GENERALIZED HUGHES COMPLEXES

AND

ACTION OF CERTAIN GROUPS ON CERTAIN MODULES AND COMPLEXES

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ABSTRACT

The aims of Chapter (I) are to provide a characterization of generalized Hughes complexes, and to give several illustrations of situations where this characterization can be exploited to good effect, and to prove that whenever our commutative ring is not Noetherian, then there is a morphism of complexes from a generalized Hughes complex of a certain type to a complex of modules of generalized fractions.

Suppose that R is a commutative ring with non-zero identity and G is a group of automorphisms of R. We use R^G to denote the fixed subring. The purposes of Chapter (II) are the following discussion: (i) under "good" conditions on R and G, what general "good" properties does R^G enjoy? (ii) Can G induce an action group on certain modules and complexes (for example local cohomology, injective envelope and Cousin complex)? Can we specify the fixed submodule and subcomplex?

§0 Introduction.

0.1 GENERAL NOTATIONS. Througout this thesis, A (respectively R) will denote a commutative Noetherian (respectively commutative) ring with non-zero identity, and $\mathcal{C}(A)$ (respectively $\mathcal{C}(R)$) will denote the category of A-modules and A-homomorphisms (respectively R-modules and R-homomorphisms). We shall use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers.

0.2 SOME HISTORY. The Cousin complex considered in Algebraic and Analytic Geometry have Also commutative algebra analogues given by Sharp in [23]. Suppose that M is an A-module. In [23], Sharp defined, $C(M)$, the Cousin complex for M. It is, in fact, the commutative algebra analogue of the Cousin complex of §2 of Chapter IV of Hartshorne [7].

Cohen-Macauly rings can be characterized in terms of the Cousin complex: A is a Cohen-Macaulay ring if and only if $C(A)$ is exact [23, (4.7)]. Weaker conditions on the $C(A)$ provide reasonable classifications of some rings which not be Cohen-Macaulay (see [26]). For example, the $C(A)$ is exact at its (-1) th, oth, $\cdots$, $(n-2)$th terms if and only if the ring A satisfies the condition (S_n) (see [26,2.2]). Also, the Cousin complex provides a natural minimal injective resolution for a Gorenstein ring (See [23, (5.4)]). For M, non-zero and finitely generated, A-module we have that M is a Cohen-Macaulay if and only if $C(M)$ is exact (See [25, (2.4)]).

Various more general Cousin complexes can be constructed. We can, for any filtration $\mathcal{F}$ [33, (1.1)] of Spec (A) that admits M, construct the Cousin complex $C(\mathcal{F}, M)$ for M with respect to $\mathcal{F}$ [33, (1.3)]. If we use for $\mathcal{F}$ the M-height filtration $\mathcal{H}(M)$ of Spec (A) [33, (1.2)], then $C(\mathcal{H}(M), M)$ is just the Cousin complex $C(M)$

which is mentioned earlier. When A is local, using the dimemsion filtration $\mathcal{D}(A)$ [33, (1.2)] of Spec (A), permits a characterization of balanced big Cohen-Macaulay A-modules (not necessarily finitely generated). A-module X is *a balanced big Cohen-Macaulay* A-module [32, P.229] if every system of parameters for A is an X-sequence. It trurns out that M is a balanced big Cohen-Macaulay A-module if and only if $C(\mathcal{D}(A), M)$ is exact and $\mathfrak{m}\mathfrak{M} \neq \mathfrak{M}$ (where $\mathfrak{m}$ denotes the maximal ideal of A).

Although Cousin complexes do provide satisfactory characterizations of various Cohen-Macaulay properties, they have the disadvantage that their constraction is rather complicated, and this perhaps makes them difficult to work with. The purpose of [21] is to show that, for an A-module M such that $Ass(M)$ has only finitely many minimal members and a filtraction $\mathcal{F}$ of Spec (A) which admits M, the Cousin complex $C(\mathcal{F}, M)$ is actually isomorphic to a complex of modules of generalized fractions in the sense of [38]; this description is perhaps simpler and easier to work with.

