GENERALIZED HUGHES COMPLEXES
AND
ACTION OF CERTAIN GROUPS ON CERTAIN MODULES AND COMPLEXES

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BIOGRAPHICAL SKETCH

The author was born in Jan. 6, 1961 in Tehran, Iran. He completed the requirement for a B.S. (respectively M.S.) degree in Mathematics in 1986 (respectively 1989) at the University of Tehran.

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ABSTRACT

The aims of Chapter (I) are to provide a characterization of generalized Hughes complexes, and to give several illustrations of situations where this characterization can be exploited to good effect, and to prove that whenever our commutative ring is not Noetherian, then there is a morphism of complexes from a generalized Hughes complex of a certain type to a complex of modules of generalized fractions.

Suppose that R is a commutative ring with non-zero identity and G is a group of automorphisms of R. We use R^G to denote the fixed subring. The purposes of Chapter (II) are the following discussion: (i) under "good" conditions on R and G, what general "good" properties does R^G enjoy? (ii) Can G induce an action group on certain modules and complexes (for example local cohomology, injective envelope and Cousin complex)? Can we specify the fixed submodule and subcomplex?

CONTENTS

§0 Introduction.

0.1 GENERAL NOTATIONS. Througout this thesis, A (respectively R) will denote a commutative Noetherian (respectively commutative) ring with non-zero identity, and $\mathcal{C}(A)$ (respectively $\mathcal{C}(R)$) will denote the category of A-modules and A-homomorphisms (respectively R-modules and R-homomorphisms). We shall use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of non-negative (respectively positive) integers.

0.2 SOME HISTORY. The Cousin complex considered in Algebraic and Analytic Geometry have Also commutative algebra analogues given by Sharp in [23]. Suppose that M is an A-module. In [23], Sharp defined, $C(M)$, the Cousin complex for M. It is, in fact, the commutative algebra analogue of the Cousin complex of §2 of Chapter IV of Hartshorne [7].

Cohen-Macauly rings can be characterized in terms of the Cousin complex: A is a Cohen-Macaulay ring if and only if $C(A)$ is exact [23, (4.7)]. Weaker conditions on the $C(A)$ provide reasonable classifications of some rings which not be Cohen-Macaulay (see [26]). For example, the $C(A)$ is exact at its (-1) th, oth, $\cdots$, $(n-2)$th terms if and only if the ring A satisfies the condition (S_n) (see [26,2.2]). Also, the Cousin complex provides a natural minimal injective resolution for a Gorenstein ring (See [23, (5.4)]). For M, non-zero and finitely generated, A-module we have that M is a Cohen-Macaulay if and only if $C(M)$ is exact (See [25, (2.4)]).

Various more general Cousin complexes can be constructed. We can, for any filtration $\mathcal{F}$ [33, (1.1)] of Spec (A) that admits M, construct the Cousin complex $C(\mathcal{F}, M)$ for M with respect to $\mathcal{F}$ [33, (1.3)]. If we use for $\mathcal{F}$ the M-height filtration $\mathcal{H}(M)$ of Spec (A) [33, (1.2)], then $C(\mathcal{H}(M), M)$ is just the Cousin complex $C(M)$

which is mentioned earlier. When A is local, using the dimemsion filtration $\mathcal{D}(A)$ [33, (1.2)] of Spec (A), permits a characterization of balanced big Cohen-Macaulay A-modules (not necessarily finitely generated). A-module X is *a balanced big Cohen-Macaulay* A-module [32, P.229] if every system of parameters for A is an X-sequence. It trurns out that M is a balanced big Cohen-Macaulay A-module if and only if $C(\mathcal{D}(A), M)$ is exact and $\mathfrak{m}\mathfrak{M} \neq \mathfrak{M}$ (where $\mathfrak{m}$ denotes the maximal ideal of A).

Although Cousin complexes do provide satisfactory characterizations of various Cohen-Macaulay properties, they have the disadvantage that their constraction is rather complicated, and this perhaps makes them difficult to work with. The purpose of [21] is to show that, for an A-module M such that $Ass(M)$ has only finitely many minimal members and a filtraction $\mathcal{F}$ of Spec (A) which admits M, the Cousin complex $C(\mathcal{F}, M)$ is actually isomorphic to a complex of modules of generalized fractions in the sense of [38]; this description is perhaps simpler and easier to work with.

K.R. Hughes [12] introduced a grade-theoretic analogue of the Cousin complex. he emploied Ree's concept of the grade of a proper ideal $\mathfrak{b}$ of A. Sharp and Yassi [36] explored relationships between Hughes's complex and the complex of the modules of generalized fractions of sharp and Zakeri and introduced the concept of generalized Hughes complexes. It was proved by Sharp and Yassi [36, Theorem 3.5] that every complex of modules of generalized fractions of a certain type is isomorphic to a generalized Hughes complex.

One of the main results of [34] is Theorem 2.3, which shows that every Cousin complex for M is (isomorphic to) a generalized Hughes complex for M. This result, and the earlier result accord a certain amount of importance to generalized Hughes complexes. Generalized Hughes complexes provide an 'umbrella concept' which covers all the algebraic Cousin complexes previously studied by sharp [23], [33], and all the complexes of modules of generalized fractions of the type described in [20, p. 420].

0.3 AN OUTLINE OF THE THESIS. Briefly, the aims of chapther (I) are to provide a characterization of generalized Hughes complexes, and to give several illustrations of situations where this characterization can be exploited to good effect, and to prove that whenever our commutative ring is not Noetherian, then there is a morphism of complexes from a generalized Hughes complex of a certain type to a complex of modules of generalized fractions.

In §1, we recall the Φ-torsion functor Γ_Φ, Φ-transform functor D_Φ, right derived functors of Γ_Φ (where Φ is a system of ideals of a commutative ring; see 1.1), and we list some properties of these functors.

We review the definition of the generalized Hughes complex in §2. For an R-module L, the generalized Hughes complex with respect to a family $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ of systems of ideals of R is denoted by $\mathcal{H}(\mathcal{S}, L)$. In §2, we answer the following questions: (i) what conditions on the terms and homomorphisms in a complex of A-modules and A-homomorphisms

$$C^\bullet : 0 \longrightarrow M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \cdots \longrightarrow C^i \xrightarrow{e^i} C^{i+1} \longrightarrow \cdots$$

are necessary and sufficient for the complex to be isomorphic (over Id_M) to the generalized Hughes complex for M with respect to a family of systems of ideals $\mathcal{S}$ of A? (ii) what conditions on the terms and homomorphisms in the complex $C^\bullet$ are necessary for there exists a homomorphism (or an epimorphism) of complexes (over Id_M) from $C^\bullet$ to the generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$ for M with respect to a family of systems of ideals $\mathcal{S}$ of A?

Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ and $\mathcal{I} = (\Theta_i)_{i\in\mathbb{N}}$ be two familes of systems of ideals of A such that $\Theta_n \subseteq \Phi_n$ for all $n \in \mathbb{N}$, and let M be an A-module. It is shown, in §3, that there is a unique morphism of complexes $\Psi = (\psi^i)_{i\geq -2}$: $\mathcal{H}(\mathcal{I}, M) \longrightarrow \mathcal{H}(\mathcal{S}, M)$ over Id_M and, under additional conditions, Ψ is an epimorphism such that if $\mathcal{H}(\mathcal{S}, M)$ is exact, then Ψ is an isomorphism. In §5, we show that these results can be viewed as a generalization of results of [35, (2.10), (2.11) and (3.6)] in which two Cousin complexes compared.

Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of A, and let $f : A \longrightarrow B$ be homomorphism of commutative Noetherian rings. In §4 and §5, we show that there is $\mathcal{S}B$, a family of systems of ideals of B, and also there is a unique morphism of complexes of B-modules and B-homomorphisms

$$\Psi = (\psi^i)_{i\geq -2} : \mathcal{H}(\mathcal{S}, M) \otimes_A B \longrightarrow \mathcal{H}(\mathcal{S}B, M \otimes_A B)$$

over $Id_{M\otimes_A B}$; if $\mathcal{H}(\mathcal{S}, M)$ is isomorphic to a Cousin complex, then Ψ is an isomorphism. In §5, under additional conditions, we prove that if $\mathcal{H}(\mathcal{S}, M)$ is isomorphic to a Cousin complex, then $\mathcal{H}(\mathcal{S}, M) \otimes_A B$ is isomorphic to a Cousin complex. This theorem is a strengthening of a result on the behaviour of Cousin complexes under ring homomorphisms established in [27, Theorem (2.6)]. Also, in §4, we show that if f is flat, then Ψ is an isomorphism.

In §5, we present a proof for [34, Theorem 2.3] which is shorter than the proof presented in [34]. This theorem shows that every Cousin complex is generalized Hughes complex. Also, we recover and extend Theorem 3.3 of [21], which shows that every two complex of A-modules and A-homomorphisms of Cousin type for M with respect to a filtration of Spec (A) which admits M are isomorphic.

In §5, we shall give an example of a generalized Hughes complex which is not isomorphic (over Id_A) to a Cousin complex for A, but we prove that if $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ is a family of systems of ideals of A and M is an A-module, then there is a descending sequence $\mathcal{F} = (F_i)_{i\in\mathbb{N}_0}$ of subsets of Spec (A), which if it were a filtration of Spec (A), then the Cousin complex $C(\mathcal{F}, M)$ is isomorphic (over Id_M) to $\mathcal{H}(\mathcal{S}, M)$. Also, we prove that if $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ is a family of systems of ideals of A and M is an A-module, then there is $\mathcal{S}' = (\Phi'_i)_{i\in\mathbb{N}}$, a family of systems of ideals of A, such that the generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$ is isomorphic (over Id_M) to $\mathcal{H}(\mathcal{S}', M)$; and for each $i \in \mathbb{N}$,

(α) for each $\mathfrak{a} \in {}'_{\mathrm{i}}$, every ideal $\mathfrak{b}$ of A with $\mathfrak{b} \supseteq \mathfrak{a}$ also belongs to Φ'_i; and

(β) $\Phi'_i \supseteq \Phi'_{i+1}$ and $V(\mathfrak{a}) \subseteq \text{Supp }(M)$ for all $\mathfrak{a} \in {}'_{\mathrm{i}}$.

Suppose that $\mathcal{U}$ is a chain of triangular subsets on R (see 6.1) and M is an R module. Then we can construct a complex of modules of generalized fractions $C(\mathcal{U}, M)$. The chain $\mathcal{U}$ determines a family $\mathcal{S}(\mathcal{U})$ of systems of ideals of R (see 6.2), and so the generalized Hughes complex $\mathcal{H}(\mathcal{S}(\mathcal{U}), M)$ for M with respect to $\mathcal{S}(\mathcal{U})$ can be contructed. One of the main results of [36] is theorem 3.5, which shows that, when R is Noetherian, there is an isomorphism of complexes

$$\Psi = (\psi^n)_{n\geq -2} : C(\mathcal{U}, M) \longrightarrow \mathcal{H}(\mathcal{S}(\mathcal{U}), M)$$

over Id_M. The proof of that theorem given in [36] using the Noetherian property of R which plays an important role: at the end of [36], it was asked whether there is any analogue of that theorem in the case when R is not necessarily Noetherian. The purpose of §6 is to response that question and is to use the methods of §2 to prove a strengthening of [36,3.5].

In §6, we prove that, in general, there is a natural homomorphism of complexes

$$\Theta = (\theta^n)_{n\geq -2} : \mathcal{H}(\mathcal{S}(\mathcal{U}), M) \longrightarrow C(\mathcal{U}, M)$$

over Id_M, Moreover, we show that, if R is Noetherian, then Θ is an isomorphism of complexes and its inverse is the isomorphism of complexes of [36, theorem 3.5] which mintioned above. In addition, we show that the class of commutative rings R for which Θ is always an isomorphism of complexes includes the N-rings studied by W. Heinger and D. Lantz in [8]. It should be noted an N-ring need not itself be Noetherian (see [8, page 122]). The final part of §6 provides an example which shows that Θ is not always an isomorphism.

In §7, the ideas of §2 are used to give a quick proof that every one of M. Kersken's Cousin complexes with respect to denominator systems over A is isomorphic to a generalized Hughes complex.

Suppose that G is a group of automorphisms of A. We use A^G to denote the fixed subring. The intuition that A^G should resemble A (the case where $G = \{Id_A\}$ is an extreme example) is often unfounded. There are many results concerning the success (and failure) of this resemblance. The purposes of Chapter (II) are the following discussion: (i) under "good" conditions on A and G, what general "good" properties does A^G enjoy? (ii) Can G induce an action group on certain modules and complexes (for example local cohomology, injective envelope and Cousin complex)? Can we specify the fixed submodule and subcomplex?

For the remainder of the introduction, M is an A-module, H is a finite group of A^G-module automorphisms of M such that $|H|$, the order of H, is invertible in A. Also, we shall let N be an A^G-module.

Let R' be a commutative ring with non-zero identity such that $|H|$ is invertible in R', and let T (respectively U) be an additive covariant (respectively contravariant) functor from $\mathcal{C}(A^G)$ to $\mathcal{C}(R')$. Then, in section 8, we show that H induces an action group on $T(M)$ (respectively $U(M)$) such that $T(M^H) \cong T(M)^H$ (as R'-modules) (respectively $U(M^H) \cong U(M)^H$ (as R'-modules)), where M^H and $T(M)^H$ (respectively $U(M)^H$) are fixed submodules. In this section, the results are true for non-Notherian rings.

Let $\mathfrak{b}$ be an ideal of A, and let $i \in \mathbb{N}_{\not\vdash}$. In section 9, under a condition on $\mathfrak{b}$, we show that H induces an action group on $H^i_{\mathfrak{b}}(N \otimes_{A^G} M)$ the i-th local cohomology for $N \otimes_{A^G} M$ with respect to $\mathfrak{b}$ such that $(H^i_{\mathfrak{b}}(N \otimes_{A^G} M))^H$ and $H^i_{\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}}}(N \otimes_{A^G} M^H)$

are A^G-isomorphic, where $(H^i_{\mathfrak{b}}(N \otimes_{A^G} M))^H$ and M^H are fixed submodules.

For the remainder of the introduction, we shall assume that G is finite and $|G|$, the order of G, is invertible in A.

Let $\mathcal{G} = (G_i)_{i \in \mathbb{N}_{\nvdash}}$ be a filtration of Spec (A) which admits $N \otimes_{A^G} A$, and let

$$F_i = \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \sigma(\mathfrak{q}) \in \mathfrak{G}_{\mathfrak{i}} \text{ for all } \sigma \in \mathfrak{G}\}$$

for all $i \in \mathbb{N}_{\nvdash}$. Then, in section 10, we show that G induces an action group on $C(\mathcal{G}, N \otimes_{A^G} A)$ (respectively $H^i(C(\mathcal{G}, N \otimes_{A^G} A))$ for all $i \in \mathbb{N}_{\nvdash}$). Also, we prove that $\mathcal{F} = (F_i)_{i \in \mathbb{N}_{\nvdash}}$ is a filtration of Spec (A^G) which admits N and the fixed subcomplex $(C(\mathcal{G}, N \otimes_{A^G} A))^G$ (respectively the fixed submodule $(H^i(C(\mathcal{G}, N \otimes_{A^G} A)))^G$ for all $i \in \mathbb{N}_{\nvdash}$) is isomorphic to the Cousin complex $C(\mathcal{F}, N)$ (respectively $H^i(C(\mathcal{F}, N))$ for all $i \in \mathbb{N}_{\nvdash}$). Also, we show that $\mathcal{F}$ is the dimension filtration of Spec (A^G) (respectively the N-height filtration of Spec (A^G)), whenenver $\mathcal{G}$ is the dimension filtration of Spec (A) (respectively the $N \otimes_{A^G} A$-height filtration of Spec (A)).

For the remainder of the indroduction, k is a field, C is a Noetherian k-algebra and K is an algebraic extension field of k for which $B = C \otimes_k K$ is a Noetherian ring. We shall let $\Gamma :=$ Gal $(K : k)$ denote the Galois group of K over k. We shall denote

$$\{a \in K : \sigma(a) = a \text{ for all } \sigma \in \Gamma\}$$

by F. We assume that $[F : k]$ is the dimension of F considered as a vector space over k.

Let $\mathfrak{p} \in$ Spec (C), and let

$$F_B(\mathfrak{p}) = \{\mathfrak{q} \in \mathfrak{Spec}\ (\mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) = \mathfrak{p}\},$$

where $f : C \longrightarrow B$ is the natural homomorphism of rings. The main result of [43] shows that Γ induces an action group on $\bigoplus_{\mathfrak{q} \in \mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q})$ such that the fixed submodule is C-isomorphic to $E(C/\mathfrak{p})$, whenever C is a Gorenstion ring and K is a finite, separable, and normal extension field of k.

In section 11, we prove that if A, when regarded as an A^G-module, is finitely generated (or A and A^G are Gorenstein rings), then G induces an action group on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E_A(A/\mathfrak{q})$ such that the fixed submodule is A^G-isomorphic to $E(A^G/\mathfrak{p})$, where $\mathfrak{p} \in$ Spec (A^G) and

$$F(\mathfrak{p}) = \{\mathfrak{q} \in \ \mathfrak{Spec}\ (\mathfrak{A}) : \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}\}.$$

In section 11, we deduce the main result of [43], but without any restriction on C, that is Γ induces an action group on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q})$ such that the fixed submodule is C-isomorphic to $E(C/\mathfrak{p})$, whenever K is a finite, separable, and normal extension field of k and $|\Gamma|$, the order of Γ, is invertible in C.

In section 12, we generalize the main result of [43]. We show that Γ induces an action group on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q})$ such that the fixed submodule is C-isomorphic to $\oplus[F : k]E(C/\mathfrak{p})$, where K is a finite extension field of k and $|\Gamma|$, the order of Γ, is invertible in C. Also, we show that Γ induces an action group on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q})$ such that the fixed submodule is C-isomorphic to $\oplus[F : k]E(C/\mathfrak{p})$, whenever C and B are Gorenstion rings. Note that, if C is a Gorenstein ring and K is a finitely generated extension field over k, then B is also a Gorenstein ring.

In section 13, we consider various questions of the following type: when does a good property of A pass to A^G? we prove that A is generalized Cohen-Macaulay (respectively Buchsbaum) ring, then A^G is generalized Cohen-Macaulay (respectively Buchsbaum) ring. In this section, we show that A is a Cohen-Macaulay ring if and if A^G is a Cohen-Macaulay ring. Also, we show that, under a certain condition, if A is a Gorenstion ring, then A^G is a Gorenstien ring; and next, we use this result to deduce, under weaker conditions, the results of K. Watanabe and R. Stanley (See [11, (2.2) and (2.4)]). Also, we show that if A is local and $N \otimes_{A^G} A$ is a balanced big Cohen-Macaulay A-module, then N is a balanced big Cohen-Macaulay A^G-module.

CHAPTER (I)
GENERALIZED HUGHES COMPLEXES

§1 Φ-torsion functor and Φ-transform functor.

Firstly, we define Φ-torsion functor and list some properties of this functor.

1.1 DEFINITION. (See [2, 2.1]) *A system of ideals of* R is a nonempty set Φ of ideals of R such that, whenever $\mathfrak{a}, \mathfrak{b} \in$, then there exists $\mathfrak{c} \in$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$.

