and $Q \cap T = P$. By [69, Exercise 37, page 44, (ii)$\Rightarrow$ (i)], the prime ideal N of T which is minimal over PT also satisfies $P_2 := N \cap D \supset P$. Note that N is uniquely determined since T is quasilocal and treed; thus, by [69, Theorem 10], $N \subseteq Q_1$. Hence, $P_2 \subseteq P_1$. Since P_1 is a quasi-$\widetilde{\star}$-prime ideal of D, it follows from [46, Lemma 4.1, Remark 4.5] that P_2 is also a quasi-$\widetilde{\star}$-prime ideal of D.

The foregoing can replace the first two sentences of the proof of [35, Proposition]. We can pick up the argument (for a while), beginning with the third sentence of that proof. In this way, we obtain elements $p_1 \in P, w \in T, r \in P_2 \setminus P$ and a positive integer mt such that $r^{mt} = p_1 w$; and a prime ideal N_1 of $D[w]$ minimal over $PD[w]$ and contained in $N \cap D[w]$. Arguing as above, we can show that $P_3 := N_1 \cap D$ satisfies $P \subseteq P_3 \subseteq P_2$ and P_3 is a quasi-$\widetilde{\star}$-prime ideal of D. Since the extension $D \subseteq D[w]$ satisfies $(\widetilde{\star}, d_{D[w]})$-GD by hypothesis, there exists a prime ideal N_2 of $D[w]$ such that $N_2 \subseteq N_1$ and $N_2 \cap D = P$. Then

$$r^{mt} = p_1 w \in PD[w] \subseteq N_2,$$

whence $r \in N_2 \cap D = P$, the desired contradiction. □

Let $\star$ be a semistar operation on a domain D. Recall that a valuation overring V of D is called a $\star$-*valuation overring of* D if $F^\star \subseteq FV$ for each $F \in f(D)$. It follows easily that if V is a $\star$-valuation overring of D, then V is a $(\star, d_V)$-linked overring of D. By [49, Theorem 3.9], V is a $\widetilde{\star}$-valuation overring of D if and only if V is a valuation overring of D_P for some quasi-$\star_f$-maximal ideal P of D.

Note that the statement of condition (3) in Theorem 5.2.14 is simplified because $\widetilde{\widetilde{\star}} = \widetilde{\star}$ for any semistar operation $\star$ [46, Corollary 3.9 (2)].

Theorem 5.2.14. *Let D be a domain and $\star$ a semistar operation on D. Then the following conditions are equivalent:*

(1) D is a $\widetilde{\star}$-GD domain;

(2) $D \subseteq D[u]$ satisfies $(\widetilde{\star}, d_{D[u]})$-GD for each u in K;

(3) $D \subseteq V$ satisfies $(\widetilde{\star}, d_V)$-GD for each $\widetilde{\star}$-valuation overring V of D;

(4) $D \subseteq T$ satisfies $(\widetilde{\star}, d_T)$-GD for each domain T containing D;

(5) $D \subseteq T$ satisfies $(\widetilde{\star}, \widetilde{\star'})$-GD whenever $\star'$ is a semistar operation on an overring T of D such that T is a $(\widetilde{\star}, \star')$-linked overring of D;

(6) $D \subseteq T$ satisfies $(\widetilde{\star}, w_T)$-GD for each overring T of D;

(7) $D \subseteq T$ satisfies $(\widetilde{\star}, w_T)$-GD for each overring T of D such that T is a $(\widetilde{\star}, w_T)$-linked overring of D.

Proof. (1) $\Rightarrow$ (2); (4) $\Rightarrow$ (3): Trivial.

(2) $\Rightarrow$ (4): We adapt the proof of the implication $(a) \Rightarrow (c)$ in [35, Theorem 1] by making the following three changes: take $P \subseteq M$ to be quasi-$\widetilde{\star}$-prime ideals of D; use Proposition 5.2.13 to conclude that $D \subseteq V$ satisfies $(\widetilde{\star}, d_V)$-GD; and reason via "transitivity" to show that $D \subseteq W$ satisfies $(\widetilde{\star}, d_W)$-GD.