K.R. Hughes [12] introduced a grade-theoretic analogue of the Cousin complex. he emploied Ree's concept of the grade of a proper ideal $\mathfrak{b}$ of A. Sharp and Yassi [36] explored relationships between Hughes's complex and the complex of the modules of generalized fractions of sharp and Zakeri and introduced the concept of generalized Hughes complexes. It was proved by Sharp and Yassi [36, Theorem 3.5] that every complex of modules of generalized fractions of a certain type is isomorphic to a generalized Hughes complex.

One of the main results of [34] is Theorem 2.3, which shows that every Cousin complex for M is (isomorphic to) a generalized Hughes complex for M. This result, and the earlier result accord a certain amount of importance to generalized Hughes complexes. Generalized Hughes complexes provide an 'umbrella concept' which covers all the algebraic Cousin complexes previously studied by sharp [23], [33], and all the complexes of modules of generalized fractions of the type described in [20, p. 420].

0.3 AN OUTLINE OF THE THESIS. Briefly, the aims of chapther (I) are to provide a characterization of generalized Hughes complexes, and to give several illustrations of situations where this characterization can be exploited to good effect, and to prove that whenever our commutative ring is not Noetherian, then there is a morphism of complexes from a generalized Hughes complex of a certain type to a complex of modules of generalized fractions.

In §1, we recall the Φ-torsion functor Γ_Φ, Φ-transform functor D_Φ, right derived functors of Γ_Φ (where Φ is a system of ideals of a commutative ring; see 1.1), and we list some properties of these functors.

We review the definition of the generalized Hughes complex in §2. For an R-module L, the generalized Hughes complex with respect to a family $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ of systems of ideals of R is denoted by $\mathcal{H}(\mathcal{S}, L)$. In §2, we answer the following questions: (i) what conditions on the terms and homomorphisms in a complex of A-modules and A-homomorphisms

$$C^\bullet : 0 \longrightarrow M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \cdots \longrightarrow C^i \xrightarrow{e^i} C^{i+1} \longrightarrow \cdots$$

are necessary and sufficient for the complex to be isomorphic (over Id_M) to the generalized Hughes complex for M with respect to a family of systems of ideals $\mathcal{S}$ of A? (ii) what conditions on the terms and homomorphisms in the complex $C^\bullet$ are necessary for there exists a homomorphism (or an epimorphism) of complexes (over Id_M) from $C^\bullet$ to the generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$ for M with respect to a family of systems of ideals $\mathcal{S}$ of A?

Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ and $\mathcal{I} = (\Theta_i)_{i\in\mathbb{N}}$ be two familes of systems of ideals of A such that $\Theta_n \subseteq \Phi_n$ for all $n \in \mathbb{N}$, and let M be an A-module. It is shown, in §3, that there is a unique morphism of complexes $\Psi = (\psi^i)_{i \geq -2}$: $\mathcal{H}(\mathcal{I}, M) \longrightarrow \mathcal{H}(\mathcal{S}, M)$ over Id_M and, under additional conditions, Ψ is an epimorphism such that if $\mathcal{H}(\mathcal{S}, M)$ is exact, then Ψ is an isomorphism. In §5, we show that these results can be viewed as a generalization of results of [35, (2.10), (2.11) and (3.6)] in which two Cousin complexes compared.