For an R-module M, set

$$\Gamma_\Phi(M) := \{m \in M : \mathfrak{a}\mathfrak{m} = \mathfrak{o} \text{ for some } \mathfrak{a} \in \}.$$

Note that $\Gamma_\Phi(M)$ is a submodule M. For a homomorphism $f : M \longrightarrow N$ of R-modules, we have $f(\Gamma_\Phi(M)) \subseteq \Gamma_\Phi(N)$; so that there is a mapping $\Gamma_\Phi(f) : \Gamma_\Phi(M) \longrightarrow \Gamma_\Phi(N)$ which agrees with f on each element of $\Gamma_\Phi(M)$.

It is clear that, if $g : M \longrightarrow N$ and $h : N \longrightarrow L$ are homomorphisms of R-modeles and $r \in R$, then $\Gamma_\Phi(hof) = \Gamma_\Phi(h)_0\Gamma_\Phi(f), \Gamma_\Phi(f+g) = \Gamma_\Phi(f) + \Gamma_\Phi(g), \Gamma_\Phi(rf) = r\Gamma_\Phi(f)$ and $\Gamma_\Phi(Id_M) = Id_{\Gamma_\Phi(M)}$. Thus, Γ_Φ is a covariant, additive and R-linear functor from $\mathcal{C}(R)$ to itself. We call Γ_Φ *the* Φ*-torsion functor*.

For $i \in \mathbb{N}_{\nvdash}$, the i-th right derived functor of Γ_Φ is denoted by H^i_Φ and is referred to as *the* i*-th generalized local cohomology functor with respect to* Φ*.*

Note that in [2], Γ_Φ is denoted by L_Φ, and also is called *the general local cohomology functor with respect to* Φ*.*

1.2 REMARK. It should be noted that if $\Phi = \{\mathfrak{a}^\mathfrak{i} : \mathfrak{i} \in \mathbb{N}\}$, for some ideal $\mathfrak{a}$ of R, then the Φ-torsion functor Γ_Φ is just the ordinary local cohomology functor with respect to $\mathfrak{a}$ (See [24]).

For $i \in \mathbb{N}_{\nvdash}$, the i-th right derived functor of $\Gamma_\mathfrak{a}$ is denoted by $H^i_\mathfrak{a}$ and is referred to as the i-th local cohomology functor with respect to $\mathfrak{a}$.

1.3 DEFINITION. Let M be an R-module and let Φ be a system of ideals of R. We shall say that M is Φ-*torsion free* precisely when $\Gamma_\Phi(M) = 0$, and that M is Φ-torsion precisely $\Gamma_\Phi(M) = M$.

If M is a Φ-torsion R-module, then all submodules and all R-homomorphic images of M are also Φ-torsion.

1.4 DEFINITION. Let $f : R \longrightarrow R'$ be a homomorphism of commutative rings. For any ideal $\mathfrak{a}$ of R, we use $\mathfrak{a}\mathfrak{R}'$ to denote the extension of $\mathfrak{a}$ to R' under f.

Also, for a system Φ of ideals of R, we use $\Phi R'$ to denote $\{\mathfrak{a}\mathfrak{R}' : \mathfrak{a} \in \}$ of R', and we refer it as the *extension* of Φ to R' *under* f.

1.5 PROPERTIES OF LOCAL COHOMOLOGY FUNCTOR. Let M be an arbitrary A-module and let Φ be a system of ideals of A. Let $i \in \mathbb{N}_{\not{\,}}$.

(1) To calculate $H^i_\Phi(M)$, one proceeds as follows. Let

$$I^\bullet : 0 \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I' \longrightarrow \cdots \longrightarrow I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

be an injective resolution for M, so that there is an A-homomorphism $\alpha : M \longrightarrow I^0$ such that the sequence

$$0 \longrightarrow M \xrightarrow{\alpha} I^0 \xrightarrow{d^0} I' \longrightarrow \cdots \longrightarrow I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

is exact. Apply the functor Γ_Φ to the complex $I^\bullet$ to obtain the complex

$$0 \xrightarrow{\Gamma_\Phi(d^{-1})} \Gamma_\Phi(I^0) \longrightarrow \cdots \longrightarrow \Gamma_\Phi(I^i) \xrightarrow{\Gamma_\Phi(d^i)} \Gamma_\Phi(I^{i+1}) \longrightarrow \cdots .$$

the i-th cohomology module $\ker\,(\Gamma_\Phi(d^i))/Im(\Gamma_\Phi(d^{i-1}))$ of the complex, which, by a standard fact of homological algebra, is independent (up to A-isomorphism) of the choice of injective resolution $I^\bullet$ of M, is $H^i_\Phi(M)$.

(2) Since Γ_Φ is covarant, additive and A-linear, it is automatic that the generalized local cohomology functor H^i_Φ is again covariant, additive and A-linear.

(3) Since Γ_Φ is left exact, H^0_Φ is naturally equivalent to Γ_Φ. Thus, loosely, we can use this natural equivalence to identify these two functors.

(4) In view of (1) and 1.3, $H^i_\Phi(M)$ is a Φ-torsion A-module.

(5) (See [2, 2.4]) Let $(M_\alpha, \mu_{\alpha\beta})$ be a direct system of A-modules and A-homomorphisms over the directed set Λ. Then

$$H^i_\Phi\left(\varinjlim_{\alpha\in\Lambda} M_\alpha\right) \cong \varinjlim_{\alpha\in\Lambda} H^i_\Phi(M_\alpha).$$

(6) (See [2, 2.5]) Suppose $f : A \longrightarrow B$ be a homomorphism of commutative Noetherian rings. Any B-module N may be regarded as an A-module by means of f. When this is the case, it is denoted by $N\lceil_A$. Note that $\lceil_A$ can be regarded as a functor from $\mathcal{C}(B)$ to $\mathcal{C}(A)$. For each $i \in \mathbb{N}_{\not\vdash}$, the functors $H^i_\Phi(\bullet\lceil_A)$ and $H^i_{\Phi B}(\bullet)\lceil_A$ from $\mathcal{C}(B)$ to $\mathcal{C}(A)$ are naturally equivalent.

(7) (See[2, 2.6]) Suppose that S is a multiplicatevely closed subset of A. Let

$$S^{-1}\Phi = \{S^{-1}\mathfrak{a} : \mathfrak{a} \in \}.$$

Then the functors $S^{-1}H^i_\Phi(\bullet)$ and $H^i_{S^{-1}\Phi}(S^{-1}(\bullet))$ (from $\mathcal{C}(A)$ to $\mathcal{C}(S^{-1}A)$) are naturally equivalent.

(8) (See[34, 1.4(i)]) If G is a Φ-torsion A-module, then every term in the minimal injective resolution of G is also Φ-torsion, so that $H^i_\Phi(G) = 0$ forall $i \in \mathbb{N}$.

(9) (See [34, 1.4(ii)]) The canonical epimorphism $\pi : M \longrightarrow M/\Gamma_\Phi(M)$ induces isomorphisms

$$H^n_\Phi(\pi) : H^n_\Phi(M) \longrightarrow H^n_\Phi(M/\Gamma_\Phi(M))$$

for all $n \in \mathbb{N}$.

1.6 NOTATION. For an ideal $\mathfrak{b}$ of A, $V(\mathfrak{b})$ denotes the set

$$\{\mathfrak{p} \in \text{ Spec } (\mathfrak{A}) : \mathfrak{b} \subseteq \mathfrak{p}\}.$$

1.7 LEMMA. Let Φ be system of ideals of A. Then an A-module M is a Φ-torsion module if and only if Supp $(M) \subseteq \bigcup_{\mathfrak{a}\in} V(\mathfrak{a})$.

Proof. Let M be Φ-torsion and $\mathfrak{p}$ $\in$Supp (M). Then there exist $0 \neq x \in M$ and $\mathfrak{b} \in$ such that $(0 :_A x) \subseteq \mathfrak{p}$ and $\mathfrak{b} \subseteq (\mathfrak{o} :_{\mathfrak{A}} \mathfrak{x})$. Thus $\mathfrak{p} \in \bigcup_{\mathfrak{a}\in} \mathfrak{V}(\mathfrak{a})$.

Conversely, suppose that Supp $(M) \subseteq \bigcup_{\mathfrak{a}\in} V(\mathfrak{a})$. Let $0 \neq x \in M$. Then

$$V((0 :_A x)) \subseteq \text{ Supp}(M) \subseteq \bigcup_{\mathfrak{a}\in} V(\mathfrak{a}).$$

Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ be the minimal prime ideals of $\mathfrak{a} =: (0 :_A x)$. Thus, for each $i = 1, \cdots, t$, there exists $\mathfrak{c}_i \in$ such that $\mathfrak{c}_i \subseteq \mathfrak{p}_i$. Therefore, in view of the fact that $r_A(\mathfrak{a}) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$, there exists $h \in \mathbb{N}$ such that

$$(\mathfrak{c}_1 \cdots \mathfrak{c}_t)^{\mathfrak{h}} \subseteq (\mathfrak{p}_1 \cdots \mathfrak{p}_t)^{\mathfrak{h}} \subseteq (\mathfrak{r}_{\mathfrak{A}}(\mathfrak{a}))^{\mathfrak{h}} \subseteq \mathfrak{a};$$

hence there exists $\mathfrak{b} \in$ such that $\mathfrak{b} \subseteq (\mathfrak{o} :_{\mathfrak{A}} \mathfrak{x})$.

1.8 REMARK. (See [34, 2.1]) Let U and W be subset of Spec (A) such that $W \supseteq$ Supp (M). The set

$$\Phi(U, W) = \{\mathfrak{a} : \mathfrak{a} \text{ an ideal of such that } \mathfrak{V}(\mathfrak{a}) \cap \mathfrak{W} \subseteq \mathfrak{U}\}$$

is a system of ideals of A. It is easy to check that if M is an A-module such that Supp $(M) \subseteq U$, then M is $\Phi(U, W)$-torsion.

1.9 LEMMA. Let Φ be a system of ideals A and $F(\Phi) = \bigcup_{\mathfrak{a}\in} V(\mathfrak{a})$. Let M be an A-module. Then

$$\text{Supp } (H^i_\Phi(M)) \subseteq F(\Phi) \cap \text{ Supp } (M)$$

for all $i \in \mathbb{N}_{\not\vdash}$.

Proof. Suppose that $\mathfrak{p} \in$ Supp $(H^i_\Phi(M))$ and $i \in \mathbb{N}_{\not\vdash}$. Then, by 1.5 (7),

$$H^i_{\Phi A_\mathfrak{p}}(M_\mathfrak{p}) \cong (H^i_\Phi(M))_\mathfrak{p} \neq 0.$$

Hence $\Phi A_\mathfrak{p} \neq \{A_\mathfrak{p}\}$ and $M_\mathfrak{p} \neq 0$. Therefore $\mathfrak{p} \in F(\Phi) \cap$ Supp (M).

1.10 LEMMA. Let $f : A \longrightarrow B$ be a homomorphism of commutative Noetherian rigns. Let G be an A-module and let N be B-module. Let U, W, X, Y be subset of Spec (A) such that $U \subseteq X$ and Supp $(G) \subseteq W$. Let

$$\Phi(U, W) = \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{a}) \cap \mathfrak{W} \subseteq \mathfrak{U}\},$$

and suppose that $\Phi(U,$Spec $(A)) = \Phi$. Then, for each $i \in \mathbb{N}_{\not\vdash}$,

(i) $H^i_{\Phi(U,W)}(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} G_\mathfrak{p}) = 0$

(ii) $H^i_{\Phi B}(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} N_\mathfrak{p}) = 0$

(iii) $H^i_{\Phi B}((\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} G_\mathfrak{p}) \otimes_A B) = 0$

Proof. (i) let $i \in \mathbb{N}_{\not\vdash}$. By 1.5 (5) and 1.5(6),

$$\begin{aligned} H^i_{\Phi(U,W)}(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} G_\mathfrak{p}) \cong \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} H^i_{\Phi(U,W)}(G_\mathfrak{p}) &\cong \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} H^i_{\Phi(U,W)A_\mathfrak{p}}(G_\mathfrak{p}) \\ &\cong \bigoplus_{\mathfrak{p}\in(\mathfrak{Y}-\mathfrak{X})\cap \text{ Supp } (\mathfrak{G})} H^i_{\Phi(U,W)A_\mathfrak{p}}(G_\mathfrak{p}). \end{aligned}$$

If $\mathfrak{p} \in W$ and $\mathfrak{p} \notin U$, then

$$\Phi(U, W)A_\mathfrak{p} = \{\mathfrak{a}\mathfrak{A}_\mathfrak{p} : \mathfrak{a} \in (\mathfrak{U}, \mathfrak{W})\} = \{\mathfrak{A}_\mathfrak{p}\}.$$

Hence

$$H^i_{\Phi(U,W)}\Big(\bigoplus_{\mathfrak{p}\in(\mathfrak{Y}-\mathfrak{X})} G_\mathfrak{p}\Big) \cong \bigoplus_{\mathfrak{p}\in(\mathfrak{Y}-\mathfrak{X})\cap \text{ Supp } (\mathfrak{G})} H^i_{\{A_\mathfrak{p}\}}(G_\mathfrak{p}) = 0.$$

(ii) By 1.5(6) and (i),

$$H^i_{\Phi B}\Big(\bigoplus_{\mathfrak{p}\in(\mathfrak{Y}-\mathfrak{X})} N_\mathfrak{p}\Big) \cong H^i_{\Phi}\Big(\bigoplus_{\mathfrak{p}\in(\mathfrak{Y}-\mathfrak{X})} N_\mathfrak{p}\Big) = 0.$$

(iii) Since tensor product commutes with direct sum,

$$\begin{aligned}
\Big(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} G_\mathfrak{p}\Big)\otimes_A B \cong \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} (G_\mathfrak{p}\otimes_A B) &\cong \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} ((G\otimes_A A_\mathfrak{p})\otimes_A B) \\
&\cong \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} ((G\otimes_A B)\otimes_A A_\mathfrak{p}) \\
&\cong \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{X}} (G\otimes_A B)_\mathfrak{p}.
\end{aligned}$$

Hence the claim follows from (ii).

1.11 LEMMA. Let $n \in \mathbb{N}_{\nvdash}$, and let $\Phi_0, \Phi_1, \cdots, \Phi_{n+1}$ be systems of ideals of A. Let M be an A-module such that M is Φ_i-torsion for every $i = 0, \ldots, n$. Then

$$\Phi := \left\{ \sum_{i=0}^{n+1} \mathfrak{a}_\mathfrak{i} : \mathfrak{a}_\mathfrak{i} \in {}_\mathfrak{i} \right\}$$

is a system of ideals of A and $H^i_\Phi(M) = H^i_{\Phi_{n+1}}(M)$ for each $i \in \mathbb{N}_{\nvdash}$.

Proof. It is easy to see that Φ is a system of ideals of A.

Let

$$I^\bullet : 0 \xrightarrow{d^{-1}} I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

be the minimal injective resolution of M. By 1.5(8), for each $i \in \mathbb{N}_{\nvdash}$, I^i is Φ_j-torsion for every $j = 0, \cdots, n$. Hence $\Gamma_\Phi(I^i) = \Gamma_{\Phi_{n+1}}(I^i)$ for all $i \in \mathbb{N}_{\nvdash}$. Hence the claim follows from 1.5(1).

For definition of Φ-transform functor, we shall need the following remark.

1.12 REMARK. (See [24, 2.2]) Let $(\Lambda, \leq)$ be a (non-empty) directed partially ordered set, and suppose that we are given an inverse system of R-modules $(M_\alpha)_{\alpha\in\Lambda}$ over Λ, with constituent R-homomorphisms $h^\alpha_\beta : M_\alpha \longrightarrow M_\beta$ (for each $(\alpha, \beta) \in \Lambda\times\Lambda$ with $\alpha \geq \beta$). Let $T : \mathcal{C}(R) \times \mathcal{C}(R) \longrightarrow \mathcal{C}(R)$ be an additive, R-linear functor of two variables which is contravariant in the first variable and covariant in the second. We show now how these data give rise to a covariant, additive and R-linear functor

$$\varinjlim_{\alpha\in\Lambda} T(M_\alpha, \bullet) : \mathcal{C}(R) \longrightarrow \mathcal{C}(R).$$

Let L, N be R-modules and let $f : L \longrightarrow N$ be an R-homomorphism. For $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, the homomorphism $h^\alpha_\beta : M_\alpha \longrightarrow M_\beta$ induces an R-homomorphism

$$T(h^\alpha_\beta, L) : T(M_\beta, L) \longrightarrow T(M_\alpha, L),$$

and the fact that T is a functor ensures that the $T(h^\alpha_\beta, L)$ turn the famiy $(T(M_\alpha, L))_{\alpha\in\Lambda}$ into a direct system of R-modules and R-homomorphisms over Λ. We may therefore form

$$\varinjlim_{\alpha\in\Lambda} T(M_\alpha, L).$$

Moreover, again for $\alpha, \beta, \in \Lambda$ with $\alpha \geq \beta$, we have a commutative diagram

$$\begin{array}{ccc} T(M_\beta, L) & \xrightarrow{\quad T(h^\alpha_\beta, L) \quad} & T(M_\alpha, L) \\ \Big\downarrow{\scriptstyle T(M_\beta, f)} & & \Big\downarrow{\scriptstyle T(M_\alpha, f)} \\ T(M_\beta, N) & \xrightarrow[\quad T(h^\alpha_\beta, N) \quad]{} & T(M_\alpha, N) \end{array} \qquad ;$$

therefore the family $T(M_\alpha, f)(\alpha \in \Lambda)$ of homomorphisms constitute a morphism of direct systems and so induce an R-homomorphism

$$\varinjlim_{\alpha\in\Lambda} T(M_\alpha, f) : \varinjlim_{\alpha\in\Lambda} T(M_\alpha, L) \longrightarrow \varinjlim_{\alpha\in\Lambda} T(M_\alpha, N).$$

It is now straightforward to check that, in this way,

$$\varinjlim_{\alpha\in\Lambda} T(M_\alpha, \bullet).$$

becomes a covariant, additive, R-linear functor from $\mathcal{C}(R)$ to itself. Observe that, since passage to direct limits preservers exactness, if T is left exact, then so too is $\varinjlim_{\alpha\in\Lambda} T(M_\alpha, \bullet)$.

1.13 DEFINITION. (See [36,1.2]) Any system of ideals of R is a non-empty partially ordered and directed set with respect to reverse inclusion. Also any system of ideals Φ of R gives rise to an inverse system of R-modules $(\mathfrak{a})_{\mathfrak{a}\in}$ over Φ, with constituent R-homomorphisms $h^{\mathfrak{a}}_{\mathfrak{b}} : \mathfrak{b} \xrightarrow{\text{inc}} \mathfrak{a}$ (for each $(\mathfrak{a}, \mathfrak{b}) \in \ \times\ $ with $\mathfrak{a} \leq \mathfrak{b}$ that is with $\mathfrak{b} \subseteq \mathfrak{a}$).