(3) $\Rightarrow$ (1): Assume (3). Our task is to show that if S be an overring of D, then $D \subseteq S$ satisfies $(\widetilde{\star}, d_S)$-GD. Suppose that $P_0 \subset P$ are quasi-$\widetilde{\star}$-prime ideals of D and $Q \in \operatorname{Spec}(S)$ such that $Q \cap D = P$. Choose a quasi-$\widetilde{\star}$-maximal ideal M of D which contains P. (Note that $\widetilde{\star}_f = \widetilde{\star}$.) Then $D_M \subseteq S_{D \setminus M}$ and $Q \cap (D \setminus M) = \emptyset$. Choose a valuation overring (V, W) of $S_{D\setminus M}$ such that $W \cap S_{D\setminus M} = QS_{D\setminus M}$. Hence, $W \cap D = P$. Since V is a valuation overring of D_M and M is a quasi-$\widetilde{\star}_f$-maximal ideal of D, it follows from [49, Theorem 3.9] that V is a $\widetilde{\star}$-valuation overring of D. Therefore, by (3), we can find $W_0 \in \operatorname{Spec}(V)$ such that $W_0 \cap D = P_0$. Thus, W_0 is the desired prime ideal of V, since $W_0 \cap S$ is contained in Q and lies over P_0.

(4) $\Rightarrow$ (5); (4) $\Rightarrow$ (6); (5) $\Rightarrow$ (7): The first two assertions are trivial, and the third assertion follows because $\widetilde{w_T} = w_T$.

(6) $\Rightarrow$ (3); (7) $\Rightarrow$ (3): By [37, Lemma 2.7], V is a $\widetilde{\star}$-valuation overring of D if and only if V is a $(\widetilde{\star}, d_V)$-linked valuation overring of D. The assertions then follow because $d_V = w_V$, the point being that any (semi)star operation of finite type on a valuation domain V coincides with d_V. $\square$

The next result generalizes the part of [1, Corollary 2.12] stating that if a domain D is such that its Nagata ring $D(X)$ is a going-down domain, then D is a going-down domain.

Corollary 5.2.15. *Let D be a domain and $\star$ a semistar operation on D. If $Na(D, \star)$ is a going-down domain, then D is a $\widetilde{\star}$-GD domain and, hence, a $\star$-GD domain.*

Proof. It is known that $Na(D, \widetilde{\star}) = Na(D, \star)$. Therefore, the first assertion follows by combining Theorem 5.1.5 and the implication (5) $\Rightarrow$ (1) in Theorem 5.2.14; then the second assertion follows from Proposition 5.2.2 (c) since $\widetilde{\star} \leq \star$. $\square$

We pause to note that Theorem 5.2.14 leads to the following alternate proof of Proposition 5.2.6. Since D is a $P\star MD$, it follows from [48, Theorem 3.1] that $Na(D,\star)$ is a Prüfer domain. Thus, $Na(D,\star)$ is a going-down domain, and so by Corollary 5.2.15, D is a $\star$-GD domain, to complete the (second) proof.

At the beginning of this section, we mentioned that the standard examples of going-down domains are Prüfer domains, one-dimensional domains and certain pullbacks. Having shown that the semistar-theoretic analogues of Prüfer domains and of one-dimensional domains are $\star$-GD domains in Propositions 5.2.6 and 5.2.2 (e), we turn next to "certain pullbacks". In particular, Corollary 5.2.17 will give a partial semistar-theoretic analogue of a result in [32] on CPI-extensions.

Recall that $P \in \text{Spec}(D)$ is called (a) *divided (prime ideal of) D* if $P = PD_P$. In [32, Theorem 2.2], D. E. Dobbs proved that a domain D is a going-down domain if and only if there exists a divided prime ideal P of D such that both D/P and D_P are going-down domains. The following result is a semistar-theoretic analogue of [32, Theorem 2.2].

Theorem 5.2.16. *Let D be a domain and $\star$ a semistar operation on D. If P is a divided quasi-$\widetilde{\star}$-prime ideal of D such that D/P is a going-down domain and D_P is a $\widetilde{\star}_P$-*GD *domain, then D is a $\widetilde{\star}$-*GD *domain.*

Proof. We modify the proof of [32, Theorem 2.2] in the natural way. For the sake of completeness, the details are provided next.

By Theorem 5.2.14, it suffices to show that $D \subseteq V$ satisfies $(\widetilde{\star}, d_V)$-GD for each $\widetilde{\star}$-valuation overring V of D. Let $P_2 \subset P_1$ be quasi-$\widetilde{\star}$-prime ideals of D and let $Q_1 \in \text{Spec}(V)$ such that $Q_1 \cap D = P_1$. Note that P is comparable to both P_1 and P_2

since P is divided in D. We next consider three cases.