Let $\mathcal{S} = (\Phi_i)_{i \in \mathbb{N}}$ be a family of systems of ideals of A, and let $f : A \longrightarrow B$ be homomorphism of commutative Noetherian rings. In §4 and §5, we show that there is $\mathcal{S}B$, a family of systems of ideals of B, and also there is a unique morphism of complexes of B-modules and B-homomorphisms

$$\Psi = (\psi^i)_{i \geq -2} : \mathcal{H}(\mathcal{S}, M) \otimes_A B \longrightarrow \mathcal{H}(\mathcal{S}B, M \otimes_A B)$$

over $Id_{M \otimes_A B}$; if $\mathcal{H}(\mathcal{S}, M)$ is isomorphic to a Cousin complex, then Ψ is an isomorphism. In §5, under additional conditions, we prove that if $\mathcal{H}(\mathcal{S}, M)$ is isomorphic to a Cousin complex, then $\mathcal{H}(\mathcal{S}, M) \otimes_A B$ is isomorphic to a Cousin complex. This theorem is a strengthening of a result on the behaviour of Cousin complexes under ring homomorphisms established in [27, Theorem (2.6)]. Also, in §4, we show that if f is flat, then Ψ is an isomorphism.

In §5, we present a proof for [34, Theorem 2.3] which is shorter than the proof presented in [34]. This theorem shows that every Cousin complex is generalized Hughes complex. Also, we recover and extend Theorem 3.3 of [21], which shows that every two complex of A-modules and A-homomorphisms of Cousin type for M with respect to a filtration of Spec (A) which admits M are isomorphic.

In §5, we shall give an example of a generalized Hughes complex which is not isomorphic (over Id_A) to a Cousin complex for A, but we prove that if $\mathcal{S} = (\Phi_i)_{i \in \mathbb{N}}$ is a family of systems of ideals of A and M is an A-module, then there is a descending sequence $\mathcal{F} = (F_i)_{i \in \mathbb{N}_{\not{0}}}$ of subsets of Spec (A), which if it were a filtration of Spec (A), then the Cousin complex $C(\mathcal{F}, M)$ is isomorphic (over Id_M) to $\mathcal{H}(\mathcal{S}, M)$. Also, we prove that if $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ is a family of systems of ideals of A and M is an A-module, then there is $\mathcal{S}' = (\Phi'_i)_{i\in\mathbb{N}}$, a family of systems of ideals of A, such that the generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$ is isomorphic (over Id_M) to $\mathcal{H}(\mathcal{S}', M)$; and for each $i \in \mathbb{N}$,

(α) for each $\mathfrak{a} \in {}'_{\mathfrak{i}}$, every ideal $\mathfrak{b}$ of A with $\mathfrak{b} \supseteq \mathfrak{a}$ also belongs to Φ'_i; and

(β) $\Phi'_i \supseteq \Phi'_{i+1}$ and $V(\mathfrak{a}) \subseteq \mathrm{Supp}\ (M)$ for all $\mathfrak{a} \in {}'_{\mathfrak{i}}$.

Suppose that $\mathcal{U}$ is a chain of triangular subsets on R (see 6.1) and M is an R module. Then we can construct a complex of modules of generalized fractions $C(\mathcal{U}, M)$. The chain $\mathcal{U}$ determines a family $\mathcal{S}(\mathcal{U})$ of systems of ideals of R (see 6.2), and so the generalized Hughes complex $\mathcal{H}(\mathcal{S}(\mathcal{U}), M)$ for M with respect to $\mathcal{S}(\mathcal{U})$ can be contructed. One of the main results of [36] is theorem 3.5, which shows that, when R is Noetherian, there is an isomorphism of complexes

$$\Psi = (\psi^n)_{n\geq -2} : C(\mathcal{U}, M) \longrightarrow \mathcal{H}(\mathcal{S}(\mathcal{U}), M)$$

over Id_M. The proof of that theorem given in [36] using the Noetherian property of R which plays an important role: at the end of [36], it was asked whether there is any analogue of that theorem in the case when R is not necessarily Noetherian. The purpose of §6 is to response that question and is to use the methods of §2 to prove a strengthening of [36,3.5].