Let Φ be a system of ideals of R. It is straightforward to adapt the ideas of 1.12 in an obvious way to produce a covariant, additive, R-linear, left exact functor

$$D_\Phi = \varinjlim_{\mathfrak{b}\in} Hom_R(\mathfrak{b}, \bullet)$$

from $\mathcal{C}(R)$ to itself; D_Φ is called the *generalized ideal transform determined by* Φ, or, more briefly, the Φ-*transform*. There is a morphism of functors

$$\eta_\Phi : Id \longrightarrow D_\Phi$$

which is such that, for each R-module G and each $g \in G$ and each $\mathfrak{a} \in$, the image $\eta_\Phi(G)(g)$ is the natural image in $D_\Phi(G)$ of the R-homomorphism $\lambda_{\mathfrak{a},\mathfrak{g}} : \mathfrak{a} \longrightarrow \mathfrak{G}$ defined by $\lambda_{\mathfrak{a},\mathfrak{g}}(r) = rg$ for all $r \in \mathfrak{a}$.

1.14 PROPERTIES OF IDEAL TRANSFORM FUNCTOR. Let Φ be a system of ideals of A and let M be an A-module.

(i) (See [36,1.2]) For each A-module G, there is an exact sequence

$$0 \longrightarrow \Gamma_\Phi(G) \longrightarrow G \stackrel{\eta_\Phi(G)}{\longrightarrow} D_\Phi(G) \longrightarrow H^1_\Phi(G) \longrightarrow 0,$$

of A-modules and A-homomorphisms, where $\eta_\Phi(G)$ is as described in 1.13. Note that

$$\text{Ker } \eta_\Phi(G) = \Gamma_\Phi(G) \text{ and Coker } \eta_\Phi(G) \cong H^1_\Phi(G).$$

(ii) (See [34,1.4 (iii)]) For all $i \in \mathbb{N}_{\nvdash}$, let $\mathcal{R}^i D_\Phi$ denote the i-th right derived functor of the left exact functor D_Φ. Note that, for each $i \in \mathbb{N}$, the functors $\mathcal{R}^i D_\Phi$ and H^{i+1}_Φ (from $\mathcal{C}(A)$ to itself) are naturally equivalent.

(iii) (See [34,1.5]) Let $\pi : M \longrightarrow M/\Gamma_\Phi(M)$ be the natural epimorphism. Then the following hold:

(1) $D_\Phi(\Gamma_\Phi(M)) = 0$;

(2) $D_\Phi(\pi) : D_\Phi(M) \longrightarrow D_\Phi(M/\Gamma_\Phi(M))$ is an isomorphism;

(3) $D_\Phi(\eta_\Phi(M)) = \eta_\Phi(D_\Phi(M)) : D_\Phi(M) \longrightarrow D_\Phi(D_\Phi(M))$ is an isomorphism;

(4) $\Gamma_\Phi(D_\Phi(M)) = 0 = H^1_\Phi(D_\Phi(M))$.

1.15 LEMMA. Let Φ be a system of ideals of A. Let $e : C \longrightarrow C'$ be an A- homorphism such that Ker e and Coker e are both Φ-torsion. Then the A-homomorphism $D_\Phi(e) : D_\Phi(C) \longrightarrow D_\Phi(C')$ is an isomorphism.

Proof. We shall need to use the exact sequences

$$0 \longrightarrow \text{ Ker } e \xrightarrow{\tau} C \xrightarrow{\lambda} Ime \longrightarrow 0$$

and

$$0 \longrightarrow Ime \xrightarrow{\rho} C' \xrightarrow{\sigma} \text{ Coker } e \longrightarrow 0$$

in which the maps are the obvious natural homomorphisms. Note that $e = \rho_0\lambda$; it is therefore enough for us to show that $D_\Phi(\rho)$ and $D_\Phi(\lambda)$ are both isomorphisms.

The first of the above exact sequences induces an exact sequence

$$0 \longrightarrow D_\Phi(\text{ Ker } e) \xrightarrow{D_\Phi(\tau)} D_\Phi(C) \xrightarrow{D_\Phi(\lambda)} D_\Phi(Im\ e) \longrightarrow \mathcal{R}^1 D_\Phi(\text{ Ker } e).$$

However, $\mathcal{R}^1 D_\Phi(\text{ Ker } e) \cong H^2_\Phi(\text{ Ker } e)$ by 1.14 (ii). By hypothesis, Ker e is Φ-torsion. Hence, by 1.14 (iii) and 1.5(8),

$$D_\Phi(\text{ Ker } e) = H^2_\Phi(\text{ Ker } e) = 0.$$

Hence $D_\Phi(\lambda)$ is an isomorphism.

Next, from the exact sequence

$$0 \longrightarrow Im\ e \xrightarrow{\rho} C' \xrightarrow{\sigma} Coker\ e \longrightarrow 0$$

we obtain an induced exact sequence

$$0 \longrightarrow D_\Phi(Im\ e) \xrightarrow{D_\Phi(\rho)} D_\Phi(C') \xrightarrow{D_\Phi(\sigma)} D_\Phi(Coker\ e).$$

However, by hypothesis, Coker e is Φ-torsion, and so D_Φ (Coker e)$= 0$, by 1.14(iii). Hence $D_\Phi(\rho)$ is an isomorphism.

1.16 LEMMA. (See [17, Theorem 7.11]) Let $f : A \longrightarrow B$ be a flat homomorphism of commutative Noetherian rigns, and let M be an A-module. Then there is a natural transformation of functors

$$\mu : Hom_A(\bullet, M) \otimes_A B \longrightarrow Hom_B((\bullet) \otimes_A B, M \otimes_A B)$$

(frome $\mathcal{C}(A)$ to $\mathcal{C}(B)$) which is such that

(i) $\mu_N(g \otimes b) = b(g \otimes Id_B)$ for each A-module N, each $b \in B$ and each $g \in Hom_A(N, M)$ and

(ii) μ_N is an isomorphism if N is a finitely generated A-module.

We shall need the following lemma which is proved in [42,4.1.7]. But, for the convenience of the reader, we have given it's proof.

1.17 LEMMA. Let $f : A \longrightarrow B$ be a flat homomorphism of commutative Noetherian rings, and let Φ be a system of ideals of A. Then the two functors $D_\Phi(\bullet) \otimes_A B$ and $D_{\Phi B}(\bullet \otimes_A B)$ (from $\mathcal{C}(A)$ to $\mathcal{C}(B)$) are naturally equivalent.

Proof. Let M be an A-module. For every two ideals $\mathfrak{a}, \mathfrak{b}$ in Φ with $\mathfrak{a} \leq \mathfrak{b}$ (that is $\mathfrak{b} \subseteq \mathfrak{a}$), the inclusion map $h^{\mathfrak{a}}_{\mathfrak{b}} : \mathfrak{b} \longrightarrow \mathfrak{a}$ gives rise to the direct systems

$$(Hom_A(\mathfrak{a}, \mathfrak{M}) \otimes_{\mathfrak{A}} \mathfrak{B})_{\mathfrak{a}\in} \text{ and } (\mathfrak{Hom}_{\mathfrak{A}}(\mathfrak{h}^{\mathfrak{a}}_{\mathfrak{b}}, \mathfrak{Id}_{\mathfrak{M}}) \otimes \mathfrak{Id}_{\mathfrak{B}})_{\substack{(\mathfrak{a}, \mathfrak{b}) \in \ \times \\ \mathfrak{a} \leq \mathfrak{b}}}$$

of B-modules and B-homomorphisms over the directed set Φ. Hence, by [1,p.33, Exercise 20], there is a natural B-isomorphism

$$w_M : (\varinjlim_{\mathfrak{a}\in} Hom_A(\mathfrak{a}, \mathfrak{M})) \otimes_{\mathfrak{A}} \mathfrak{B} \longrightarrow \varinjlim_{\mathfrak{a}\in} (\mathfrak{Hom}_{\mathfrak{A}}(\mathfrak{a}, \mathfrak{M}) \otimes_{\mathfrak{A}} \mathfrak{B}).$$

For any $\mathfrak{a}, \mathfrak{b}$ in Φ with $\mathfrak{b} \subseteq \mathfrak{a}$, let $h^{\mathfrak{a}\mathfrak{B}}_{\mathfrak{b}\mathfrak{B}} : \mathfrak{b}\mathfrak{B} \longrightarrow \mathfrak{a}\mathfrak{B}$ be the inclusion map. Recall from [17, Theorem 7.7] that there exists a B-isomorphism $\lambda_{\mathfrak{a}} : \mathfrak{a}\mathfrak{B} \longrightarrow \mathfrak{a} \otimes_{\mathfrak{A}} \mathfrak{B}$, for any ideal $\mathfrak{a}$ of A, such that the diagram

$$\begin{array}{ccc}
\mathfrak{a}\mathfrak{B} & \xrightarrow{\quad \lambda_{\mathfrak{a}} \quad} & \mathfrak{a} \otimes_{\mathfrak{A}} \mathfrak{B} \\
{\scriptstyle h^{\mathfrak{a}\mathfrak{B}}_{\mathfrak{b}\mathfrak{B}}} \Big\uparrow & & \Big\uparrow {\scriptstyle h^{\mathfrak{a}}_{\mathfrak{b}} \otimes Id_B} \\
\mathfrak{b}\mathfrak{B} & \xrightarrow[\quad \lambda_{\mathfrak{b}} \quad]{} & \mathfrak{b} \otimes_{\mathfrak{A}} \mathfrak{B}
\end{array}$$

commutes for any $\mathfrak{a}, \mathfrak{b}$ in Φ with $\mathfrak{a} \leq \mathfrak{b}$. Therefore, by 1.16, the diagram

$$\begin{array}{ccc}
Hom_A(\mathfrak{a}, \mathfrak{M}) \otimes_{\mathfrak{A}} \mathfrak{B} & \xrightarrow{\quad Hom_B(\lambda_{\mathfrak{a}}, Id_{M\otimes_A B})_0 \mu_{\mathfrak{a},\mathfrak{M}} \quad} & Hom_B(\mathfrak{a}B, M \otimes_A B) \\
{\scriptstyle Hom_A(h^{\mathfrak{a}}_{\mathfrak{b}}, Id_M) \otimes Id_B} \Big\downarrow & & \Big\downarrow {\scriptstyle Hom_B(h^{\mathfrak{a}\mathfrak{B}}_{\mathfrak{b}\mathfrak{B}}, Id_{M\otimes_A B})} \\
Hom_A(\mathfrak{b}, \mathfrak{M}) \otimes_{\mathfrak{A}} \mathfrak{B} & \xrightarrow[\quad Hom_B(\lambda_{\mathfrak{b}}, Id_{M\otimes_A B})_0 \mu_{\mathfrak{b},\mathfrak{M}} \quad]{} & Hom_B(\mathfrak{b}\mathfrak{B}, \mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B})
\end{array}$$

commutes for any $\mathfrak{a}, \mathfrak{b}$ in Φ with $\mathfrak{a} \leq \mathfrak{b}$. Therefor there exists a B-isomorphism

$$\gamma_M : \varinjlim_{\mathfrak{a}\in} (Hom_A(\mathfrak{a}, \mathfrak{M}) \otimes_{\mathfrak{A}} \mathfrak{B}) \longrightarrow \varinjlim_{\mathfrak{a}\in} \mathfrak{Hom}_{\mathfrak{B}}(\mathfrak{a}\mathfrak{B}, \mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}).$$

It is easy to see that there exists a B-isomorphism

$$\pi_M : \varinjlim_{\mathfrak{a}\in} Hom_B(\mathfrak{a}\mathfrak{B}, \mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}) \longrightarrow \varinjlim_{\mathfrak{a}\mathfrak{B}\in\mathfrak{B}} \mathfrak{Hom}_{\mathfrak{B}}(\mathfrak{a}\mathfrak{B}, \mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}).$$

We denote $\pi_M 0 \gamma_M 0 w_M$ by θ_M. Hence, if

$$[\ \ ] : Hom_A(\mathfrak{a}, \mathfrak{M}) \longrightarrow \mathfrak{D}(\mathfrak{M}) \text{ and } [\ \ ]' : \mathfrak{Hom}_{\mathfrak{B}}(\mathfrak{a}\mathfrak{B}, \mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}) \longrightarrow \mathfrak{D}_{\mathfrak{B}}(\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B})$$

are the cononical homomorphisms for any $\mathfrak{a} \in$, then

$$\theta_M : D_\Phi(M) \otimes_A B \longrightarrow D_{\Phi B}(M \otimes_A B)$$

is a B-isomorphism and, for all $\mathfrak{a} \in$, $g \in Hom_A(\mathfrak{a}, \mathfrak{M})$ and $\alpha \in B$,

$$\theta_M([g] \otimes \alpha) = [\alpha(g \otimes Id_B)_0 \lambda_{\mathfrak{a}}]'.$$

Also if N is a second A-module and $\psi : M \to N$ is an A-homomorphism, the diagram

$$\begin{array}{ccc}
D_\Phi(M) \otimes_A B & \xrightarrow{\theta_M} & D_{\Phi B}(M \otimes_A B) \\
{\scriptstyle D_\Phi(\psi) \otimes Id_B} \downarrow & & \downarrow {\scriptstyle D_{\Phi B}(\psi \otimes Id_B)} \\
D_\Phi(N) \otimes_A B & \xrightarrow[\theta_N]{} & D_{\Phi B}(N \otimes_A B)
\end{array}$$

commutes. This completes the proof.

We shall need the following lemma for 1.19.

1.18 LEMMA. Let Φ be a system of ideals of A and let $\mathfrak{a}$ be an ideal of A. Let M be an A-module. Then we can identify $Hom_A(\mathfrak{a}, \mathfrak{d}(\mathfrak{M}))$ with a submodule of $Hom_A(\mathfrak{a}, \mathfrak{M})$: when this is done,

$$Hom_A(\mathfrak{a}, \mathfrak{d}(\mathfrak{M})) = \mathfrak{d}(\mathfrak{Hom}_{\mathfrak{A}}(\mathfrak{a}, \mathfrak{M})).$$

Proof. Suppose that $g \in \Gamma_\Phi(Hom_A(\mathfrak{a}, \mathfrak{M}))$. Then there exists $\mathfrak{b} \in$ such that $\mathfrak{b}\mathfrak{g} = \mathfrak{o}$. Hence Im $g \subseteq \Gamma_\Phi(M)$; hence $g \in Hom_A(\mathfrak{a}, \mathfrak{d}(\mathfrak{M}))$.

On the other hand, suppose that $g \in Hom_A(\mathfrak{a}, \mathfrak{d}(\mathfrak{M}))$ and $\mathfrak{a} = \mathfrak{A}\alpha_\mathfrak{1} + \cdots + \mathfrak{A}\alpha_\mathfrak{t}$ where $\alpha_1, \cdots, \alpha_t \in A$. Then, for each $1 \leq i \leq t$, there exists $\mathfrak{b}_\mathfrak{i} \in$ such that $\mathfrak{b}_\mathfrak{i}\mathfrak{g}(\alpha_\mathfrak{i}) = \mathfrak{o}$. Next there exists $\mathfrak{c} \in$ such that $\mathfrak{c} \subseteq \mathfrak{b}_\mathfrak{i}$ for each $i = 1, \cdots, t$. Hence $\mathfrak{c}\mathfrak{g} = \mathfrak{o}$; so that $g \in \Gamma_\Phi(Hom_A(\mathfrak{a}, \mathfrak{M}))$.

1.19 LEMMA. Let Φ and Θ be two systems of ideals of A, and let M be an A-module. Then

$$\Gamma_\Phi(D_\Theta(M)) \cong D_\Theta(\Gamma_\Phi(M)).$$

Proof. By 1.5(5) and 1.18,

$$\begin{aligned}\Gamma_\Phi(D_\Theta(M)) &= \Gamma_\Phi(\varinjlim_{\mathfrak{a}\in\mathfrak{f}} Hom_A(\mathfrak{a}, \mathfrak{M})) \\ &\cong \varinjlim_{\mathfrak{a}\in\mathfrak{f}} \Gamma_\Phi(Hom_A(\mathfrak{a}, \mathfrak{M})) \\ &= \varinjlim_{\mathfrak{a}\in\mathfrak{f}} Hom_A(\mathfrak{a}, \mathfrak{d}(\mathfrak{M})) = \mathfrak{D}_\mathfrak{f}(\mathfrak{d}(\mathfrak{M})).\end{aligned}$$

1.20 LEMMA. Let M be an A-module, and let Φ be a system of ideals of A. Then Supp $(D_\Phi(M)) \subseteq$ Supp (M).

Proof. Let $\mathfrak{p} \in \text{Supp } (D_\Phi(M))$. Since the natural homomorphism $A \to A_\mathfrak{p}$ is flat, it follows from 1.17 that,

$$0 \neq (D_\Phi(M))_\mathfrak{p} \cong D_\Phi(M) \otimes_A A_\mathfrak{p} \cong D_{\Phi A\mathfrak{p}}(M \otimes_A A_\mathfrak{p}) \cong D_{\Phi A\mathfrak{p}}(M_\mathfrak{p}).$$

Hence $M_\mathfrak{p} \neq 0$, and so $\mathfrak{p} \in \text{Supp } (M)$.

1.21 LEMMA. Let $n \in \mathbb{N}_{\nvdash}$ and let $\Phi_0, \Phi_1, \cdots, \Phi_{n+1}$ be systems of ideals of A. Let M be an A-module such that M is Φ_i-torsion forall $i = 0, \cdots, n$. Then

$$\Phi = \left\{ \sum_{i=0}^{n+1} \mathfrak{a}_\mathfrak{i} : \mathfrak{a}_\mathfrak{i} \in {}_\mathfrak{i} \right\}$$

is a system of ideals of A and there is an A-isomorphism $\psi : D_\Phi(M) \longrightarrow D_{\Phi_{n+1}}(M)$ such that the diogram

$$\begin{array}{ccc} M & \xrightarrow{\eta_\Phi(M)} & D_\Phi(M) \\ & \searrow^{\eta_{\Phi_{n+1}}(M)} & \downarrow \psi \\ & & D_{\Phi_{n+1}}(M) \end{array}$$

commutes.

Proof. Application of the morphism of functors $\eta_{\Phi_{n+1}} : Id \longrightarrow D_{\Phi_{n+1}}$ to the modules and homomorphism in

$$\eta_\Phi(M) : M \longrightarrow D_\Phi(M)$$

yields a commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{\eta_{\Phi}(M)} & D_{\Phi}(M) \\
\Big\downarrow{\scriptstyle \eta_{\Phi_{n+1}}(M)} & & \Big\downarrow{\scriptstyle \eta_{\Phi_{n+1}}(D_{\Phi}(M))} \\
D_{\Phi_{n+1}}(M) & \xrightarrow[D_{\Phi_{n+1}}(\eta_{\Phi}(M))]{} & D_{\Phi_{n+1}}(D_{\Phi}(M))
\end{array}$$

Thus, in order to prove the lemma, it is enough to show that $D_{\Phi_{n+1}}(\eta_{\Phi}(M))$ and $\eta_{\Phi_{n+1}}(D_{\Phi}(M))$ are A-isomorphisms.

By, 1.14 (i) and 1.5(4), Ker $\eta_{\Phi}(M)$ and Coker $\eta_{\Phi}(M)$ are Φ-torsion, and so it is immediate that Ker $\eta_{\Phi}(M)$ and Coker $\eta_{\Phi}(M)$ are Φ_{n+1}-torsion. Hence, by 1.15, $D_{\Phi_{n+1}}(\eta_{\Phi}(M))$ is a an A-isomorphism.