Case 1: $P \subseteq P_2$. By [32, Lemma 2.1], $D \subseteq V$ satisfies "going-down to P"; i.e., there exists $J \in \mathrm{Spec}(V)$ such that $J \subseteq Q_1$ and $J \cap D = P$. Observe that the extension of domains $D/P \subseteq V/J$ satisfies GD, since D/P is a *GD* domain. As $Q_1/J \cap D/P = P_1/P$, there exists $Q_2 \in \mathrm{Spec}(V)$ such that $J \subseteq Q_2 \subseteq Q_1$ and $Q_2/J \cap D/P = P_2/P$. Thus, $Q_2 \cap D = P_2$, as desired.

Case 2: $P_2 \subseteq P \subseteq P_1$. By [32, Lemma 2.1], $D \subseteq V$ satisfies "going-down to P", and so there exists $I \in \mathrm{Spec}(V)$ such that $I \subseteq Q_1$ and $I \cap D = P$. Observe that the extension of domains $D_P \subseteq V_I$ satisfies $(\widetilde{\star}_P, d_{V_I})$-GD, since D_P is a $\widetilde{\star}_P$-GD domain.

We claim that PD_P is a quasi-$\widetilde{\star}_P$-prime ideal of D_P. Indeed, since $P^{\widetilde{\star}} \cap D = P$ and localization commutes with intersection, we have that $P^{\widetilde{\star}}D_P \cap D_P = PD_P$. Therefore, since $PD_P = P$, it follows that

$$PD_P \subseteq (PD_P)^{\widetilde{\star}_P} \cap D_P = (PD_P)^{\widetilde{\star}} \cap D_P = P^{\widetilde{\star}} \cap D_P \subseteq PD_P.$$

Hence, $PD_P = (PD_P)^{\widetilde{\star}_P} \cap D_P$, thus proving the above claim.

By [46], it follows that P_2D_P is also a quasi-$\widetilde{\star}_P$-prime ideal. As $IV_I \cap D_P = PD_P$, there exists $Q_2 \in \mathrm{Spec}(V)$ such that $Q_2 \subseteq I$ and $Q_2V_I \cap D_P = P_2D_P$. Thus, $Q_2 \cap D = P_2$, as desired.

Case 3: $P_1 \subseteq P$. Argue as in Case 2, using the fact that $D_P \subseteq V_{D \setminus P}$ satisfies $(\widetilde{\star}_P, d_{V_{D\setminus P}})$-GD to produce a suitable Q_2. □

The next two results concern CPI-extensions, in the sense of [16]. Recall that if P is a prime ideal of a domain R, then *the CPI-extension of R with respect to P* is $R + PR_P$, which may, of course, be viewed as the pullback $R_P \times_{R_P/PR_P} R/P$. Note that $(R + PR_P)_{PR_P} = R_P$ and $(R + PR_P)/PR_P \cong R/P$ (cf. [31, Lemma 2.2(a)]).

Corollary 5.2.17. *([32, Corollary 2.3]) Let R be a domain, $P \in \mathrm{Spec}(R)$, and $\star$ a semistar operation on $R + PR_P$. Suppose that PR_P is a quasi-$\widetilde{\star}$-prime ideal of $R + PR_P$, that R/P is a going-down domain, and that R_P is a $\widetilde{\star}_P$-GD domain. Then $R + PR_P$ is a $\widetilde{\star}$-GD domain.*

Proof. Consider $D := R + PR_P$ and $Q := PR_P$. By [16, Proposition 2.5, Theorem 2.4] (cf. also [31, Lemma 2.2 (a)]), the quasi-$\widetilde{\star}$-prime ideal Q of D is also a divided prime ideal of D. Moreover, as noted above, $D/Q \cong R/P$ and $D_Q = R_P$. In addition, it is easy to check that the semistar operations $\widetilde{\star}_P$ on R_P and $\widetilde{\star}_Q$ on D_Q coincide, and so the hypothesis that R_P is a $\widetilde{\star}_P$-GD domain implies (actually, is equivalent to the fact) that D_Q is a $\widetilde{\star}_Q$-GD domain. Therefore, by Theorem 5.2.16, $D = R + PR_P$ is a $\widetilde{\star}$-GD domain. □