In §6, we prove that, in general, there is a natural homomorphism of complexes

$$\Theta = (\theta^n)_{n\geq -2} : \mathcal{H}(\mathcal{S}(\mathcal{U}), M) \longrightarrow C(\mathcal{U}, M)$$

over Id_M, Moreover, we show that, if R is Noetherian, then Θ is an isomorphism of complexes and its inverse is the isomorphism of complexes of [36, theorem 3.5] which mintioned above. In addition, we show that the class of commutative rings R for which Θ is always an isomorphism of complexes includes the N-rings studied by W. Heinger and D. Lantz in [8]. It should be noted an N-ring need not itself be Noetherian (see [8, page 122]). The final part of §6 provides an example which shows that Θ is not always an isomorphism.

In §7, the ideas of §2 are used to give a quick proof that every one of M. Kersken's Cousin complexes with respect to denominator systems over A is isomorphic to a generalized Hughes complex.

Suppose that G is a group of automorphisms of A. We use A^G to denote the fixed subring. The intuition that A^G should resemble A (the case where $G = \{Id_A\}$ is an extreme example) is often unfounded. There are many results concerning the success (and failure) of this resemblance. The purposes of Chapter (II) are the following discussion: (i) under "good" conditions on A and G, what general "good" properties does A^G enjoy? (ii) Can G induce an action group on certain modules and complexes (for example local cohomology, injective envelope and Cousin complex)? Can we specify the fixed submodule and subcomplex?

For the remainder of the introduction, M is an A-module, H is a finite group of A^G-module automorphisms of M such that $|H|$, the order of H, is invertible in A. Also, we shall let N be an A^G-module.

Let R' be a commutative ring with non-zero identity such that $|H|$ is invertible in R', and let T (respectively U) be an additive covariant (respectively contravariant) functor from $\mathcal{C}(A^G)$ to $\mathcal{C}(R')$. Then, in section 8, we show that H induces an action group on $T(M)$ (respectively $U(M)$) such that $T(M^H) \cong T(M)^H$ (as R'-modules) (respectively $U(M^H) \cong U(M)^H$ (as R'-modules)), where M^H and $T(M)^H$ (respectively $U(M)^H$) are fixed submodules. In this section, the results are true for non-Notherian rings.

Let $\mathfrak{b}$ be an ideal of A, and let $i \in \mathbb{N}_{\not{0}}$. In section 9, under a condition on $\mathfrak{b}$, we show that H induces an action group on $H^i_{\mathfrak{b}}(N \otimes_{A^G} M)$ the i-th local cohomology for $N \otimes_{A^G} M$ with respect to $\mathfrak{b}$ such that $(H^i_{\mathfrak{b}}(N \otimes_{A^G} M))^H$ and $H^i_{\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}}}(N \otimes_{A^G} M^H)$

are A^G-isomorphic, where $(H^i_{\mathfrak{b}}(N \otimes_{A^G} M))^H$ and M^H are fixed submodules.

For the remainder of the introduction, we shall assume that G is finite and $|G|$, the order of G, is invertible in A.

Let $\mathcal{G} = (G_i)_{i\in\mathbb{N}_{\not{\mathscr{k}}}}$ be a filtration of Spec (A) which admits $N \otimes_{A^G} A$, and let

$$F_i = \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \sigma(\mathfrak{q}) \in \mathfrak{G}_{\mathfrak{i}} \text{ for all } \sigma \in \mathfrak{G}\}$$