By, 1.19 and the hypothesis, $D_{\Phi}(M)$ is Φ_i-torsion for all $i = 0, \cdots, n$. Hence, by 1.11 and the part (4) of 1.14 (iii),

$$H^1_{\Phi_{n+1}}(D_{\Phi}(M)) = H^1_{\Phi}(D_{\Phi}(M)) = 0 \text{ and } \Gamma_{\Phi_{n+1}}(D_{\Phi}(M)) = \Gamma_{\Phi}(D_{\Phi}(M)) = 0.$$

Hence, in view of 1.14 (i), $\eta_{\Phi_{n+1}}(D_{\Phi}(M))$ is an isomorphism.

Now, we present another proof for 1.21.

For each $\mathfrak{a}'_{\mathfrak{n}+1} := \mathfrak{a}_{\mathfrak{o}} + \mathfrak{a}_{1} + \cdots + \mathfrak{a}_{\mathfrak{n}+1} \in$, let

$[\ \ ]' : Hom_A(\mathfrak{a}'_{\mathfrak{n}+1}, \mathfrak{M}) \to \mathfrak{D}(\mathfrak{M})$ and $[\ \ ] : \mathfrak{Hom}_{\mathfrak{A}}(\mathfrak{a}_{\mathfrak{n}+1}, \mathfrak{M}) \to \mathfrak{D}_{\mathfrak{n}+1}(\mathfrak{M})$ be the cononical homorphisms.

For each $\mathfrak{a}'_{\mathfrak{n}+1} := \mathfrak{a}_{\mathfrak{o}} + \cdots + \mathfrak{a}_{\mathfrak{n}+1} \in$, we define

$$\psi_{\mathfrak{a}'_{\mathfrak{n}+1}} : Hom_A(\mathfrak{a}'_{\mathfrak{n}+1}, \mathfrak{M}) \longrightarrow \mathfrak{D}_{\mathfrak{n}+1}(\mathfrak{M})$$

by $\psi_{\mathfrak{a}'_{n+1}}(f) = [f|_{\mathfrak{a}_{n+1}}]$ where $f \in Hom_A(\mathfrak{a}'_{n+1}, \mathfrak{M})$. It is clear that $\psi_{\mathfrak{a}'_{n+1}}$ is an A-homorphism.

Let $\mathfrak{a}'_{n+1} := \mathfrak{a}_0 + \mathfrak{a}_1 + \cdots + \mathfrak{a}_{n+1}, \mathfrak{b}'_{n+1} := \mathfrak{b}_0 + \mathfrak{b}_1 + \cdots + \mathfrak{b}_{n+1} \in$ with $\mathfrak{a}'_{n+1} \subseteq \mathfrak{b}'_{n+1}$, we show that the diagram

$$\begin{array}{ccc} Hom_A(\mathfrak{a}'_{n+1}, M) & \xrightarrow{\psi_{\mathfrak{a}'_{n+1}}} & D_{\Phi_{n+1}}(M) \\ \downarrow & \nearrow_{\psi_{\mathfrak{b}'_{n+1}}} & \\ Hom_A(\mathfrak{b}'_{n+1}, M) & & \end{array} \qquad (*)$$

in which the vertical map is the restriction homomorphism, is commutative. Let $f \in Hom_A(\mathfrak{a}'_{n+1}, \mathfrak{M})$. There exists $\mathfrak{c} \in {}_{n+1}$ such that $\mathfrak{c} \subseteq \mathfrak{a}_{n+1}$ and $\mathfrak{c} \subseteq \mathfrak{b}_{n+1}$. Hence $(f|_{\mathfrak{a}_{n+1}})|_{\mathfrak{c}} = (f|_{\mathfrak{b}_{n+1}})|_{\mathfrak{c}}$. Therefore $\psi_{\mathfrak{a}'_{n+1}}(f) = \psi_{\mathfrak{b}'_{n+1}}(f|_{\mathfrak{b}'_{n+1}})$.

Hence there exists an A-homomorphism $\psi : D_{\Phi}(M) \longrightarrow D_{\Phi_{n+1}}(M)$ which such that $\psi([f]') = [f|_{\mathfrak{a}_{n+1}}]$, for each $\mathfrak{a}'_{n+1} = \mathfrak{a}_0 + \cdots + \mathfrak{a}_{n+1} \in$ and $f \in Hom_A(\mathfrak{a}'_{n+1}, \mathfrak{M})$. We show that ψ is injective and surjective.

It follows from 1.5(8) that there exists an injective A-module E such that $M \subseteq E$ and E is Φ_i-torsion A-module for all $i = 0, \cdots, n$.

Let $\mathfrak{a}'_{n+1} = \mathfrak{a}_0 + \cdots + \mathfrak{a}_{n+1} \in$ and $f \in Hom_A(\mathfrak{a}'_{n+1}, \mathfrak{M})$ such that $\psi([f]') = 0$. Then there exists $\mathfrak{b}_{n+1} \in {}_{n+1}$ such that $\mathfrak{b}_{n+1} \subseteq \mathfrak{a}_{n+1}$ and $f|_{\mathfrak{b}_{n+1}} = 0$. There exists an A-homomorphism $g : A \to E$ such that $g|_{\mathfrak{a}'_{n+1}} = f$. Assume that $g(1_A) = e$. since E is Φ_i-torsion and Φ_i is a system of ideals of A for all $i = 0, \cdots, n$, there are $\mathfrak{b}_i \in {}_i$ such that $\mathfrak{b}_i \mathfrak{e} = \mathfrak{o}$ and $\mathfrak{b}_i \subseteq \mathfrak{a}_i$ for all $i = 0, 1, \cdots, n$. Let $\mathfrak{b}'_{n+1} := \mathfrak{b}_0 + \cdots + \mathfrak{b}_{n+1}$. Then

$$\mathfrak{b}'_{n+1} \subseteq \mathfrak{a}'_{n+1} \text{ and } \mathfrak{f}|_{\mathfrak{b}'_{n+1}} = \mathfrak{g}|_{\mathfrak{b}'_{n+1}} = \mathfrak{g}|_{\mathfrak{b}_{n+1}} = \mathfrak{f}|_{\mathfrak{b}_{n+1}} = \mathfrak{o}.$$

Therefor $[f]' = 0$.

It follows from last pargragh that ψ is injective. Now, we show that ψ is surjective.

Suppose that $\mathfrak{a}_{\mathfrak{n}+\mathfrak{1}} \in {}_{\mathfrak{n}+\mathfrak{1}}$ and $h \in Hom_A(\mathfrak{a}_{\mathfrak{n}+\mathfrak{1}}, \mathfrak{M})$. Then there exists an A-homomorphism $\theta : A \to E$ such that $\theta|_{\mathfrak{a}_{\mathfrak{n}+\mathfrak{1}}} = h$. Assume that $\theta(1_A) = x$. Then, since E is Φ_i-torsion for all $i = 0, \cdots, n$, there are $\mathfrak{a}_\mathfrak{i} \in {}_\mathfrak{i}$ such that $\mathfrak{a}_\mathfrak{i}\mathfrak{x} = \mathfrak{o}$ for all $i = 0, \cdots, n$. Let $\mathfrak{a}'_{\mathfrak{n}+\mathfrak{1}} := \mathfrak{a}_\mathfrak{o} + \cdots + \mathfrak{a}_{\mathfrak{n}+\mathfrak{1}}$. Then $\theta|_{\mathfrak{a}'_{\mathfrak{n}+\mathfrak{1}}} = \theta|_{\mathfrak{a}_{\mathfrak{n}+\mathfrak{1}}} = h$. Hence $\theta|_{\mathfrak{a}'_{\mathfrak{n}+\mathfrak{1}}} \in Hom_A(\mathfrak{a}'_{\mathfrak{n}+\mathfrak{1}}, \mathfrak{M})$ and $\psi[(\theta|_{\mathfrak{a}'_{\mathfrak{n}+\mathfrak{1}}}]') = [h]$.

§2 *A* Characterization of generalized Hughes complexes.

2.1 DEFINITION. Let M be an R-module, and let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of R. The *generalized Hughes complex for M with respect to $\mathcal{S}$* is writen as

$$0 \longrightarrow M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} K^1 \longrightarrow \cdots \longrightarrow K^i \xrightarrow{h^i} K^{i+1} \longrightarrow \cdots \quad (*)$$

and is described as follows. Set $K^{-2} = 0, K^{-1} = M$, and let $h^{-2} : K^{-2} \longrightarrow K^{-1}$ denote the zero homomorphism. For all $n \in \mathbb{N}_{\nvdash}$, let $K^n := D_{\Phi_{n+1}}$ (Coker h^{n-2}), and let $h^{n-1} : K^{n-1} \longrightarrow K^n$ be the composition of the natural epimorphism from K^{n-1} to Coker h^{n-2} and the homomorphism $\eta_{\Phi_{n+1}}$ (Coker h^{n-2}): Coker $h^{n-2} \longrightarrow D_{\Phi_{n+1}}$ (Coker h^{n-2})$= K^n$. Then complex $(*)$ is denoted by $\mathcal{H}(\mathcal{S}, M)$ and is a generalization of are constructed by K.R. Hughes in [12].

2.2 DEFINITION. To say that the two complexes $C^\bullet_\alpha = (C^i_\alpha)_{i\geq 2} (\alpha = 1, 2)$ of A-modules and A-homomorphisms of the form

$$0 \xrightarrow{d^{-2}_{C^\bullet_\alpha}} M \xrightarrow{d^{-1}_{C^\bullet_\alpha}} C^0_\alpha \xrightarrow{d^{0}_{C^\bullet_\alpha}} C^1_\alpha \longrightarrow \cdots \longrightarrow C^n_\alpha \xrightarrow{d^{n}_{C^\bullet_\alpha}} C^{n+1}_\alpha \longrightarrow \cdots$$

are *isomorphic over Id_M* is to say that there is an isomorphism of complexes

$$\Psi = (\psi^i)_{i\geq -2} : C^\bullet_1 \longrightarrow C^\bullet_2$$

which is such that $\psi^{-1} : M \to M$ is the identity mapping Id_M. Similarly to say that a morphism of complexes $\Theta = (\theta^i)_{i\geq -2} : C^\bullet_1 \longrightarrow C^\bullet_2$ is *over Id_M* is simply to say that $\theta^{-1} = Id_M$.

Throughout this section, M is an A-module and $\mathcal{S} = (\Phi_i)_{i\in\mathbf{N}}$ denotes a family of systems of ideals of A. Let

$$\Phi_0 := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } A \text{ such that } V(\mathfrak{a}) \subseteq \text{Supp } (M)\}.$$

Note that Φ_0 is a system of ideals of A and, for every $x \in M$, $(0 :_A x) \in \Phi_0$ thus M is Φ_0-torsion.

As a preparation for our charactorization of generalized Hughes complexes, we begin by noting some properties of such complexes.

2.3 REMARKS. (i) For each $n \in \mathbb{N}_{\nvdash}$,

$$\text{Coker } h^{n-1} \cong \text{ Coker } (\eta_{\Phi_{n+1}}(\text{Coker } h^{n-2})) \cong H^1_{\Phi_{n+1}}(\text{Coker } h^{n-2})$$

(by 1.14 (i)), and so Coker h^{n-1} is Φ_{n+1}-torsion (by 1.5(4)).

(ii) For each $n \in \mathbb{N}_{\nvdash}$,

$$H^{n-1}(\mathcal{H}(\mathcal{S}, M)) \cong \text{ Ker } (\eta_{\Phi_{n+1}}(\text{Coker } h^{n-2})) \cong \Gamma_{\Phi_{n+1}}(\text{Coker } h^{n-2})$$

(by 1.14 (i)), and so $H^{n-1}(\mathcal{H}(\mathcal{S}, M))$ is Φ_{n+1}-torsion (by 1.5(4)).

(iii) For each $n \in \mathbb{N}_{\nvdash}$,

$$\eta_{\Phi_{n+1}}(K^n) : K^n \longrightarrow D_{\Phi_{n+1}}(K^n)$$

is an isomorphism (by 1.5(4) and 1.14 (iii)).

2.4 LEMMA. (i) For each $n \in \mathbb{N}_{\nvdash}$, Coker h^{n-2} is Φ_i-torsion for every $i = 0, \cdots, n$.

(ii) For each $n \in \mathbb{N}_{\nvdash}, \mathbb{H}^{\ltimes - \Vdash}(\mathcal{H}(\mathcal{S}, \mathbb{M}))$ is Φ_i-torsion for every $i = 0, \cdots n+1$.

(iii) for each $n \in \mathbb{N}_{\nvdash}, \mathbb{K}^{\ltimes}$ is Φ_i-torsion for every $i = 0, \cdots, n$.

Proof. (i) We prove this part of lemma by induction on n. The case in which $n = 0$ is immediate by the fact that Coker $h^{-2} = M$.

Now suppose, inductively, that $j > 0$ and the result has been proved for $j-1$. By 2.3 (i), it is enough to show that Coker h^{j-2} is a Φ_i-torsion A-module for all $i = 0, 1, \cdots, j-1$. It follows immediatly from the inductive hypothesis that Coker h^{j-3} is a Φ_i-torsion A-module for all $i = 0, 1, \cdots, j-1$. Therefore it follows from 1.19 and $K^{j-1} = D_{\Phi_j}$ (Coker h^{j-3}) that K^{j-1} is a Φ_i-torsion A-module for all $i = 0, 1, \cdots, j-1$. Hence, by 1.3, Coker h^{j-2} is a Φ_i-torsion A-module for all $i = 0, \cdots, j-1$; and this completes the inductive step.

(ii) This follows from (i) and 2.3 (ii).

(iii) This follows from (i) and 1.19

We can deduce the following lemma from 2.4 and 1.7, but we present here a direct proof.

2.5 LEMMA. Let $F(\Phi_n) = \bigcup_{\mathfrak{a} \in \mathfrak{n}} \vee(\mathfrak{a})$ and $F_n = \bigcap_{i=0}^{n} F(\Phi_i)$ for each $n \in \mathbb{N}_{\not\vdash}$. Then

(i) for each $n \in \mathbb{N}_{\not\vdash}$, Supp (Coker h^{n-2}) $\subseteq F_n$.

(ii) for each $n \in \mathbb{N}_{\not\vdash}$, Supp $H^{n-1}(\mathcal{H}(\mathcal{S}, M)) \subseteq F_{n+1}$.

(iii) for each $n \in \mathbb{N}_{\not\vdash}$, Supp $(K^n) \subseteq F_n$.

Proof. (i) We prove this by induction on n. To begin, note that, in the case when $n = 0$, the claim is immediate from the fact that if $\mathfrak{p}, \mathfrak{q} \in$Spec (A) such that $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in$ Supp (M), then $\mathfrak{q} \in$ Supp (M).

Now suppose, inductively, that $j > 0$ and the result has been proved for $j-1$. It immediately follows from this inductive hypothesis that Supp (Coker h^{j-3}) $\subseteq F_{j-1}$. Hence, by 2.3 (i) and 1.9,

$$\text{Supp (Coker } h^{j-2}) \subseteq \text{ Supp (Coker } h^{j-3}) \cap F(\Phi_j) \subseteq F_{j-1} \cap F(\Phi_j) = F_j.$$

The inductive step is therefore complete.

(ii) This follows from 2.3(ii), 1.9 and (i).

(iii) This follows from (i) and 1.20.

We can now prove the main results of this section.

2.6 THEOREM. Let $F(\Phi_n) = \bigcup_{\mathfrak{a} \in \mathfrak{n}} \vee(\mathfrak{a})$. For each $n \in \mathbb{N}_{\not{\circ}}$, and let

$$0 \xrightarrow{e^{-2}} M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \cdots \longrightarrow C^n \xrightarrow{e^n} C^{n+1} \longrightarrow \cdots$$

be a complex $C^{\bullet}$ of A-modules and A-homomorphisms such that

(a) Supp (Coker e^{n-2}) $\subseteq F(\Phi_n)$ (or equivalently Coker e^{n-2} is Φ_n-torsion),

(b) Supp $(H^{n-1}(C^{\bullet})) \subseteq F(\Phi_{n+1})$ (or equivalently $H^{n-1}(C^{\bullet})$ is Φ_{n+1}-torsion),

for each $n \in \mathbb{N}_{\not{\circ}}$. Then there is a morphism of complexes

$$\Psi = (\psi^n)_{n \geq -2} : C^{\bullet} \longrightarrow \mathcal{H}(\mathcal{S}, M)$$

over Id_M.

In fact, we can give the following information about the constituent homomorphisms of Ψ. Of course, $\psi^{-1} = Id_M$. For $n \in \mathbb{N}_{\not{\circ}}$, let

$$\overline{\psi^{n-1}} : \text{ Coker } e^{n-2} \longrightarrow \text{ Coker } h^{n-2}$$

be the homomorphism induced by ψ^{n-1}, and let $\overline{e^{n-1}}$: Coker $e^{n-2} \longrightarrow C^n$ be the homomorphism induced by e^{n-1}. Then ($D_{\Phi_{n+1}}(\overline{e^{n-1}})$ is an isomorphism and)

$$\psi^n = D_{\Phi_{n+1}}(\overline{\psi^{n-1}})_0 D_{\Phi_{n+1}}(\overline{e^{n-1}})^{-1} \;{}_0\eta_{\Phi_{n+1}}(C^n).$$

Moreover, if $\eta_{\Phi_{n+1}}(C^n) : C^n \longrightarrow D_{\Phi_{n+1}}(C^n)$ is an isomorphism for every $n \in \mathbb{N}_{\not{\circ}}$, then the morphism Ψ is an isomorphism of complexes.

Proof. It will be convenient to write $C^{-2} = 0$ and $C^{-1} = M$. Let $\psi^{-2} : C^{-2} \to K^{-2}$ be the zero map and $\psi^{-1} : C^{-1} \to K^{-1}$ be the identity mapping on M. These provide a basis for the following induction.

Let $n \in \mathbb{N}_0$ and suppose that we have already construceted A-homomorphisms $\psi^i : C^i \longrightarrow K^i$ for $i = -2, -1, 0, \cdots, n-1$ such that the diagram

$$\begin{array}{ccccccccccc}
C^{-2} & \xrightarrow{e^{-2}} & C^{-1} & \xrightarrow{e^{-1}} & C^{0} & \xrightarrow{e^{0}} & \cdots & \longrightarrow & C^{n-2} & \xrightarrow{e^{n-2}} & C^{n-1} \\
\downarrow \psi^{-2} & & \downarrow \psi^{-1} & & \downarrow \psi^{0} & & & & \downarrow \psi^{n-2} & & \downarrow \psi^{n-1} \\
K^{-2} & \xrightarrow{h^{-2}} & K^{-1} & \xrightarrow{h^{-1}} & K^{0} & \xrightarrow{h^{0}} & \cdots & \longrightarrow & K^{n-2} & \xrightarrow{h^{n-2}} & K^{n-1}
\end{array}$$

comumutes, and suppose that it has also been shown that $\psi^0, \cdots, \psi^{n-1}$ must all be isomorphisms if $\eta_{\Phi_{j+1}}(C^j) : C^j \longrightarrow D_{\Phi_{j+1}}(C^j)$ is an isomorphism for all $j = 0, \cdots, n-1$.