Remark 5.2.18. Let D be a domain and $\star$ a semistar operation on D. Although it follows from Proposition 5.2.5 that if D is a $\star$-GD domain, then D_P is a $\star_P$-GD domain for all $P \in \mathrm{Spec}(D)$, it need not be the case that if D is a $\star$-GD domain, then D/P is a going-down domain, even if P is a quasi-$\star$-prime ideal. (This should be contrasted with the fact [30, Remark 2.11] that if D is a going-down domain and $P \in \mathrm{Spec}(D)$, then D/P is also a going-down domain.) For a specific example, let K be a field; X, Y, and Z algebraically independent indeterminates over K; $D := K[X, Y, Z]$; and Δ the set of prime ideals P of D such that $\mathrm{ht}(P) = 1$. As in Example 5.2.3, let $\star := \star_\Delta$, the spectral semistar operation associated to Δ. Then $\mathrm{QMax}^{\widetilde{\star}}(D) = \Delta$, and so $\widetilde{\star}$-$\dim(D) = 1$. Therefore, by Proposition 5.2.2 (e), D is a $\star$-GD domain. Consider $P := (X) \in \Delta$. Then $D/P \cong K[Y, Z]$ is not a going-down domain, since D/P is a Noetherian domain of (Krull) dimension 2: see [28, Proposition 7].

Let D be a domain with quotient field K. Recall that the *global transform of* D is the overring of D defined by $D^g := \{x \in K \mid xM_1 \cdots M_k \subseteq R$ for some $M_i \in \text{Max}(D)\}$. (See [72] for D^g in the more general setting of rings possibly with nontrivial zero-divisors). Matijevic has shown that if D is a Noetherian domain, then any ring between D and D^g is also a Noetherian domain [72, Corollary]. It is easy to verify that if D is a one-dimensional Noetherian domain, then $D^g = K$, and so Matijevic's result generalizes the Krull-Akizuki Theorem (in the form given in [69, Theorem 93]). In Proposition 5.2.19, we will give a semistar-theoretic version of Matijevic's result.

Let $\star$ be a semistar operation on D. As the semistar-theoretic analogue of the global transform, one defines the $\star$-*global transform of* D to be the set $D^{\star g} = \{x \in K \mid xP_1 \cdots P_k \subseteq D$ for some $P_i \in \text{QMax}^{\star}(D)\}$. (See [19, page 227].) Arguing as in the proof of [23, Lemma 3.2(3)], one can show that $Na(D, \star)^g \cap K = D^{\star g}$.

Proposition 5.2.19. *Let D be a domain and $\star$ a semistar operation on D such that D is a $\widetilde{\star}$-Noetherian domain. Let T be an overring of D such that $T \subseteq D^{\star g}$. Let $\star'$ be a semistar operation on T such that T is a $(\star, \star')$-linked overring of D. Then T is a $\widetilde{\star'}$-Noetherian domain.*

Proof. Note that $Na(D, \star)$ is a Noetherian domain by [80, Theorem 3.6]; and $Na(D, \star) \subseteq T[X]_{N_\star} \subseteq Na(D, \star)^g$ by the above comments. Hence, by Matijevic's result [72, Corollary], $T[X]_{N_\star}$ is Noetherian. Since T being $(\star, \star')$-linked over D implies that $N_\star \subseteq N_{\star'}$,we have that $Na(T, \star') = (T[X]_{N_\star})_{N_{\star'}}$ is also Noetherian. Consequently, T is a $\widetilde{\star'}$-Noetherian domain by [80, Theorem 3.6]. □

Corollary 5.2.20. *Let D be a domain which is not a field, and let $\star$ be a semistar operation on D. Then the following conditions are equivalent:*

(1) D is a $\widetilde{\star}$-Noetherian domain and $\widetilde{\star}$-$\dim(D) = 1$*;*

(2) Each $(\star, \star')$-linked overring of D is a $\widetilde{\star'}$-Noetherian domain;

(3) $Na(D, \star)$ is a one-dimensional Noetherian domain;

(4) Each overring of $Na(D, \star)$ is a Noetherian domain.

Proof. We adapt the proof given in [19, Corollary 3.7] for the star operation case.

(1) $\Leftrightarrow$ (3): This follows by combining [80, Theorem 3.6] and Remark 1.1.

(3) $\Rightarrow$ (2): By (3) and the above remarks, $Na(D, \star)^g$ is the quotient field of $Na(D, \star)$, namely, $K(X)$. As we know that $Na(D, \star)^g \cap K = D^{\star g}$, it follows that $D^{\star g} = K$. Hence, by Proposition 5.2.19, each $(\star, \star')$-linked overring of D is a $\widetilde{\star'}$-Noetherian domain.

(3) $\Leftrightarrow$ (4): It is well known that a domain E is Noetherian and of dimension at most 1 if and only if each overring of E (including E itself) is Noetherian: cf. [69, Theorem 93 and Exercise 20, page 64].