for all $i \in \mathbb{N}_{\not{\mathscr{k}}}$. Then, in section 10, we show that G induces an action group on $C(\mathcal{G}, N \otimes_{A^G} A)$ (respectively $H^i(C(\mathcal{G}, N \otimes_{A^G} A))$ for all $i \in \mathbb{N}_{\not{\mathscr{k}}}$). Also, we prove that $\mathcal{F} = (F_i)_{i\in\mathbb{N}_{\not{\mathscr{k}}}}$ is a filtration of Spec (A^G) which admits N and the fixed subcomplex $(C(\mathcal{G}, N \otimes_{A^G} A))^G$ (respectively the fixed submodule $(H^i(C(\mathcal{G}, N \otimes_{A^G} A)))^G$ for all $i \in \mathbb{N}_{\not{\mathscr{k}}}$) is isomorphic to the Cousin complex $C(\mathcal{F}, N)$ (respectively $H^i(C(\mathcal{F}, N))$ for all $i \in \mathbb{N}_{\not{\mathscr{k}}}$). Also, we show that $\mathcal{F}$ is the dimension filtration of Spec (A^G) (respectively the N-height filtration of Spec (A^G)), whenenver $\mathcal{G}$ is the dimension filtration of Spec (A) (respectively the $N \otimes_{A^G} A$-height filtration of Spec (A)).

For the remainder of the indroduction, k is a field, C is a Noetherian k-algebra and K is an algebraic extension field of k for which $B = C \otimes_k K$ is a Noetherian ring. We shall let $\Gamma :=$ Gal $(K : k)$ denote the Galois group of K over k. We shall denote

$$\{a \in K : \sigma(a) = a \text{ for all } \sigma \in \Gamma\}$$

by F. We assume that $[F : k]$ is the dimension of F considered as a vector space over k.

Let $\mathfrak{p} \in$ Spec (C), and let

$$F_B(\mathfrak{p}) = \{\mathfrak{q} \in \ \mathfrak{Spec}\ (\mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) = \mathfrak{p}\},$$

where $f : C \longrightarrow B$ is the natural homomorphism of rings. The main result of [43] shows that Γ induces an action group on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q})$ such that the fixed submodule is C-isomorphic to $E(C/\mathfrak{p})$, whenever C is a Gorenstion ring and K is a finite, separable, and normal extension field of k.

In section 11, we prove that if A, when regarded as an A^G-module, is finitely generated (or A and A^G are Gorenstein rings), then G induces an action group on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E_A(A/\mathfrak{q})$ such that the fixed submodule is A^G-isomorphic to $E(A^G/\mathfrak{p})$, where $\mathfrak{p} \in \text{Spec } (A^G)$ and

$$F(\mathfrak{p}) = \{\mathfrak{q} \in \mathfrak{Spec}\ (\mathfrak{A}) : \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}\}.$$

In section 11, we deduce the main result of [43], but without any restriction on C, that is Γ induces an action group on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q})$ such that the fixed submodule is C-isomorphic to $E(C/\mathfrak{p})$, whenever K is a finite, separable, and normal extension field of k and $|\Gamma|$, the order of Γ, is invertible in C.

In section 12, we generalize the main result of [43]. We show that Γ induces an action group on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q})$ such that the fixed submodule is C-isomorphic to $\oplus[F : k]E(C/\mathfrak{p})$, where K is a finite extension field of k and $|\Gamma|$, the order of Γ, is invertible in C. Also, we show that Γ induces an action group on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q})$ such that the fixed submodule is C-isomorphic to $\oplus[F : k]E(C/\mathfrak{p})$, whenever C and B are Gorenstion rings. Note that, if C is a Gorenstein ring and K is a finitely generated extension field over k, then B is also a Gorenstein ring.

In section 13, we consider various questions of the following type: when does a good property of A pass to A^G? we prove that A is generalized Cohen-Macaulay (respectively Buchsbaum) ring, then A^G is generalized Cohen-Macaulay (respectively Buchsbaum) ring. In this section, we show that A is a Cohen-Macaulay ring if and if A^G is a Cohen-Macaulay ring. Also, we show that, under a certain condition, if A is a Gorenstion ring, then A^G is a Gorenstien ring; and next, we use this result to deduce, under weaker conditions, the results of K. Watanabe and R. Stanley (See [11, (2.2) and (2.4)]). Also, we show that if A is local and $N \otimes_{A^G} A$ is a balanced big Cohen-Macaulay A-module, then N is a balanced big Cohen-Macaulay A^G-module.