From our inductive assumptions we obtain a commutative diagram

$$\begin{array}{ccc}
 & & C^{n} \\
 & & \uparrow \overline{e^{n-1}} \\
C^{n-1} & \xrightarrow{\pi'_{n-1}} & \text{Coker } e^{n-2} \\
\downarrow \psi^{n-1} & & \downarrow \overline{\psi^{n-1}} \\
K^{n-1} & \xrightarrow{\pi_{n-1}} & \text{Coker } h^{n-2}
\end{array}$$

in which π_{n-1}, π'_{n-1} and $\overline{\psi^{n-1}}$ are the canonical homomorphisms. Now, using the morphism of functores $\eta_{\Phi_{n+1}} : Id \to D_{\Phi_{n+1}}$ of 1.13; we obtain a commutative diagram

$$\begin{array}{ccccc}
 & & C^n & \xrightarrow{\eta_{\Phi_{n+1}}(C^n)} & D_{\Phi_{n+1}}(C^n)\\
 & & \uparrow \overline{e^{n-2}} & & \uparrow D_{\Phi_{n+1}}(\overline{e^{n-1}})\\
C^{n-1} & \xrightarrow{\pi'_{n-1}} & \text{Coker } e^{n-2} & \xrightarrow{\eta_{\Phi_{n+1}}\ (\text{Coker } e^{n-2})} & D_{\Phi_{n+1}}\ (\text{Coker } e^{n-2})\\
\downarrow \psi^{n-1} & & \downarrow \overline{\psi^{n-1}} & & \downarrow D_{\Phi_{n+1}}(\overline{\psi^{n-1}})\\
K^{n-1} & \xrightarrow{\pi_{n-1}} & \text{Coker } h^{n-2} & \xrightarrow{\eta_{\Phi_{n+1}}\ (\text{Coker } h^{n-2})} & D_{\Phi_{n+1}}\ (\text{Coker } h^{n-2}) := K^n.
\end{array}$$

Ker $\overline{e^{n-1}}$ and Coker $\overline{e^{n-1}}$ are Φ_{n+1}-torsion, because Ker $\overline{e^{n-1}} = H^{n-1}(C^\bullet)$ and Coker $\overline{e^{n-1}}$ = Coker e^{n-1}. Hence, by 1.15, $D_{\Phi_{n+1}}(\overline{e^{n-1}})$ is an A-isomorphism. Now, let $\psi^n : C^n \longrightarrow K^n$ to be $D_{\Phi_{n+1}}(\overline{\psi^{n-1}})_0\ D_{\Phi_{n+1}}(\overline{e^{n-1}})^{-1}\ {}_0\eta_{\Phi_{n+1}}(C^n)$. Then Ψ^n is an A-homomorphism and the diagram

$$\begin{array}{ccc}
C^{n-1} & \xrightarrow{e^{n-1}} & C^n\\
\psi^{n-1}\downarrow & & \downarrow \psi^n\\
K^{n-1} & \xrightarrow{h^{n-1}} & K^n
\end{array}$$

commutes. ψ^n must be an isomorphism if $\eta_{\Phi_{j+1}}(C^j) : C^j \to D_{\Phi_{j+1}}(C^j)$ is an isomorphism for all $j = 0, \cdots, n$. This completes the inductive step and the result follows by induction.

2.7 THEOREM. Let the situation and notation be as in 2.6. Then the morphism of complexes

$$\Psi = (\psi^i)_{i \geq -2} : C^\bullet \longrightarrow \mathcal{H}(\mathcal{S}, M)$$

over Id_M is unique.

Proof. We prove this by induction on n.

Let $n \in \mathbb{N}_{\nvdash}$ and suppose that we have proved that there is exactly one family of A-homorphisms $(\psi^i)_{-2 \leq i \leq n-1}$ such that

$(a) \psi^i : C^i \to K^i$ for each $i = -2, \cdots, n-1$,

$(b) \psi^{-1} : M \to M$ is the identity map, and

(c) the diagram

$$\begin{array}{ccccccccc}
C^{-2} & \xrightarrow{e^{-2}} & C^{-1} & \xrightarrow{e^{-1}} & C^0 & \xrightarrow{e^0} & \cdots \longrightarrow & C^{n-2} & \xrightarrow{e^{n-2}} & C^{n-1} \\
\downarrow \psi^{-2} & & \downarrow \psi^{-1} & & \downarrow \psi^0 & & & \downarrow \psi^{n-2} & & \downarrow \psi^{n-1} \\
K^{-2} & \xrightarrow{h^{-2}} & K^{-1} & \xrightarrow{h^{-1}} & K^0 & \xrightarrow{h^0} & \cdots \longrightarrow & K^{n-2} & \xrightarrow{h^{n-2}} & K^{n-1}
\end{array}$$

commutes.

This is certainly the case when $n = 0$.

Let $\varphi : C^n \longrightarrow K^n$ be an A-homomorphism such that the diagram

$$\begin{array}{ccc}
C^{n-1} & \xrightarrow{e^{n-1}} & C^n \\
\psi^{n-1} \downarrow & & \downarrow \varphi \\
K^{n-1} & \xrightarrow{h^{n-1}} & K^n
\end{array}$$

commutes. Then we obtain a commutative diagram

$$
\begin{array}{ccc}
\text{Coker } e^{n-2} & \xrightarrow{\overline{e^{n-1}}} & C^n \\
\Big\downarrow \overline{\psi^{n-1}} & & \Big\downarrow \varphi \\
\text{Coker } h^{n-2} & \xrightarrow{\eta_{\Phi_{n+1}}\ (\text{Coker } h^{n-2})} & K^n := D_{\Phi_{n+1}}\ (\text{Coker } h^{n-2}).
\end{array}
$$

we now use the morphism of functors $\eta_{\Phi_{n+1}} : Id \to D_{\Phi_{n+1}}$ of 1.13 and use the functor $D_{\Phi_{n+1}}$; we obtain diagrams

$$
\begin{array}{ccc}
C^n & \xrightarrow{\eta_{\Phi_{n+1}}(C^n)} & D_{\Phi_{n+1}}(C^n) \\
\Big\downarrow \varphi & & \Big\downarrow D_{\Phi_{n+1}}(\varphi) \\
K^n & \xrightarrow{\eta_{\Phi_{n+1}}(K^n)} & D_{\Phi_{n+1}}(K^n)
\end{array}
$$

and

$$
\begin{array}{ccc}
D_{\Phi_{n+1}}\ (\text{Coker } e^{n-2}) & \xrightarrow{D_{\Phi_{n+1}}(\overline{e^{n-1}})} & D_{\Phi_{n+1}}(C^n) \\
D_{\Phi_{n+1}}(\overline{\psi^{n-1}}) \Big\downarrow & & \Big\downarrow D_{\Phi_{n+1}}(\varphi) \\
D_{\Phi_{n+1}}\ (\text{Coker } h^{n-2}) & \xrightarrow{D_{\Phi_{n+1}}(\eta_{\Phi_{n+1}}\ (\text{Coker } h^{n-2}))} & D_{\Phi_{n+1}}(K^n)
\end{array}
$$

in which commute. It should be noted that $D_{\Phi_{n+1}}(\overline{e^{n-1}})$ is an isomorphism, by 2.6, and that, by part (3) of 1.14 (iii),

$$D_{\Phi_{n+1}}(\eta_{\Phi_{n+1}}(\text{Coker } h^{n-2})) = \eta_{\Phi_{n+1}}(D_{\Phi_{n+1}}(\text{Coker } h^{n-2})) = \eta_{\Phi_{n+1}}(K^n)$$

and $\eta_{\Phi_{n+1}}(K^n)$ is an A-isomorphism. Thus

$$\begin{aligned}\varphi &= (\eta_{\Phi_{n+1}}(K^n))^{-1}\ {}_0D_{\Phi_{n+1}}(\varphi)_0\eta_{\Phi_{n+1}}(C^n)\\ &= (\eta_{\Phi_{n+1}}(K^n))^{-1}\ {}_0D_{\Phi_{n+1}}(\eta_{\Phi_{n+1}}(\mathrm{Coker}h^{n-2}))_0D_{\Phi_{n+1}}(\psi^{n-1})_0(D_{\Phi_{n+1}}(\overline{e^{n-1}}))^{-1}\ {}_0\eta_{\Phi_{n+1}}(C^n)\\ &= D_{\Phi_{n+1}}(\psi^{n-1})_0(D_{\Phi_{n+1}}(\overline{e^{n-1}}))^{-1}\ {}_0\eta_{\Phi_{n+1}}(C^n) = \psi^n.\end{aligned}$$

It therefore follows that there is at most one A-homomorphism $\psi^n : C^n \to K^n$ such that the diagram

$$\begin{array}{ccc} C^{n-1} & \xrightarrow{e^{n-1}} & C^n \\ \Big\downarrow{\scriptstyle \psi^{n-1}} & & \Big\downarrow{\scriptstyle \psi^n} \\ K^{n-1} & \xrightarrow{h^{n-1}} & K^n \end{array}$$

commutes. The inductive step is therefore complete.

2.8 COROLLARY. Let $F(\Phi_n) = \bigcup\limits_{\mathfrak{a}\in\mathfrak{n}} V(\mathfrak{a})$ and $F_n = \bigcap\limits_{i=0}^{n} F(\Phi_i)$ for each $n \in \mathbb{N}_0$. Let

$$0 \xrightarrow{e^{-2}} M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \cdots \longrightarrow C^n \xrightarrow{e^n} C^{n+1} \longrightarrow \cdots$$

be a complex $C^\bullet$ of A-modules and A-homorphisms. Then the following statements are equivalent:

(i) there is a unique isomorphism of complexes over Id_M from $C^\bullet$ to $\mathcal{H}(\mathcal{S}, M)$.

(ii) For each $n \in \mathbb{N}_0$,

(a) Supp (Coker e^{n-2})$\subseteq F_n$ (or equivalently Coker e^{n-2} is Φ_i-torsion for all $i = 0, \cdots, n$),

(b) Supp $(H^{n-1}(C^\bullet)) \subseteq F_{n+1}$ (or equivalently $H^{n-1}(C^\bullet)$ is Φ_i-torsion for all $i = 0, \cdots, n+1$),

(c) $\eta_{\Phi_{n+1}}(C^n) : C^n \longrightarrow D_{\Phi_{n+1}}(C^n)$ is an A-isomorphism.

(iii) For each $n \in \mathbb{N}_{\nvdash}$,

(a) Supp (Coker e^{n-2}) $\subseteq F(\Phi_n)$ (or equivalently Coker e^{n-2} is Φ_n-torsion),

(b) Supp $(H^{n-1}(C^\bullet)) \subseteq F(\Phi_{n+1})$ (or equivalently $H^{n-1}(C^\bullet)$ is Φ_{n+1}-torsion),

(c) $\eta_{\Phi_{n+1}}(C^n) : C^n \longrightarrow D_{\Phi_{n+1}}(C^n)$ is an A-isomorphism.

Proof. *(i)* $\Rightarrow$ *(ii)* This follows from 2.4, 2.5 and part (3) of 1.14 (iii).

(ii) $\Rightarrow$ *(iii)* this is clear.

(iii) $\Rightarrow$ *(i)* This follows from 2.6 and 2.7

2.9 THEOREM. Let $F(\Phi_n) = \bigcup_{\mathfrak{a} \in \mathfrak{n}} V(\mathfrak{a})$ for each $n \in \mathbb{N}_{\nvdash}$, and let

$$0 \xrightarrow{e^{-2}} M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \cdots \longrightarrow C^n \xrightarrow{e^n} C^{n+1} \longrightarrow \cdots$$

be a complex $C^\bullet$ of A-modules and A-homomorphisms such that

(a) Supp (Coker e^{n-2}) $\subseteq F(\Phi_n)$ (or equivalently Coker e^{n-2} is Φ_n-torsion),

(b) Supp $(H^{n-1}(C^\bullet)) \subseteq F(\Phi_{n+1})$ (or equivalently $H^{n-1}(C^\bullet)$ is Φ_{n+1}-torsion),

so that, by 2.6 and 2.7, there is a unique morphism of complexes $\Psi = (\psi^n)_{n \geq -2}$: $C^\bullet \longrightarrow \mathcal{H}(\mathcal{S}, M)$ over Id_M. Assume that

(c) $e^n(\Gamma_{\Phi_{n+1}}(C^n)) \subseteq \Gamma_{\Phi_{n+2}}(C^{n+1})$ for all $n \in \mathbb{N}_{\nvdash}$, and

(d) $H^1_{\Phi_{n+1}}(C^n) = 0$ for all $n \in \mathbb{N}_{\nvdash}$.

Then ψ^n is an epimorphism with kernel $\Gamma_{\Phi_{n+1}}(C^n)$ for each $n \in \mathbb{N}_{\nvdash}$; so that $\mathcal{H}(\mathcal{S}, M)$ is isomorphic (over Id_M) to the quotient complex $\mathcal{H}(\mathcal{I}, M)/Y^\bullet$, where $Y^\bullet = (Y^n)_{n \geq -2}$ denotes the subcomplex

$$0 \longrightarrow 0 \xrightarrow{u^{-1}} \Gamma_{\Phi_1}(C^0) \xrightarrow{u^0} \cdots \longrightarrow \Gamma_{\Phi_{n+1}}(C^n) \xrightarrow{u^n} \Gamma_{\Phi_{n+2}}(C^{n+1}) \longrightarrow \cdots$$

of $C^\bullet$.

Proof. Write the quotient complex $C^\bullet/Y^\bullet$ as

$$Z^\bullet : 0 \xrightarrow{f^{-2}} M \xrightarrow{f^{-1}} Z^0 \xrightarrow{f^0} Z^1 \longrightarrow \cdots \longrightarrow Z^i \xrightarrow{f^i} Z^{i+1} \longrightarrow \cdots$$

We show that $\mathcal{H}(\mathcal{S}, M)$ is isomorphic (over Id_M) to the $Z^\bullet$.

Let $n \in \mathbb{N}_{\nvdash}$. Firstly, we show that Supp (Coker f^{n-2}) $\subseteq F(\Phi_n)$. We have the exact sequence

$$0 \longrightarrow \frac{Ime^{n-2} + Y^{n-1}}{Ime^{n-2}} \longrightarrow \frac{C^{n-1}}{Ime^{n-2}} \longrightarrow \frac{C^{n-1}}{Ime^{n-2} + Y^{n-1}} \longrightarrow 0$$

weher Coker $e^{n-2} = C^{n-1}/Ime^{n-2}$ and Coker $f^{n-2} = Z^{n-1}/Imf^{n-2} \cong C^{n-1}/Ime^{n-2} + Y^{n-1}$. Hence

$$\text{Supp (Coker } f^{n-2}) \subseteq \text{Supp (Coker } e^{n-2}) \subseteq F(\Phi_n).$$

Now, we show that Supp $(H^{n-1}(Z^\bullet)) \subseteq F(\Phi_{n+1})$. It is clear that there is a morphism of complexes $\Pi = (\pi^n)_{n\geq -2} : C^\bullet \longrightarrow Z^\bullet$ over Id_M such that π^n is canonical epimorphism, for each $n \in \mathbb{N}_{\nvdash}$. We have a commutative diagram

$$\begin{array}{ccc} \text{Coker } e^{n-2} & \xrightarrow{\overline{e^{n-1}}} & C^n \\ \downarrow {\scriptstyle \overline{\pi^{n-1}}} & & \downarrow {\scriptstyle \pi^n} \\ \text{Coker } f^{n-2} & \xrightarrow{\overline{f^{n-1}}} & Z^n \end{array}$$

in which $\overline{\pi^{n-1}}, \overline{e^{n-1}}$ and $\overline{f^{n-1}}$ are induced homomorphisms. We have exact sequences

$$(\overline{\pi^{n-1}})^{-1}(\text{ Ker } \overline{f^{n-1}}) \xrightarrow{\overline{\pi^{n-1}}} \text{ Ker } \overline{f^{n-1}} \longrightarrow 0$$

and

$$0 \longrightarrow (\text{ Ker } \overline{e^{n-1}}) \cap (\overline{e^{n-1}})^{-1}(Y^n) \longrightarrow (\overline{e^{n-1}})^{-1}(Y^n) \xrightarrow{\overline{e^{n-1}}} Ime^{n-1} \cap Y^n \longrightarrow 0.$$

Since Ker $\overline{f^{n-1}} = H^{n-1}(Z^\bullet)$, Ker $\overline{e^{n-1}} = H^{n-1}(C^\bullet)$ and $(\overline{\pi^{n-1}})^{-1}($ Ker $\overline{f^{n-1}}) = (\overline{e^{n-1}})^{-1}(Y^n)$, we have

$$\begin{aligned}
\text{Supp } (H^{n-1}(Z^\bullet)) &\subseteq \text{Supp } ((\overline{e^{n-1}})^{-1}(Y^n)) \\
&= \text{Supp } (Ime^{n-1} \cap Y^n) \cup \text{ Supp } (H^{n-1}(C^\bullet) \cap (\overline{e^{n-1}})^{-1}(Y^n)) \\
&\subseteq \text{Supp } (Y^n) \cup \text{ Supp } (H^{n-1}(C^\bullet)) \\
&\subseteq \text{Supp } (Y^n) \cup F(\Phi_{n+1}).
\end{aligned}$$

It follows from 1.9 that Supp $(Y^n) =$ Supp $(\Gamma_{\Phi_{n+1}}(C^n)) \subseteq F(\Phi_{n+1})$. Hence

$$\text{Supp}(H^{n-1}(Z^\bullet)) \subseteq F(\Phi_{n+1}).$$

We can therefore apply theorems 2.6 and 2.7 to the complex $Z^\bullet$ to see that there is a unique morphism of complexes

$$\overline{\Psi} = (\overline{\psi^n})_{n \geq -2} : Z^\bullet \longrightarrow \mathcal{H}(\mathcal{S}, M)$$

over Id_M. By Theorem 2.6, in order to show that $\overline{\Psi}$ is an isomorphism of complexes, it is enough for us to show that, for each $n \in \mathbb{N}_{\not\vdash}$,

$$\eta_{\Phi_{n+1}}(C^n/Y^n) : C^n/Y^n \longrightarrow D_{\Phi_{n+1}}(C^n/Y^n)$$

is a isomorphism. However, $Y^n = \Gamma_{\Phi_{n+1}}(C^n)$, and so, since $\Gamma_{\Phi_{n+1}}(C^n/\Gamma_{\Phi_{n+1}}(C^n)) = 0$ and, by 1.5(9),

$$H^1_{\Phi_{n+1}}(C^n/\Gamma_{\Phi_{n+1}}(C^n)) \cong H^1_{\Phi_{n+1}}(C^n),$$

it is an easy consequence of 1.14 (i) and the hypothesis that $H^1_{\Phi_{n+1}}(C^n) = 0$.

It is clear that $\overline{\psi^n}_0\pi^n$ is an epimorphism with kernel $Y^n = \Gamma_{\Phi_{n+1}}(C^n)$, for each $n \in \mathbb{N}_{\not\vdash}$. The uniquenes aspect of Ψ implies that $\overline{\Psi}_0\Pi$ must be Ψ. Hence ψ^n is an epimorphism with kernel $Y^n = \Gamma_{\Phi_{n+1}}(C^n)$.

2.10 REMARK. Let the situation and notation be as in 2.9. Next, we present another proof for this statement Supp $(H^{n-1}(Z^\bullet)) \subseteq F(\Phi_{n+1})$ for all $n \in \mathbb{N}_{\not 0}$.