(2) $\Rightarrow$ (1): Assume (2). As D is $(\star, \star)$-linked over itself, it follows that D is a $\widetilde{\star}$-Noetherian domain. If the assertion fails, $\widetilde{\star}$-$\dim(D) \geq 2$. If this is so, we can choose $P_1 \subset P_2$ to be quasi-$\widetilde{\star}$-prime ideals of D with P_2 a quasi-$\widetilde{\star}$-maximal ideal of D. By [58, Corollary 19.7 (1)], there exists a valuation overring V of D with prime ideals $Q_1 \subset Q_2$ such that $Q_i \cap D = P_i$, $i = 1, 2$, and Q_2 is the unique maximal ideal of V. We will show that V is $(\star, d_V)$-linked over D. If this is not the case, there exists a nonzero finitely generated ideal F of D such that $F^\star = D^\star$ and $FV \neq V$. These two conditions imply, respectively, that $F \nsubseteq P_2$ (by a comment in the Introduction) and $FV \subseteq Q_2$. Therefore, $F \subseteq FV \cap D \subseteq Q_2 \cap D = P_2$, a contradiction. This proves

that V is $(\star, d_V)$-linked over D. Hence, by (2), V is a d_V-Noetherian domain, i.e., a Noetherian domain. As V is also a valuation domain, $\dim(V) \leq 1$, the desired contradiction. $\square$

Our final result will show that for $\widetilde{\star}$-Noetherian domains D, the "$\widetilde{\star}$-GD domain" property for D is equivalent to the ordinary "going-down domain" property for the associated $\star$-Nagata domain.

Theorem 5.2.21. *Let D be a domain and $\star$ a semistar operation on D. Suppose also that D is a $\widetilde{\star}$-Noetherian domain. Then the following conditions are equivalent:*

(1) $\widetilde{\star}\text{-}\dim(D) \leq 1$*;*

(2) If T is any overring of D and $\star'$ is a semistar operation on T such that T is a $(\star, \star')$-linked overring of D, then T is a $\widetilde{\star'}$-GD *domain;*

(3) D is a $\widetilde{\star}$-GD *domain;*

(4) $Na(D, \star)$ is a going-down domain.

Proof. (1) $\Rightarrow$ (2): Assume (1). Let $T, \star'$ be such that T is a $(\star, \star')$-linked overring of D. Then by Corollary 5.2.20, T is a $\widetilde{\star'}$-Noetherian domain. Using the implication (2) $\Rightarrow$ (1) in Corollary 5.2.20 and the fact that linkedness is a transitive relation [36, Lemma 3.1 (b)], we conclude that $\widetilde{\star'}\text{-}\dim(T) \leq 1$. Hence, T is a $\widetilde{\star'}$-GD domain by Proposition 5.2.2 (e).

(2) $\Rightarrow$ (3): This is trivial since D is a $(\star, \star)$-linked overring of itself;

(3) $\Rightarrow$ (1): We will prove the contrapositive. Suppose that $\mathrm{ht}(P) > 1$ for some quasi-$\widetilde{\star}$-prime ideal P of D. Since D is a $\widetilde{\star}$-Noetherian domain, D_P is Noetherian by

[37, Proposition 3.8]. By a celebrated result of Chevalley [22], there is a DVR (hence one-dimensional) overring V of D_P such that the maximal ideal of V lies over P. Then $D \subseteq V$ does not satisfy $(\widetilde{\star}, d_V)$-GD, and so D is not a $\widetilde{\star}$-GD domain.

(1) $\Leftrightarrow$ (4): Note that $\widetilde{\star}\text{-}\dim(D) = \dim(Na(D, \star))$ by Remark 5.0.5; and $Na(D, \star)$ is a Noetherian domain by [80, Theorem 3.6]. Recall that a Noetherian domain is a going-down domain if and only if it is of (Krull) dimension at most 1 [28, Proposition 7]. Therefore, $Na(D, \star)$ is a going-down domain if and only if $\dim(Na(D, \star)) \leq 1$, that is, if and only if $\widetilde{\star}\text{-}\dim(D) \leq 1$. $\square$

Note that Theorem 5.2.21 provides a partial converse for Corollary 5.2.15. There are natural limits on possible stronger converses. In other words, there exist examples of a domain D and a semistar operation $\star$ on D such that D is a $\widetilde{\star}$-GD domain but $Na(D, \star)$ is not a going-down domain. In fact, such examples exist with $\star = d_D$, for according to [1, Corollary 2.12 and Theorem 2.7], it suffices to produce a going-down domain D whose integral closure is not a Prüfer domain. To that end, one need only consider any one-dimensional integrally closed domain D which is not a Prüfer domain (cf. [1, Remark 2.11]). In the next section we provide a complete answer to the converse for Corollary 5.2.15.