Let $n \in \mathbb{N}_{\not 0}$, and let $\mathfrak{p} \in$spec(A) and $\mathfrak{p} \notin \mathfrak{F}(_{\mathfrak{n}+1})$. Then $(H^{n-1}(C^\bullet))_\mathfrak{p} = 0$ (1). In view of 1.9, it is clear that

$$\text{Supp}(H^n(Y^\bullet)) \subseteq \text{ Supp (Ker } (u^n)) \subseteq \text{ Supp } (Y^n) \subseteq F(\Phi_{n+1}).$$

Hence $(H^n(Y^\bullet))_\mathfrak{p} = 0$ (2). It therefore follows from the long exact sequence of cohomology modules resulting from the exact sequence of complexes

$$0 \longrightarrow Y^\bullet \longrightarrow C^\bullet \longrightarrow Z^\bullet \longrightarrow 0$$

that there is an exact sequence of A-modules

$$H^{n-1}(C^\bullet) \longrightarrow H^{n-1}(Z^\bullet) \longrightarrow H^n(Y^\bullet).$$

Hence there is an exact sequence of A_p-modules

$$(H^{n-1}(C^\bullet))_\mathfrak{p} \longrightarrow (H^{n-1}(Z^\bullet))_\mathfrak{p} \longrightarrow (H^n(Y^\bullet))_\mathfrak{p} \ \ (3).$$

It follows from (1), (2) and (3) that $(H^{n-1}(Z^\bullet))_\mathfrak{p} = 0$. Hence $\mathfrak{p} \notin$ Supp $(H^{n-1}(Z^\bullet))$.

§3 Morphisms between generalized Hughes complexes.

Throughout this section, M will denote an A-module. We shall let

$$\Phi_0 := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{a}) \subseteq \text{ Supp } (\mathfrak{M})\}.$$

3.1 PROPOSITION. Let $\mathcal{S} = (\Phi_i)_{i \in \mathbb{N}}$ and $\mathcal{I} = (\Theta_i)_{i \in \mathbb{N}}$ be two families of systems of ideals of A.

(i) If, for each $n \in \mathbb{N}$ and each $\mathfrak{b} \in \mathfrak{f}_{\mathfrak{n}}$, there exists $\mathfrak{a} \in {}_{\mathfrak{n}}$ such that $\mathfrak{a} \subseteq \mathfrak{b}$, then there is a unique morphism of complexes

$$\Psi : \mathcal{H}(\mathcal{I}, M) \longrightarrow \mathcal{H}(\mathcal{S}, M)$$

over Id_M.

In particular, this conclusion applies of $\Theta_n \subseteq \Phi_n$ for all $n \in \mathbb{N}$.

(ii) If, for each $n \in \mathbb{N}$, the system Θ_n is a cofinal subset of Φ_n, then the unique morphism of complexes $\Psi : \mathcal{H}(\mathcal{I}, M) \longrightarrow \mathcal{H}(\mathcal{S}, M)$ over Id_M (of (i)) is an isomorphism.

Proof. Write the generalized Hughes complex $\mathcal{H}(\mathcal{I}, M)$ as

$$0 \longrightarrow M \xrightarrow{w^{-1}} L^0 \xrightarrow{w^0} L^1 \longrightarrow \cdots \longrightarrow L^i \xrightarrow{w^i} L^{i+1} \longrightarrow \cdots$$

(i) It is easy to see that $\bigcup_{\mathfrak{a} \in \mathfrak{f}_{\mathfrak{n}}} V(\mathfrak{a}) \subseteq \bigcup_{\mathfrak{a} \in_{\mathfrak{n}}} \mathfrak{V}(\mathfrak{a})$, for each $n \in \mathbb{N}$. Hence, by 2.5, Supp (Coker w^{n-2}) $\subseteq \bigcup_{\mathfrak{a} \in_{\mathfrak{n}}} V(\mathfrak{a})$ and Supp $(H^{n-1}(\mathcal{H}(\mathcal{I}, M))) \subseteq \bigcup_{\mathfrak{a} \in_{\mathfrak{n}+1}} V(\mathfrak{a})$, for each $n \in \mathbb{N}_{\not\vdash}$. The result is therefore immediate from 2.6 and 2.7.

(ii) It follows from the assumption that $L^n := D_{\Theta_{n+1}}$ (Coker w^{n-2}) is isomorphic to $D_{\Phi_{n+1}}$ (Coker w^{n-2}), for each $n \in \mathbb{N}_{\not\vdash}$. Hence, by the part (3) of 1.14 (iii),

$\eta_{\Phi_{n+1}} : L^n \longrightarrow D_{\Phi_{n+1}}(L^n)$ is an isomorphism, for each $n \in \mathbb{N}_{\nvdash}$. The claim follows from 2.6

It is easy to deduce the following theorem from 2.9 and the proof of 3.1.

3.2 THEOREM. Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ and $\mathcal{I} = (\Theta_i)_{i\in\mathbb{N}}$ be two families of systems of ideals of A such that $\Theta_n \subseteq \Phi_n$ for all $n \in \mathbb{N}$; so that, by 3.1, there is a unique morphism of complexes $\Psi = (\psi^n)_{n\geq -2} : \mathcal{H}(\mathcal{I}, M) \longrightarrow \mathcal{H}(\mathcal{S}, M)$ over Id_M.

Write the generalized Hughes complex $\mathcal{H}(\mathcal{I}, M)$ as

$$0 \longrightarrow M \xrightarrow{w^{-1}} L^0 \xrightarrow{w^0} L^1 \longrightarrow \cdots \longrightarrow L^i \xrightarrow{w^i} L^{i+1} \longrightarrow \cdots$$

Assume that

$(a) w^n(\Gamma_{\Phi_{n+1}}(L^n)) \subseteq \Gamma_{\Phi_{n+2}}(L^{n+1})$ for all $n \in \mathbb{N}_{\nvdash}$, and
$(b) H^1_{\Phi_{n+1}}(L^n) = 0$ for all $n \in \mathbb{N}_{\nvdash}$.

Then ψ^n is an epimorphism with kernel $\Gamma_{\Phi_{n+1}}(L^n)$, for each $n \in \mathbb{N}_{\nvdash}$, so that $\mathcal{H}(\mathcal{S}, M)$ is isomorphic (over Id_M) to the quotient complex $\mathcal{H}(\mathcal{I}, M)/Y^\bullet$, where $Y^\bullet = (Y^n)_{n\geq -2}$ denotes the subcomplex

$$0 \longrightarrow 0 \longrightarrow \Gamma_{\Phi_1}(L^0) \longrightarrow \cdots \longrightarrow \Gamma_{\Phi_{n+1}}(L^n) \longrightarrow \Gamma_{\Phi_{n+2}}(L^{n+1}) \longrightarrow \cdots$$

of $\mathcal{H}(\mathcal{I}, M)$.

3.3 LEMMA. Let the situation and notation be as in 3.2. write the generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$ as

$$0 \xrightarrow{h^{-2}} M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} K^1 \longrightarrow \cdots \longrightarrow K^i \xrightarrow{h^i} K^{i+1} \longrightarrow \cdots$$

Also set $L^{-3} = K^{-3} = 0, L^{-2} = K^{-2} = 0, L^{-1} = K^{-1} = M$, and use $w^{-3} = h^{-3} : K^{-3} \longrightarrow K^{-2}$ and $w^{-2} = h^{-2} : K^{-2} \longrightarrow K^{-1}$ to denote the zero homomorphisms.

Let $n \in \mathbb{N}_{\not{0}}$. If $\mathcal{H}(\mathcal{S}, M)$ is exact at $K^{-2}, K^{-1}, \cdots, K^{n-2}$, then $\mathcal{H}(\mathcal{I}, M)$ is also exact at $L^{-2}, L^{-1}, \cdots, L^{n-2}$, and ψ^i is an isomorphism for all $i = -2, -1, \cdots, n-1$.

In particular, if $\mathcal{H}(\mathcal{S}, M)$ is exact, then $\mathcal{H}(\mathcal{I}, M)$ is also exact and Ψ is an isomorphism of complexes.

Proof. We prove this by induction on n. the claims are certainly the case when $n = 0$.

Now suppose, inductively, that $l > 0$ and the result has been proved for $l - 1$. Assume that $\mathcal{H}(\mathcal{S}, M)$ is exact at $K^{-2}, K^{-1}, \cdots, K^{l-2}$. It is enough to prove that ψ^i is an isomorphism for all $i = -2, -1, \cdots, l-1$. It immediately follows from the inductive hypothesis that $\psi^{-2}, \psi^{-1}, \cdots, \psi^{l-2}$ are all isomorphisms. This implies that Coker $w^{l-3} \cong$ Coker h^{l-3}: Now, by 2.3 (ii),

$$H^{l-2}(\mathcal{H}(\mathcal{S}, M)) \cong \Gamma_{\Phi_l} \text{ (Coker } h^{l-3}),$$

and this is zero since $\mathcal{H}(\mathcal{S}, M)$ is eact at K^{l-2}. Therefore Γ_{Φ_l} (Coker $w^{l-3}) = 0$. Hence we can use 1.19 to deduce that

$$\Gamma_{\Phi_l}(L^{l-1}) = \Gamma_{\Phi_l}(D_{\Theta_l}(\text{Coker } w^{l-3})) \cong D_{\Theta_l}(\Gamma_{\Phi_l}(\text{Coker } w^{l-3})) = 0.$$

Therefore ψ^{l-1} is an isomorphism, by 3.2, and the inductive step is complete.

§4 Generalized Hughes complexes and ring homomorphisms.

4.1 LEMMA. Let G be an A-module, and let $f : A \to B$ be homomorphism of commutative Noetherian rings. Let L be a B-module. Then, for each $i \in \mathbb{N}_{\not{\mathrm{r}}}$,

$$\mathrm{Supp}_B(Tor_i^A(G,L)) \subseteq \{\mathfrak{q} \in \ \mathrm{Spec}\ (\mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \ \mathrm{Supp}\ _{\mathfrak{A}}(\mathfrak{G})\}.$$

Proof. Let $i \in \mathbb{N}_{\not{\mathrm{r}}}$. Assume that $\mathfrak{q} \in \ \mathrm{Supp}_{\mathfrak{B}}(\mathfrak{Tor}_{\mathfrak{i}}^{\mathfrak{A}}(\mathfrak{G},\mathfrak{L}))$, and set $\mathfrak{p} := \mathfrak{f}^{-1}(\mathfrak{q})$. Then there exists $0 \neq x \in Tor_i^A(G,L)$ such that $(0 :_B x) \subseteq \mathfrak{q}$. Hence $(0 :_A x) \subseteq f^{-1}(\mathfrak{q})$. Consequently, $f^{-1}(\mathfrak{q}) \in \mathrm{Supp}\ _A(Tor_i^A(G,L))$. Since, by [16,3.E, page 21],

$$Tor_i^{A_{\mathfrak{p}}}(G_{\mathfrak{p}}, L_{\mathfrak{p}}) \cong (Tor_i^A(G,L))_{\mathfrak{p}},$$

it follows that $\mathfrak{p} \in \mathrm{Supp}\ _A(G)$.

4.2 THEOREM. Let M be an A-module, and let

$$0 \xrightarrow{e^{-2}} M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \cdots \longrightarrow C^n \xrightarrow{e^n} C^{n+1} \longrightarrow \cdots$$

be a complex $C^{\bullet}$ of A-modules and A- homomorphisms. Let $f : A \longrightarrow B$ be homomorphism of commutative Noetherian rings, and let L be a B-module. write the complex $C^{\bullet} \otimes_A L$ as

$$0 \longrightarrow M \otimes_A L \xrightarrow{e^{-1}\otimes Id_L} C^0 \otimes_A L \longrightarrow \cdots \longrightarrow C^n \otimes_A L \xrightarrow{e^n \otimes Id_L} C^{n+1} \otimes_A L \longrightarrow \cdots .$$

Then, for each $n \in \mathbb{N}_{\not{\mathrm{r}}}$,
$\mathrm{Supp}\ _B(\mathrm{Coker}\ (e^{n-2} \otimes Id_L)) = \mathrm{Supp}_B(\mathrm{Coker}\ e^{n-2} \otimes_A L)$ and $\mathrm{Supp}_B(H^{n-1}(C^{\bullet} \otimes_A L)) \subseteq \mathrm{Supp}_B(H^{n-1}(C^{\bullet}) \otimes_A L) \cup \ \mathrm{Supp}\ _B(Tor_1^A(\mathrm{Coker}\ e^{n-1}, L))$.

Proof. Let $n \in \mathbb{N}_{\not{\mathrm{r}}}$. It will be convenient to abbreviate $H^{n-1}(C^{\bullet})$ by H^{n-1} in the remainder of proof; also, we interpret C^{-1} as M and C^{-2} as zero. The complex

$C^\bullet$ induces, in an obvious way, exact sequences

$$0 \longrightarrow H^{n-1} \xrightarrow{\tau} \text{Coker} e^{n-2} \xrightarrow{\lambda} Ime^{n-1} \longrightarrow 0 \qquad (1)$$

and

$$0 \longrightarrow Ime^{n-1} \xrightarrow{\rho} C^n \xrightarrow{\sigma} \text{Coker } e^{n-1} \longrightarrow 0. \qquad (2)$$

Also, there is the commutative diagram

$$\begin{array}{ccccccc}
C^{n-2} & \xrightarrow{e^{n-2}} & C^{n-1} & \xrightarrow{\pi} & \text{Coker } e^{n-2} & \longrightarrow & 0 \\
 & & & \searrow^{e^{n-1}} & \downarrow{\scriptstyle \rho_0\lambda} & & \\
 & & & & C^n & &
\end{array} \qquad (3)$$

in which π is the canonical epimorphism. Apllication of the right exact function $\bullet \otimes_A L$ to the modules and homomorphisms in the diagram (3) yields a commutative diagram

$$\begin{array}{ccccccc}
C^{n-2} \otimes_A L & \xrightarrow{e^{n-2} \otimes_A Id_L} & C^{n-1} \otimes_A L & \xrightarrow{\pi \otimes Id_L} & \text{Coker } e^{n-2} \otimes_A L & \longrightarrow & 0 \\
 & & & \searrow^{e^{n-1} \otimes Id_L} & \downarrow{\scriptstyle (\rho_0\lambda) \otimes Id_L} & & \\
 & & & & C^n \otimes_A L & &
\end{array} \qquad (4)$$

such that its row is an exact sequence. Let $\overline{\pi \otimes Id_L}$: Coker $(e^{n-2} \otimes Id_L) \longrightarrow$ Coker $e^{n-2} \otimes_A L$ and $\overline{e^{n-1} \otimes Id_L}$: Coker $(e^{n-2} \otimes Id_L) \longrightarrow C^n \otimes_A L$ be the homomorphisms induced by $\pi \otimes Id_L$ and $e^{n-1} \otimes Id_L$. Then, by (4), $\overline{\pi \otimes Id_L}$ is an isomorphism of B-modules and the diagram

$$
\begin{array}{ccc}
\text{Coker } (e^{n-2} \otimes Id_L) & \xrightarrow[\cong]{\overline{\pi \otimes Id_L}} & \text{Coker } e^{n-2} \otimes_A L \\
& \searrow^{\overline{e^{n-1} \otimes Id_L}} & \downarrow (\rho_0\lambda) \otimes Id_L \\
& & C^n \otimes_A L
\end{array} \qquad (5)
$$

commutes. Hence, by (5), there is an isomorphism of B-modules

$$H^{n-1}(C^\bullet \otimes_A L) = \text{ Ker } (\overline{e^{n-1} \otimes Id_L}) \xrightarrow{\cong} \text{ Ker } ((\rho_0\lambda) \otimes Id_L). \qquad (6)$$

It follows from (5) that Supp_B (Coker $(e^{n-2} \otimes Id_L)) = \text{Supp}_B$ (Coker $e^{n-2} \otimes_A L$).

It is easy to see that Ker $((\rho_0\lambda) \otimes Id_L) = (\lambda \otimes Id_L)^{-1}($ Ker $(\rho \otimes Id_L))$. Let $X = (\lambda \otimes Id_L)^{-1}($ Ker $(\rho \otimes Id_L))$. Then sequence

$$0 \longrightarrow \text{ Ker } (\lambda \otimes Id_L) \cap X \longrightarrow X \xrightarrow{\lambda \otimes Id_L} \text{ Ker } (\rho \otimes Id_L) \cap Im(\lambda \otimes Id_L) \longrightarrow 0$$

is exact. Hence, by (6),

$$\text{Supp}_B(H^{n-1}(C^\bullet \otimes_A L)) = \text{Supp}_B(X) \subseteq \text{Supp}_B(\text{ Ker } (\lambda \otimes Id_L)) \cup \text{Supp}_B(\text{ Ker } (\rho \otimes Id_L)). \quad (7)$$

The exact sequences (1) and (2) induce exact sequences

$$H^{n-1} \otimes_A L \xrightarrow{\tau \otimes Id_L} (\text{Coker } e^{n-2} \otimes_A L) \xrightarrow{\lambda \otimes Id_L} (Ime^{n-1}) \otimes_A L \longrightarrow 0$$

and

$$Ime^{n-1} \otimes_A L \xrightarrow{\rho \otimes Id_L} C^n \otimes_A L \xrightarrow{\lambda \otimes Id_L} \longrightarrow \begin{array}{c} (\text{Coker } e^{n-1}, L) \xrightarrow{\mu} \\ (\text{Coker } e^{n-1} \otimes_A L) \longrightarrow 0. \end{array}$$

Hence, by (7),

$$\begin{aligned} \text{Supp}_B(H^{n-1}(C^\bullet \otimes_A L)) &\subseteq \text{Supp}_B(Im(\tau \otimes Id_L)) \cup \text{Supp}_B(Im\mu) \\ &\subseteq \text{Supp}_B(H^{n-1} \otimes_A L) \cup \text{Supp}_B(Tor_1^A(\text{Coker } e^{n-1}, L)). \end{aligned}$$

4.3 THEOREM. Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of A, and let M be an A-module. Write the generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$ as

$$0 \xrightarrow{h^{-2}} M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} K^1 \longrightarrow \cdots \longrightarrow K^i \xrightarrow{h^i} K^{i+1} \longrightarrow \cdots$$

Let $f : A \longrightarrow B$ be a homomorphism of commutative Noetherian rings. Let $\mathcal{S}B = (\Phi_i B)_{i\in\mathbb{N}}$ (we are using notation introduced in 1.4), a family of systems of ideals of B. Then there is a unique morphism of complexes of B-modules and B-homomorphisms

$$\Psi = (\psi^n)_{n\geq -2} : \mathcal{H}(\mathcal{S}, M) \otimes_A B \longrightarrow \mathcal{H}(\mathcal{S}B, M \otimes_A B)$$

over $Id_{M\otimes_A B}$.

Furthermore, Ψ is an isomorphism of complexes if and only if

$$\eta_{\Phi_{n+1}B}(K^n \otimes_A B) : K^n \otimes_A B \longrightarrow D_{\Phi_{n+1}B}(K^n \otimes_A B)$$

is an isomorphism for every $n \in \mathbf{N}_0$.

Proof. Set

$$\Phi_0 := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{a}) \subseteq \text{ Supp}(\mathfrak{M})\}.$$

By 2.6, it is enough for us to show that, for each $n \in \mathbb{N}_{\not\vdash}$,

$$\text{Supp}_B(\text{Coker } h^{n-2} \otimes Id_B) \subseteq \bigcup_{\mathfrak{b}\in_{\mathfrak{n}}\mathfrak{B}} V(\mathfrak{b})$$

and

$$\text{Supp}_B(H^{n-1}(\mathcal{H}(\mathcal{S}, M) \otimes_A B)) \subseteq \bigcup_{\mathfrak{b}\in\Phi_{n+1}B} V(\mathfrak{b}).$$

It follows from 2.5, 4.1 and 4.2 that

$$\begin{aligned}
\text{Supp}_B(\text{Coker } (h^{n-2} \otimes Id_B)) &= \text{Supp}_B(\text{Coker } h^{n-2} \otimes_A B) \\
&\subseteq \{\mathfrak{q} \in \text{Spec}(\mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \text{Supp}_{\mathfrak{A}}(\text{Coker}\mathfrak{h}^{\mathfrak{n}-2})\} \\
&\subseteq \{\mathfrak{q} \in \text{Spec}(\mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \bigcup_{\mathfrak{a}\in_{\mathfrak{n}}} \mathfrak{V}(\mathfrak{a})\} \subseteq \bigcup_{\mathfrak{b}\in_{\mathfrak{n}}\mathfrak{B}} \mathfrak{V}(\mathfrak{b}),
\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{Supp}_B(H^{n-1}(\mathcal{H}(\mathcal{S},M)\otimes_A B)) &\subseteq \mathrm{Supp}_B(H^{n-1}(\mathcal{H}(\mathcal{S},M)\otimes_A B)) \cup \mathrm{Supp}_B(Tor_1^A(\text{Coker } h^{n-1},B)) \\
&\subseteq \{\mathfrak{q}\in \mathrm{Spac}(\mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \mathrm{Supp}_{\mathfrak{A}}(\mathfrak{H}^{\mathfrak{n}-1}(\mathcal{H}(\mathcal{S},\mathfrak{M}))\cup \mathrm{Supp}_A(\text{Coker } h^{n-1})\} \\
&\subseteq \{\mathfrak{q}\in \mathrm{Spac}(\mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \bigcup_{\mathfrak{a}\in_{\mathfrak{n}+1}} \mathfrak{V}(\mathfrak{a})\} \\
&\subseteq \bigcup_{\mathfrak{b}\in_{\mathfrak{n}+1}\mathfrak{B}} V(\mathfrak{b})
\end{aligned}
$$

We shall treat two important cases in which the morphism of complexes Ψ of theorem 4.3 is actually an isomorphism, namely a case where $\mathcal{H}(\mathcal{S},M)$ is isomorphic (over Id_M) to a Cousin complex (we shall describe this case in 5.1 9), and the case where the ring homomorphism $f : A \longrightarrow B$ is flat.

4.4 THEOREM. Let the sitaution and notation be as in 4.3, and let f be flat. Then the unique morphism of complexes of B-modules and B-homomorphisms

$$\Psi : \mathcal{H}(\mathcal{S},M)\otimes_A B \longrightarrow \mathcal{H}(\mathcal{S}B, M\otimes_A B)$$

over $Id_{M\otimes_A B}$ of 4.3 is an isomorphism.

Proof. By Theorem 4.3 and 1.14 (i), it is enough for us to show that

$$\Gamma_{\Phi_{n+1}B}(K^n\otimes_A B) = H^1_{\Phi_{n+1}B}(K^n\otimes_A B) \text{ for all } n\in\mathbb{N}_{\not\vdash}.$$

However we can use Lemma 1.17 and 2.3(iii) to see that, for all $n\in\mathbb{N}_{\not\vdash}$,

$$D_{\Phi_{n+1}B}(K^n\otimes_A B)\cong D_{\Phi_{n+1}}(K^n)\otimes_A B\cong K^n\otimes_A B \text{ (as } B-modules).$$

Hence, by the part (4) of 1.14 (iii), the proof is complete.

§5 Generalized Hughes complexes and Cousin complexes.

Throughout this section, M will denote an A-module. We shall need some properties of Cousin complexes and we list them for the reader convenience.

5.1 DEFINITION. (i) A *filtration* of Spec (A) [33, 1.1] is a descending sequence $\mathcal{F} = (F_i)_{i\in\mathbb{N}_{\not\,\circ}}$ of subsets of Spec (A), so that

$$\mathrm{Spec}(A) \ \supseteq F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots ,$$

with the property that, for each $i \in \mathbb{N}_{\not\,\circ}$, each member of $\partial F_i = F_i \backslash F_{i+1}$ is a minimal member of F_i with respect to inclusion. We say that $\mathcal{F}$ *admits* M if Supp $(M) \subseteq F_0$.

(ii) Let $\mathcal{F}$ be a filtration of Spec (A) which admits M. the *Cousin complex* $C(\mathcal{F}, M)$ *for* M *with respect to* $\mathcal{F}$ has the form

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots ,$$

Where for each $n \in \mathbb{N}_{\not\,\circ}$,

$$M^n = \bigoplus_{\mathfrak{p}\in\partial\mathfrak{F}_\mathfrak{n}} (\mathrm{Coker}\ d^{n-2})_\mathfrak{p}.$$

The homomorphisms of the complex have the following properties: for $m \in M$ and $\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{o}$, the component of $d^{-1}(m)$ in $M_\mathfrak{p}$ is $m/1$; for $n > 0, x \in M^{n-1}$ and $\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{n}$, the component of $d^{n-1}(x)$ in (Coker $d^{n-2})_\mathfrak{p}$ is $\pi(x)/1$, where $\pi : M^{n-1} \longrightarrow$ Coker d^{n-2} is the cononical epimorphism. The fact that such a complex can be constructed is explained in [33,1.3] and relies on arguments from [23, section 2]. Such complexes are algebraic analogues of the Cousin complex studied earlier by Hartshorne in [7, chapter IV].

We shall maintoin the above notation for $C(\mathcal{F}, M)$ throughout this section.

5.2 REMARK, (See [33,1.4 (i)] and [34,1.1 (iii)]) Let $\mathcal{F} = (F_i)_{i\in\mathbb{N}_0}$ be a filtration of Spec (A) which admits M, and write the Cousin complex $C(\mathcal{F}, M)$ as

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots .$$

Then, for each $n \in \mathbb{N}_0$,

Supp $(M^n) \subseteq$ Supp (Coker $d^{n-2}) \subseteq F_n \cap$ Supp (M) and

Supp $(H^{n-1}(C(\mathcal{F}, M)) \subseteq F_{n+1} \cap$ Supp (M).

5.3 DEFINITION and LEMMA. (See [35, (2.2) and (2.3)]) Let $\mathcal{F} = (F_i)_{i\in\mathbb{N}_0}$ be a filtration of Spec (A) which admits M. For each $\mathfrak{p} \in$ Spec (A), we define the $\mathcal{F}$-height $\mathfrak{p}$, denoted by $\mathrm{ht}_{\mathcal{F}}\mathfrak{p}$, as follows.

If $\mathfrak{p} \notin \mathfrak{F}_0$, then we set $\mathrm{ht}_{\mathcal{F}}\mathfrak{p} = -1$. If $\mathfrak{p} \in \bigcap_{i\in\mathbb{N}_0} \mathfrak{F}_i$, then we set $\mathrm{ht}_{\mathcal{F}}\mathfrak{p} = \infty$. If neither of these conditions is satisfied, then the set $\{i \in \mathbb{N}_0 : \mathfrak{p} \in \mathfrak{F}_i\}$ has a greatest member, n say, and we set $\mathrm{ht}_{\mathcal{F}}\mathfrak{p} = \mathfrak{n}$.

Let $\mathfrak{p}, \mathfrak{q} \in$ Supp (M) with $\mathfrak{p} \subset \mathfrak{q}$ (the symbol '$\subset$' is reseved to denote the strict inclusion). Then $\mathrm{ht}_{\mathcal{F}}\mathfrak{p} \leq \mathrm{ht}_{\mathcal{F}}\mathfrak{q}$.

Also, for every $\mathfrak{p} \in$ Supp $(\mathfrak{M})$, we have $\mathrm{ht}_M\mathfrak{p} \leq \mathrm{ht}_{\mathcal{F}}\mathfrak{p}$. (Here, for $\mathfrak{p} \in$ Supp (M), the notation $\mathrm{ht}_M\mathfrak{p}$ means the M- height of $\mathfrak{p}$, that is the dimension of the $A_{\mathfrak{p}}$-modules $M_{\mathfrak{p}}$; the dimension of a non-zero module is the supremum of lengths of chains of prime ideals in its support if this supremum exists and ∞ otherwise)

5.4 NOTATION, DEFINITION AND REMARKS. The Cousin complex $C(M)$ for M is described in [23, section 2]: it is actually the Cousin complex $C(\mathcal{H}(M), M)$ for M with respect to the M-height filtration $\mathcal{H}(M) = (H_i)_{i\in\mathbb{N}_0}$ of

Spec (A), where

$$H_i = \{\mathfrak{p} \in \text{ Supp } (\mathfrak{M}) : \text{ht}_{\mathfrak{M}}\mathfrak{p} \geq \mathfrak{i}\} \text{ for all } \mathfrak{i} \in \mathbb{N}_{\not\prec}.$$

We shall call $C(\mathcal{H}(M), M)$ *the basic Cousin complex for* M.

Cohen-Macaulay modules can be characterized in terms of the Cousin complex: a non-zero finitely generated A-module G is Cohen-Macoulay if and only if $C(\mathcal{H}(G), G)$ is exact [25,2.4]. Also, A is a Gorenstein ring if and only if $C(\mathcal{H}(A), A)$ provides an injective resolution (resp. minimal injective resolution) for A [23, 5.4].

5.5 THEOREM. Let $\mathcal{F} = (F_i)_{i\in\mathbb{N}_{\not\prec}}$ and $\mathcal{G} = (G_i)_{i\in\mathbb{N}_{\not\prec}}$ be filtrations of Spec(A) which admits M. Suppose that $F_i \subseteq G_i$ for all $i \in \mathbb{N}_{\not\prec}$; so that, on use of 5.3, $\text{ht}_M\mathfrak{p} \leq \text{ ht}_{\mathcal{F}}\mathfrak{p} \leq \text{ ht}_{\mathcal{G}}\mathfrak{p}$ for all $\mathfrak{p} \in$ Supp $(\mathfrak{M})$. Write the Cousin complex $C(\mathcal{F}, M)$ as

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots .$$

Let $\mathcal{H}(M) = (H_i)_{i\in\mathbb{N}_{\not\prec}}$ be the M-height filtration of Spec (A) given by $H_i = \{\mathfrak{p} \in$ Supp $(\mathfrak{M}) : \text{ ht}_{\mathfrak{M}}\mathfrak{p} \geq \mathfrak{i}\}$ for all $i \in \mathbb{N}_{\not\prec}$.

(i) (See [35, 2.9]) for each $n \in \mathbb{N}_{\not\prec}$,

$$M^n = \bigoplus_{\mathfrak{p}\in\partial\mathfrak{F}_\mathfrak{n}\cap\partial\mathfrak{H}_\mathfrak{n}} (\text{Coker } d^{n-2})_{\mathfrak{p}}.$$

(ii) (See [35, 2.10]) for each $n \in \mathbb{N}_{\not\prec}$, let

$$S^n = \bigoplus_{\mathfrak{p}\in\partial\mathfrak{F}_\mathfrak{n}\cap\partial\mathfrak{H}_\mathfrak{n}\setminus\partial\mathfrak{G}_\mathfrak{n}} (\text{Coker } d^{n-2})_{\mathfrak{p}}.$$

Then $d^n(S^n) \subseteq S^{n+1}$ for all $n \in \mathbb{N}_{\not\prec}$, and so, if u^n denotes the restriction of d^n to S^n (for each $n \geq -2$) (interpert $S^{-2} = S^{-1} = 0$), then

$$0 \xrightarrow{u^{-2}} 0 \xrightarrow{u^{-1}} S^0 \xrightarrow{u^0} S^1 \longrightarrow \cdots \longrightarrow S^n \xrightarrow{u^n} S^{n+1} \longrightarrow \cdots .$$

is a subcomplex of $C(\mathcal{F}, M)$. We denote this subcomplex by $S(\mathcal{F}, \mathcal{G}, M)$.

The quotient complex $C(\mathcal{F}, M)/S(\mathcal{F}, \mathcal{G}, M)$ is isomorphic to the Cousin complex $C(\mathcal{G}, M)$.

(iii) (See [35, 2.11]) There is a morphism of complexes

$$\Psi = (\psi^i)_{i\geq -2} : C(\mathcal{F}, M) \longrightarrow C(\mathcal{G}, M)$$

which has the following properties:

(a) $\psi^{-1} : M \longrightarrow M$ is the identity mapping;

(b) for each $n \in \mathbb{N}_{\not{\circ}}$, the map ψ^n is an epimorphism whose kernel is a direct summand of M^n;

(c) for each $n \in \mathbb{N}_{\not{\circ}}$ and each $\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{n} \cap \partial\mathfrak{H}_\mathfrak{n} \backslash \partial\mathfrak{G}_\mathfrak{n}$, the restriction of ψ^n to the direct summand $(\mathrm{Coker}\ d^{n-2})_\mathfrak{p}$ of M^n is zero.

(iv) (See [35, 3.6]) If $C(\mathcal{G}, M)$ is exact, then

(a) $C(\mathcal{F}, M)$ is exact;

(b) the morphism of complexes $\Psi = (\psi^i)_{i\geq -2}$ of (iii) is an isomorphism.

(c) both the complexes $C(\mathcal{G}, M)$ and $C(\mathcal{F}, M)$ are isomorphic to the basic Cousin complex $C(\mathcal{H}(M), M)$; so that all three complexes are exact.

5.6 REMARK. (i) Let U be a subset of Spec (A), and let

$$\Phi = \Phi(U,\ \text{Spec }(A)) = \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of A such that } V(\mathfrak{a}) \subseteq \mathfrak{U}\}.$$

Then $\bigcup_{\mathfrak{a}\in} V(\mathfrak{a}) \subseteq \mathfrak{U}$. Moreover, let $V(\mathfrak{p}) \subseteq \mathfrak{U}$ for every $\mathfrak{p} \in \mathfrak{U}$. Then $\mathfrak{p} \in$ for every $\mathfrak{p} \in \mathfrak{U}$. Hence $\bigcup_{\mathfrak{a}\in} V(\mathfrak{a}) = \mathfrak{U}$.

(ii) Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of A. Define

$$\Phi_0 = \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{a}) \subseteq \text{ Supp } (\mathfrak{M})\}$$

a system of ideals of A. Set $F(\Phi_n) := \bigcup_{\mathfrak{a}\in_{\mathfrak{n}}} V(\mathfrak{a})$ and $F_n := \bigcap_{i=0}^{n} F(\Phi_i)$ for all $n \in \mathbb{N}_{\nvdash}$. Then

$$\text{Supp }(M) = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots .$$

Let

$$\Phi'_n = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{F}_\mathfrak{n}\}$$

for each $n \in \mathbb{N}_{\nvdash}$. It is easy to see that whenever, for each $n \in \mathbb{N}_{\nvdash}$, $\mathfrak{p}, \mathfrak{q} \in \text{Spec }(A)$ are such that $\mathfrak{p} \in \mathfrak{F}_\mathfrak{n}$ and $\mathfrak{p}\ \mathfrak{subseteqq}$, then $\mathfrak{q} \in \mathfrak{F}_\mathfrak{n}$. Hence, by (i), $\bigcup_{\mathfrak{a}\in'_{\mathfrak{n}}} V(\mathfrak{a}) = \mathfrak{F}_\mathfrak{n}$ for all $n \in \mathbb{N}_{\nvdash}$.

(iii) Let $\mathcal{F}' = (F'_i)_{i\in\mathbb{N}_{\nvdash}}$ be a filtration of Spec (A) which admits M. Let

$$\Phi'_n = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{F}'_\mathfrak{n}\} \text{ and}$$

$$\Phi''_n = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{F}'_\mathfrak{n} \cap \text{ Supp}(\mathfrak{M})\}$$

for each $n \in \mathbb{N}_{\nvdash}$. It is easy to see that $\Phi''_n \subseteq \Phi'_n$, for each $n \in \mathbb{N}_{\nvdash}$. By 5.3, for each $n \in \mathbb{N}_{\nvdash}, \mathfrak{q} \in \mathfrak{F}'_\mathfrak{n} \cap$ Supp $(\mathfrak{M})$, whenever $\mathfrak{p} \in \mathfrak{F}'_\mathfrak{n} \cap$ Supp (M) and $\mathfrak{q} \in$ Spec (A) are such that $\mathfrak{p} \subseteq \mathfrak{q}$. Hence $F'_n \cap$ Supp $(M) \subseteq \Phi''_n$, for each $n \in \mathbb{N}_{\nvdash}$. Thus, by (i),

$$\text{Supp }(M) \cap F'_n = \bigcup_{\mathfrak{a}\in''_{\mathfrak{n}}} V(\mathfrak{a}) \subseteq \bigcup_{\mathfrak{a}\in'_{\mathfrak{n}}} \mathfrak{V}(\mathfrak{a}) \text{ and } \bigcup_{\mathfrak{a}\in'_{\mathfrak{n}}} \mathfrak{V}(\mathfrak{a}) \subseteq \mathfrak{F}'_\mathfrak{n},$$

for each $n \in \mathbb{N}_{\nvdash}$. Hence $F'_n \cap$ Supp $(M) = \bigcup_{\mathfrak{a}\in'_{\mathfrak{n}}} V(\mathfrak{a}) \cap$ Supp $(\mathfrak{M})$, for each $n \in \mathbb{N}_{\nvdash}$.

5.7 REMARK. The method of proof of theorem 2.6 has some similarities with that of [34, theorem 2.3], which shows that every Cousin complex is a generalized Hughes complex. In fact, we can deduce that result very quickly from 2.6, as follows.

Let $\mathcal{F} = (F_i)_{i\in\mathbb{N}_{\nvdash}}$ be a filtration of Spec (A) which admits M, and write the Cousin complex $C(\mathcal{F}, M)$ as

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots .$$

Let W be a subset of Spec (A) such taht $W \supseteq$ Supp (M). For each $n \in \mathbb{N}_{\nvdash}$ the set

$$\Phi_n(\mathcal{F}, W) = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \cap \mathfrak{W} \subseteq \mathfrak{F}_{\mathfrak{n}}\}$$

is a system of ideals of A. Let the family of systems of ideals $(\Phi_n(\mathcal{F}, W))_{n\in\mathbb{N}}$ be denoted by $\mathcal{S}(\mathcal{F}, W)$.

It follows from 5.2 and 1.8 that, for all $n \in \mathbb{N}_{\nvdash}$, Coker d^{n-2} is $\Phi_n(\mathcal{F}, W)$-torsion and $H^{n-1}(C(\mathcal{F}, M))$ is $\Phi_{n+1}(\mathcal{F}, W)$-torsion (or it follows from 5.2 and 5.6 (iii) and the fact that $\Phi_n(\mathcal{F},$ Spec $(A)) \subseteq \Phi_n(\mathcal{F}, W)$ for all $n \in \mathbb{N}_{\nvdash}$ that

$$\text{Supp (Coker } d^{n-2}) \subseteq \bigcup_{\mathfrak{a}\in_{\mathfrak{n}}(\mathcal{F},\text{Spec}(\mathfrak{A}))} V(\mathfrak{a}) \subseteq \bigcup_{\mathfrak{a}\in_{\mathfrak{n}}(\mathcal{F},\mathfrak{W})} \mathfrak{V}(\mathfrak{a}) \text{ and}$$

$$\text{Supp}(H^{n-1}(C(\mathcal{F}, M))) \subseteq \bigcup_{\mathfrak{a}\in_{\mathfrak{n}}(\mathcal{F},\text{Spec}(\mathfrak{A}))} V(\mathfrak{a}) \subseteq \bigcup_{\mathfrak{a}\in_{\mathfrak{n}}(\mathcal{F},\mathfrak{W})} \mathfrak{V}(\mathfrak{a})$$

for all $n \in \mathbb{N}_{\nvdash}$). Also, by 1.10(i),

$$\Gamma_{\Phi_{n+1}(\mathcal{F},W)}(M^n) = H^1_{\Phi_{n+1}(\mathcal{F},W)}(M^n) = 0 \text{ for all } n \in \mathbb{N}_{\nvdash}.$$

It therefore follows from 2.6, 2.7 and 1.14 (i) that there is a unique isomorphism of complexes $C(\mathcal{F}, M) \xrightarrow{\cong} \mathcal{H}(\mathcal{S}(\mathcal{F}, W), M)$ over Id_M. Note that the uniqueness ensures that this isomorphism must be the same as the inverse of the one constructed in the proof of theorem 2.3 of [34].