5.3 $\star$-Nagata domains and the GD property

In the last section we characterize going-down (treed) $\star$-Nagata domains. Recall that D is said to be a $\star$-quasi-Prüfer domain, in case, if Q is a prime ideal in $D[X]$, and $Q \subseteq P[X]$, for some $P \in \mathrm{QSpec}^{\star}(D)$, then $Q = (Q \cap D)[X]$. This notion is the semistar analogue of the classical notion of the quasi-Prüfer domains [47, Section 6.5] (that is among other equivalent conditions, the domain D is said to be a quasi-Prüfer domain if it has Prüfer integral closure). Let D be an integral domain and $\star$ a semistar operation on D.

One can consider the contraction map $h : \mathrm{Spec}(Na(D,\star)) \to \mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$. Indeed if N is a prime ideal of $Na(D,\star)$, then there exists a quasi-$\widetilde{\star}$-maximal ideal M of D, such that $N \subseteq MNa(D,\star)$. So that

$$h(N) = N \cap D \subseteq MNa(D,\star) \cap D = MNa(D,\star) \cap K \cap D = M^{\widetilde{\star}} \cap D = M.$$

The third equality holds by [49, Proposition 3.4 (3)]. So that $h(N) \in \mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$, since it is contained in M and [46, Lemma 4.1 and Remark 4.5]. Note that if $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$, then

$$h(PNa(D,\star)) = PNa(D,\star) \cap D = PNa(D,\star) \cap K \cap D = P^{\widetilde{\star}} \cap D = P.$$

Therefore $h(\mathrm{Spec}(Na(D,\star))) = \mathrm{QSpec}^{\widetilde{\star}}(D) \cup \{0\}$. In fact using [21, Theorem 2.16], the map h is an isomorphism if and only if D is a $\star_f$-quasi-Prüfer domain.

Lemma 5.3.1. *Let D be an integral domain and $\star$ a semistar operation on D. Then the canonical map $h : \mathrm{Spec}(Na(D,\star)) \to Q\,\mathrm{Spec}^{\widetilde{\star}}(D) \cup \{0\}$ is a bijective if and only if D is a $\widetilde{\star}$-quasi-Prüfer domain.*

Lemma 5.3.2. *Let D be an integral domain and $\star$ a semistar operation on D such that $Na(D,\star)$ is treed. Then:*

(1) D is $\widetilde{\star}$-treed.

(2) *The canonical contraction map* $h : \operatorname{Spec}(Na(D,\star)) \to Q\operatorname{Spec}^{\widetilde{\star}}(D) \cup \{0\}$ *is a bijective.*

Proof. (1) The proof is identical with [1, Proposition 2.2 (a)].

(2) If the assertion fails, there exists a quasi-$\widetilde{\star}$-prime ideal P of D and a prime ideal N of $Na(D,\star)$ such that $h(N) = P$ and $N \neq PNa(D,\star)$. Then $Q = N \cap D[X]$ is an upper of P. Choose a maximal ideal $MNa(D,\star)$ of $Na(D,\star)$ (for a suitable quasi-$\widetilde{\star}$-maximal ideal M of D). Consider the inclusions $P[X] \subset Q \subset M[X]$. For polynomials over the domain D/P, we have $\overline{Q} = Q/P[X]$ is an upper to zero contained in $\overline{M}[X]$, where $\overline{M} = M/P$. By [1, Lemma 2.1 (d)], $\overline{M}[X]$ contains infinitely many (incomparable) uppers to zero. Hence $M[X]$ in $D[X]$ contains infinitely many incomparable primes. Passing to $Na(D,\star)$, we find that $MNa(D,\star)$ contains infinitely many incomparable primes. Thus $Na(D,\star)$ can not be a treed domain, a contradiction. $\square$

Theorem 5.3.3. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

(1) $Na(D,\star)$ *is treed.*

(2) D *is* $\widetilde{\star}$*-treed, and the canonical map* $h : \operatorname{Spec}(Na(D,\star)) \to Q\operatorname{Spec}^{\widetilde{\star}}(D) \cup \{0\}$ *is a bijective.*

(3) D *is* $\widetilde{\star}$*-treed, and a* $\widetilde{\star}$*-quasi-Prüfer domain.*

Proof. (1) $\Rightarrow$ (2) holds by Lemma 5.3.2.

(2) $\Rightarrow$ (1). Deny. Then some quasi-$\widetilde{\star}$-maximal ideal M of D is such that $MNa(D,\star)$ contains incomparable primes, $\mathfrak{Q}_1, \mathfrak{Q}_2 \in \operatorname{Spec}(Na(D,\star))$. Put $Q_i := \mathfrak{Q}_i \cap$

$D[X]$ and $P_i := Q_i \cap D = \mathfrak{Q}_i \cap D$. So that $P_iNa(D,\star) \subseteq Q_iNa(D,\star) \subseteq MNa(D,\star)$. We have

$$h(Q_iNa(D,\star)) = P_i = h(P_iNa(D,\star)),$$

and therefore $Q_iNa(D,\star) = P_iNa(D,\star)$ since h is injective. So that $\mathfrak{Q}_i = Q_iNa(D,\star) = P_iNa(D,\star)$, which implies that P_1 and P_2 are incomparable. Now consider

$$P_i = Q_i \cap D \subseteq MNa(D,\star) \cap D = M^{\widetilde{\star}} \cap D = M,$$

which is contradicts the hypothesis that D is $\widetilde{\star}$-treed.

(2) $\Leftrightarrow$ (3) is true by Lemma 5.3.1. $\square$

Proposition 5.3.4. *Let D be a domain which is not a field, and let $\star$ be a semistar operation on D. Suppose that D is a $\widetilde{\star}$-Noetherian domain. Then D is $\widetilde{\star}$-treed (if and) only if $\widetilde{\star}$-$\dim(D) = 1$.*

Proof. Suppose that D is a $\widetilde{\star}$-treed domain. Assume that $\widetilde{\star}$-$\dim(D) > 1$. So there exists a quasi-$\widetilde{\star}$-maximal ideal M of D such that $\mathrm{ht}(M) > 1$. Note that D_M is treed, and by [37, Proposition 3.8] D_M is a Noetherian domain. This contradicts the hypothesis of [73, Theorem 31.2]. $\square$

Proposition 5.3.5. *Let D be an integral domain and $\star$ a semistar operation on D. The following then are equivalent:*

(1) D is a $\widetilde{\star}$-GD domain.

(2) D_P is a GD domain for all $P \in \mathrm{QSpec}^{\widetilde{\star}}(D)$.

(3) D_M is a GD domain for all $M \in \mathrm{QMax}^{\widetilde{\star}}(D)$.

Proof. (1) $\Rightarrow$ (2). Let T be an overring of D_P. Suppose that $P_1D_P \subset P_2D_P$ are prime ideals of D_P and Q_2 is a prime ideal of T such that $Q_2 \cap D_P = P_2D_P$. Since $P_1 \subset P_2$ are quasi-$\widetilde{\star}$-prime ideals of D (since they are contained in P and [46, Lemma 4.1 and Remark 4.5]) and $Q_2 \cap D = P_2$ and the fact that D is a $\widetilde{\star}$-GD domain, there exists a prime ideal Q_2 of T satisfying both $Q_1 \subseteq Q_2$ and $Q_1 \cap D = P_1$. So that $Q_1 \cap D_P = P_1D_P$. Therefore D_P is a going-down domain.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let that T be an overring of D. Suppose that $P_1 \subset P_2$ are quasi-$\widetilde{\star}$-prime ideals of D and Q_2 is a prime ideal of T such that $Q_2 \cap D = P_2$. There exists a quasi-$\widetilde{\star}$-maximal ideal M of D such that contains P_2. So we have $P_1D_M \subset P_2D_M$ and $Q_2 \cap D_M = P_2D_M$. Since by hypothesis D_M is a GD domain, there exists a prime ideal Q_1 in T satisfying both $Q_1 \subseteq Q_2$ and $Q_1 \cap D_M = P_1D_M$. Thus we obtain that $Q_1 \cap D = P_1$. This ends the proof. □

The following theorem is the main result of this section.

Theorem 5.3.6. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

(1) $Na(D, \star)$ is a GD *domain.*

*(2) D is $\widetilde{\star}$-*GD*, and the canonical map $h : \mathrm{Spec}(Na(D, \star)) \to Q\,\mathrm{Spec}^{\widetilde{\star}}(D) \cup \{0\}$ is a bijective.*

*(3) D is $\widetilde{\star}$-*GD*, and a $\widetilde{\star}$-quasi-Prüfer domain.*

Proof. (1) $\Rightarrow$ (2). Note that by [29, Theorem 2.2], we have $Na(D, \star)$ is a treed domain and therefore by Theorem 5.3.3, h is bijective. Also D is $\widetilde{\star}$-GD by Corollary 5.2.15.

(2) $\Rightarrow$ (3) is true by Lemma 5.3.1.

(3) $\Rightarrow$ (1). Assume that (3) holds true. We plan to show that localizations of $Na(D,\star)$ at every maximal ideal is a GD domain. Since D is a $\widetilde{\star}$-quasi-Prüfer domain, by [21, Theorem 2.16] we have $Na(D,\star)$ is a quasi-Prüfer domain. Let N be a maximal ideal of $Na(D,\star)$. So by [49, proposition 3.1 (3)] there is an $M \in Q\operatorname{Max}^{\widetilde{\star}}(D)$ such that $N = MNa(D,\star)$. Since $Na(D,\star)$ is a quasi-Prüfer domain, using [21, Theorem 1.1], $Na(D,\star)_N = D_M(X)$ is a quasi-Prüfer domain. Another use of [21, Theorem 1.1] yields us that D_M itself is a quasi-Prüfer domain. On the other hand by Proposition 5.3.5, we have D_M is a GD domain. Hence we showed that D_M is both a GD and a quasi-Prüfer domain. So that by [1, Corollary 2.12], $D_M(X)$ is a GD domain. Now $Na(D,\star)$ is a GD domain by [29, Lemma 2.1] or by Proposition 5.3.5 with $\star = d$ the identity semistar operation. $\square$

Corollary 5.3.7. *Let D be an integral domain. The following are equivalent:*

(1) $Na(D,v)$ is a GD *domain.*

*(2) D is w-*GD*, and the canonical map $h : \operatorname{Spec}(Na(D,v)) \to Q\operatorname{Spec}^w(D) \cup \{0\}$ is a bijective.*

*(3) D is w-*GD*, and a w-quasi-Prüfer domain.*

The next corollary gives several new characterizations of P$\star$MDs. Recall that a domain D, with a semistar operation $\star$ on D, is a P$\star$MD if and only if $Na(D,\star)$ is a Prüfer domain. By a characterization of Prüfer domains (cf. [71, Theorem 1]), this is equivalent to $Na(D,\star)$ being an integrally closed treed finite-conductor domain. One point of Corollary 5.3.8 is that, domains of the form $Na(D,\star)$ are so special that, for them, the "finite-conductor domain" condition becomes redundant.

Corollary 5.3.8. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

(1) D is a $P\star MD$.

(2) $D^{\widetilde{\star}}$ is integrally closed and $Na(D,\star)$ is a GD *domain.*

(3) $D^{\widetilde{\star}}$ is integrally closed and $Na(D,\star)$ is a treed domain.

(4) $Na(D,\star)$ is integrally closed and GD.

(5) $Na(D,\star)$ is integrally closed and treed.

Proof. (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are trivial, since GD domains are treed.

(1) $\Rightarrow$ (2). If D is a $P\star MD$ then by [48, Theorem 3.1], $Na(D,\star)$ is a Prüfer domain. So that $Na(D,\star)$ is integrally closed. Thus $D^{\widetilde{\star}} = Na(D,\star)\cap K$ is also integrally closed.

(3) $\Rightarrow$ (1). Since $Na(D,\star)$ is a treed domain, we have D is a $\widetilde{\star}$-quasi-Prüfer domain by Theorem 5.3.3. Thus $Na(D,\star)$ is a quasi-Prüfer domain by [21, Theorem 2.16]. Since $D^{\widetilde{\star}}$ is integrally closed and $Na(D,\star)$ is a quasi-Prüfer domain, we obtain that $Na(D,\star)$ is a Prüfer domain by [21, Corollary 2.17]. Hence D is a $P\star MD$ by [48, Theorem 3.1].

(1) $\Rightarrow$ (4). Since D is a $P\star MD$, then by [48, Theorem 3.1], $Na(D,\star)$ is a Prüfer domain. So that $Na(D,\star)$ is integrally closed.

(5) $\Rightarrow$ (3). Since $Na(D,\star)$ is integrally closed, we have $D^{\widetilde{\star}} = Na(D,\star)\cap K$ is integrally closed. $\square$

The following is a new characterization of $PvMD$s.

Corollary 5.3.9. *Let D be an integral domain. The following are equivalent:*

(1) D is a $PvMD$.

(2) D is integrally closed and $Na(D, v)$ is a GD *domain.*

(3) D is integrally closed and $Na(D, v)$ is a treed domain.

(4) $Na(D, v)$ is integrally closed and GD*.*

(5) $Na(D, v)$ is integrally closed and treed.

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