Also, there is a unique isomorphism of complexes

$$\Pi : C(\mathcal{F}, M) \longrightarrow \mathcal{H}(\mathcal{S}(\mathcal{F}, \text{ Spec } (A)), M)$$

over Id_M, where $\mathcal{S}(\mathcal{F}$, Spec $(A))$ is the family $(\Phi_n)_{n\in\mathbb{N}}$ of systems of ideals of A for which

$$\Phi_n := \Phi_n(\mathcal{F}, \text{ Spec } (A)) = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{F}_{\mathfrak{n}}\}$$

for all $n \in \mathbb{N}$.

5.8 REMARK. We can deduce [34,2.2] from 5.2 and 1.10. Note that, the proof of 1.10 in this way is much easier than the proof presented in [34].

We show that we can deduce the theorem [21,3.3] from 2.6 and 2.7 and we need the following results.

5.9 PROPOSITION. (See [23, (2.3) and (2.5)]) Suppose $U \subseteq Y$ are subsets of Spec (A) such that each element of $Y - U$ is a minimal (with respect to inclusion) element in Y. Let G be an A-module such that Supp $_A(G) \subseteq Y$. Then there is an A-homomorphism

$$\alpha(G) : G \longrightarrow \bigoplus_{\mathfrak{p} \in \mathfrak{Y} - \mathfrak{U}} G_{\mathfrak{p}}$$

for which, for $g \in G$ and $\mathfrak{p} \in \mathfrak{Y} - \mathfrak{U}$, the component of $\alpha(G)(g)$in the direct summand $G_{\mathfrak{p}}$ is $g/1$; furthermore,

$$\text{Supp (Ker } \alpha(G)) \subseteq U \text{ and Supp (Coker } \alpha(G)) \subseteq U.$$

5.10 LEMMA. Let $f : A \longrightarrow B$ be a homomorphism of commutative noetherian rings. Let $U \subseteq Y$ be subsets of Spec (A) such that each element of $Y - U$ be minimal (with respect to inclusion) in Y, and let N be a B-module such that Supp $_A(N) \subseteq Y$. Let

$$\alpha(N) : N \longrightarrow \bigoplus_{\mathfrak{p} \in \mathfrak{Y} - \mathfrak{U}} N_{\mathfrak{p}}$$

be the natural A-homomorphism such that, for $x \in N$ and $\mathfrak{p} \in \mathfrak{Y} - \mathfrak{U}$, the component of $\alpha(N)(x)$ in the direct summand $N_{\mathfrak{p}}$ is $x/1$ (it follows from 5.9 that there is such an A-homomorphism). Let Φ be the following system of ideals of A:

$$\Phi(U, \text{ Spec } (A)) = \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{a}) \subseteq \mathfrak{U}\}.$$

Then

(i) There is an A-isomorphism

$$\varphi : \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}} \longrightarrow D_{\Phi}(N)$$

such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\alpha(N)} & \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}} \\ & \searrow^{\eta_{\Phi}(N)} & \downarrow \varphi \\ & & D_{\Phi}(N) \end{array}$$

commutes.

(ii) There is a B-isomorphism

$$\psi : \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}} \longrightarrow D_{\Phi B}(N)$$

such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\alpha(N)} & \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}} \\ & \searrow^{\eta_{\Phi_B}(N)} & \downarrow \psi \\ & & D_{\Phi B}(N) \end{array}$$

commutes.

Proof. (i) Application of the morphism of functors $\eta_{\Phi} : Id \longrightarrow D_{\Phi}$ to the modules and A-homomorphism

$$\alpha(N) : N \longrightarrow \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}}$$

yields a commutative diagram

$$\begin{array}{ccc}
N & \xrightarrow{\alpha(N)} & \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}} \\
\Big\downarrow{\scriptstyle \eta_\Phi(N)} & & \Big\downarrow{\scriptstyle \eta_\Phi(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}})} \\
D_\Phi(N) & \xrightarrow[D_\Phi(\alpha(N))]{} & D_\Phi(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}}).
\end{array}$$

It is enough for us to show that $\eta_\Phi(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}})$ and $D_\Phi(\alpha(N))$ are A-isomorphisms.

By 1.10 (i), $H^i_\Phi(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}}) = 0$ for $i = 0, 1$, and so it clear from 1.14 (i) that $\eta_\Phi(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}})$ is an A-isomorphism.

By 1.8 and 5.9, Ker $\alpha(N)$ and Coker $\alpha(N)$ are Φ-torsion. Hence by 1.15, $D_\Phi(\alpha(N))$ is an A-isomorphism.

(ii) It is easy to check that $\alpha(N)$ is a B-homomorphism. Appllication of the functors $\eta_{\Phi B} : Id \longrightarrow D_{\Phi B}$ to the modules and B-homomorphism

$$\alpha(N) : N \longrightarrow \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}}$$

yields a commutative diagram

$$\begin{array}{ccc}
N & \xrightarrow{\alpha(N)} & \bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}} \\
\Big\downarrow{\scriptstyle \eta_{\Phi B}(N)} & & \Big\downarrow{\scriptstyle \eta_{\Phi B}(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}})} \\
D_{\Phi B}(N) & \xrightarrow[D_{\Phi B}(\alpha(N))]{} & D_{\Phi B}(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}}).
\end{array}$$

It is enough for us to show that $\eta_{\Phi B}(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}})$ and $D_{\Phi B}(\alpha(N))$ are B-isomorphisms.

By 1.10 (ii), $H^i_{\Phi B}(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}}) = 0$ for $i = 0, 1$, and so it is clear from 1.14 (i) that $\eta_{\Phi B}(\bigoplus_{\mathfrak{p}\in\mathfrak{Y}-\mathfrak{U}} N_{\mathfrak{p}})$ is a B-isomorphism.

It is immediate that Ker $\alpha(N)$ and Coker $\alpha(N)$ are ΦB-torsion since, by (i), Ker $\alpha(N)$ and Coker $\alpha(N)$ are Φ-torsion. Hence, by 1.15, $D_{\Phi B}(\alpha(N))$ is a B-isomorphism.

5.11 THEOREM. Let $\mathcal{F} = (F_i)_{i\in\mathbb{N}_{\not\vdash}}$ be a filtration of spec (A) which admits M, and let $C(\mathcal{F}, M)$ be the Cousin complex for M with respect to $\mathcal{F}$. Let

$$0 \xrightarrow{e^{-2}} M \xrightarrow{e^{-1}} C^0 \xrightarrow{e^0} C^1 \longrightarrow \cdots \longrightarrow C^n \xrightarrow{e^n} C^{n+1} \longrightarrow \cdots .$$

be a complex $C^\bullet$ of A-modules and A-homomorphisms such that , for each $n \in \mathbb{N}_{\not\vdash}$, Supp (Coker e^{n-2}) is subset of F_n and Supp $(H^{n-1}(C^\bullet))$ is subset of F_{n+1}. Then there is a unique morphism of complexes

$$\Psi = (\psi^i)_{i\geq -2} : C^\bullet \longrightarrow C(\mathcal{F}, M)$$

over Id_M.

Moreover the morphism Ψ is an isomorphism of complexes if and only if, for each $n \in \mathbb{N}_{\not\vdash}$, Supp $(C^n) \subseteq F_n$ and the A-homomorphism $\alpha(C^n) : C^n \longrightarrow \bigoplus_{\mathfrak{p}\in\partial\mathfrak{F}_{\mathfrak{n}}} (C^n)_{\mathfrak{p}}$ such that, for $x \in C^n$ and $\mathfrak{p} \in \partial\mathfrak{F}_{\mathfrak{n}}$, the component of $\alpha(C^n)(x)$ in the direct summand $(C^n)_{\mathfrak{p}}$ is $x/1$ (it follows from 5.9 that there is such an A-homomorphism), is an isomorphism.

Proof. Let Φ_n be the system of ideals of A

$$\Phi_n(\mathcal{F}, \text{ Spec } (A)) = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{F}_{\mathfrak{n}}\},$$

for each $n \in \mathbb{N}_0$. Let the family of systems of ideals $(\Phi_n)_{n\in\mathbb{N}}$ be denoted by $\mathcal{S}$. It follows from 1.8 that for each $n \in \mathbb{N}_0$, Coker e^{n-2} is Φ_n-torsion and $H^{n-1}(C^\bullet)$ is Φ_{n+1}-torsion, Hence, by 2.6 and 2.7, there is a unique morphism of complexes from $C^\bullet$ to $\mathcal{H}(\mathcal{S}, M)$ over Id_M. Hence, in view of 5.7, there is unique morphism of complexes

$$\Psi = (\psi^i)_{i\geq -2} : C^\bullet \longrightarrow C(\mathcal{F}, M)$$

over Id_M.

Let Ψ be an isomorphism of complexes. Then, by 5.2, Supp $(C^n) \subseteq F_n$, for each $n \in \mathbb{N}_0$. Also, by 2.8 and 5.7, $\eta_{\Phi_{n+1}}(C^n) : C^n \longrightarrow D_{\Phi_{n+1}}(C^n)$ is an isomorphism, for each $n \in \mathbb{N}_0$. It therefore follows from 5.10 (i) that $\alpha(C^n) : C^n \longrightarrow \bigoplus_{\mathfrak{p}\in\partial\mathfrak{F}_n} (C^n)_{\mathfrak{p}}$ is an isomorphism, for each $n \in \mathbb{N}_0$.

Let Supp $(C^n) \subseteq F_n$ and $\alpha(C^n) : C^n \longrightarrow \bigoplus_{\mathfrak{p}\in\partial\mathfrak{F}_n} (C^n)_{\mathfrak{p}} = 0$ be an isomorphism, for each $n \in \mathbb{N}_0$. Then, by 5.10 (i), $\eta_{\Phi_{n+1}}(C^n) : C^n \longrightarrow D_{\Phi_{n+1}}(C^n)$ is an isomorphism, for each $n \in \mathbb{N}_0$. It therefore follows from 2.8 that Ψ is an isomorphism of complexes.

It is easy to deduce the following theorem from 5.11.

5.12 DEFINITION AND THEOREM. (See [21, (3.1) and (3.3)]) Let $\mathcal{F} = (F_i)_{i\in\mathbb{N}_0}$ be a filtration of Spec (A) that admits M. A complex $X^\bullet = (X^i)_{i\geq -2}$ of A-modules and A-homomorphis is said to be of *Cousin type for M with respect to $\mathcal{F}$* if it has the form

$$0 \xrightarrow{d_{X^\bullet}^{-2}} M \xrightarrow{d_{X^\bullet}^{-1}} X^0 \longrightarrow \cdots \longrightarrow X^i \xrightarrow{d_{X^\bullet}^{i}} X^{i+1} \longrightarrow \cdots .$$

and satisfies the following for each $n \in \mathbb{N}_0$:

(i) Supp $(X^n) \subseteq F_n$;

(ii) Supp (Coker $d_{X^\bullet}^{n-1}$) $\subseteq F_n$;

(iii) Supp $(H^{n-1}(X^\bullet)) \subseteq F_n$;

(iv) the natural A-homorphism $\alpha(X^n) : X^n \longrightarrow \bigoplus_{\mathfrak{p}\in\partial\mathfrak{F}_n} (X^n)_\mathfrak{p}$ such that, for $x \in X^n$ and $\mathfrak{p} \in \partial F_n$, the component of $\alpha(X^n)(x)$ in the summand $(X^n)_\mathfrak{p}$ is $x/1$ (it follows from condition (i) and 5.9 that there is such an A-homomorphism), is an isomorphism.

Let $X^\bullet = (X^i)_{i\geq -2}$ and $Y^\bullet = (Y^i)_{i\geq -2}$ be complexes of Cousin type for M with respect to $\mathcal{F}$. Then there is exactly one isomorphism of complexes

$$\Psi = (\psi^i)_{i\geq -2} : X^\bullet \longrightarrow Y^\bullet$$

over Id_M.

One of the main results of [34] is theorem 2.3, which shows that every Cousin complex is a generalized Hughes complex. It is not the case that every generalized Hughes complex is a Cousin complex, as the following example shows.

5.13 EXAMPLE. Suppose that the ring A is not Cohen-Macaulay.Consider Hughes' grade-theoretic analogue of the Cousin complex [12] which motivated the definition of generalized Hughes complex in [36]. This is ([36, 1.4]) just $\mathcal{H}(\mathcal{G}, A)$, where $\mathcal{G}$ denotes the family of systems of ideals of A given by $\mathcal{G} = (\mathcal{G}(n))_{n\in\mathbb{N}}$, where, for each $n \in \mathbb{N}$, the system $\mathcal{G}(n)$ is the set of all ideals of A of grade at last n (we adopt the convention where by grade $A = \infty$).

By [12], Hughes' complex $\mathcal{H}(\mathcal{G}, A)$ is exact. If it were isomorphic (over Id_A) to a Cousin complex for A with respect to some filtration of Spec (A), then, by 5.5 (iii), the basic Cousin complex $C(\mathcal{H}(A), A)$ for A would be exact, so that A would be Cohen-Macaulay by 5.4. Thus, since A is not Cohen-Macaulay, $\mathcal{H}(\mathcal{G}, A)$ is an example of a generalized Hughes complex which is not isomorphic (over Id_A) to a Cousin complex for A.

The next few remarks are concerned with the following question: given a generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$ for M with respect to a family of systems of ideals $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ of A, can we identify a descending sequence $\mathcal{F} = (F_i)_{i\in\mathbb{N}_{\not\vdash}}$ of subsets of Spec (A) which, if it were a filtration of Spec (A), would of necessity be such that Cousin complex $C(\mathcal{F}, M)$ is isomorphic (over Id_M) to $\mathcal{H}(\mathcal{S}, M)$?

5.14 LEMMA. Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of A. Define

$$\Phi_0 := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{a}) \subseteq \text{ Supp } (\mathfrak{M})\},$$

a system of ideals of A.

For each $n \in \mathbb{N}_{\not\vdash}$, let

$$\Theta_n = \left\{ \sum_{i=0}^{n} \mathfrak{a}_\mathfrak{i} : \mathfrak{a}_\mathfrak{i} \in {}_\mathfrak{i} \right\}.$$

Then $\mathcal{I} = (\Theta_i)_{i\in\mathbb{N}}$ is a family of systems of ideals of A and there is a unique isomorphism of complexes $\Psi : \mathcal{H}(\mathcal{I}, M) \longrightarrow \mathcal{H}(\mathcal{S}, M)$ over Id_M.

Proof. It is easy to check that each $\Theta_n (n \in \mathbb{N})$ is a system of ideals of A.

By 3.1 (i), there is a unique morphism of complexes $\Psi : \mathcal{H}(\mathcal{I}, M) \longrightarrow \mathcal{H}(\mathcal{S}, M)$ over Id_M. Write the generalized Haghes complex $\mathcal{H}(\mathcal{I}, M)$ as

$$0 \xrightarrow{w^{-2}} M \xrightarrow{w^{-1}} L^0 \xrightarrow{w^0} L^1 \longrightarrow \cdots \longrightarrow L^i \xrightarrow{w^i} L^{i+1} \longrightarrow \cdots .$$

By theorem 2.6 and 1.14 (i), it is enough for us to show that

$$\Gamma_{\Phi_{n+1}}(L^n) = H^1_{\Phi_{n+1}}(L^n) = 0 \text{ for all } n \in \mathbb{N}_{\not\vdash}.$$

By 2.4(iii) for every $n \in \mathbb{N}_{\not\vdash}, \mathbb{L}^{\ltimes}$ is Θ_n-torsion. Hence L^n is Φ_i-torsion for every $i = 0, \cdots, n$. It then follows from 1.21 and 2.3(iii) that

$$D_{\Phi_{n+1}}(L^n) \cong D_{\Theta_{n+1}}(L^n) \cong L^n \text{ for all } n \in \mathbb{N}_{\not\vdash}.$$

Therefore, by the part (4) of 1.14 (iii), the proof is complete.

5.15 PROPOSITION. Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of A. Define

$$\Phi_0 := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{a}) \subseteq \text{ Supp (M)}\},$$

a system of ideals of A.

Set $F(\Phi_n) := \bigcup_{\mathfrak{a}\in_{\mathfrak{n}}} V(\mathfrak{a})$, $F_n := \bigcap_{i=0}^{n} F(\Phi_i)$ and

$$\Phi'_n := \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{F}_\mathfrak{n}\},$$

a system of ideals of A, for all $n \in \mathbb{N}_{\nvdash}$. Let $\mathcal{S}' = (\Phi'_i)_{i\in\mathbb{N}}$. Then

(i) Supp $(M) = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots$ and $\Phi'_n \supseteq \Phi'_{n+1}$ for all $n \in \mathbb{N}$;

(ii) $F_n = \bigcup_{\mathfrak{b}\in'_\mathfrak{n}} V(\mathfrak{b})$ for all $n \in \mathbb{N}_{\nvdash}$;

(iii) there is a (unique) isomorphism of complexes $\Psi : \mathcal{H}(\mathcal{S}', M) \longrightarrow \mathcal{H}(\mathcal{S}, M)$ over Id_M;

(iv) if $\mathcal{F} = (F_i)_{i\in\mathbb{N}_{\nvdash}}$ is a filtrution of Spec (A), then there is a unique isomorphism of complexes $\mathcal{H}(\mathcal{S}, M) \longrightarrow C(\mathcal{F}, M)$ over Id_M.

Proof. (i) This is clear.

(ii) This is follows from 5.6 (ii).

(iii) For each $n \in \mathbb{N}_{\nvdash}$, let

$$\Theta_n := \left\{ \sum_{i=0}^{n} \mathfrak{a}_i : \mathfrak{a}_\mathfrak{i} \in {}_\mathfrak{i} \right\}.$$

Then, by 3.1 (ii) and 5.14, it is enough for us to show that the system Θ_n is a cofinal subset Φ'_n, for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. It is clear that $\Theta_n \subseteq \Phi'_n$. Now, let $\mathfrak{b} \in {}'_{\mathfrak{n}}$ and $i \in \mathbb{N}_{\nvdash}$ such that $0 \leq i \leq n$. Then $V(\mathfrak{b}) \subseteq F(\Phi_i)$. We can assume that $\mathfrak{b}$ is proper. Let $\mathfrak{p}_1, \cdots \mathfrak{p}_t$ be the minimal prime ideals of $\mathfrak{b}$. Then $\mathfrak{p}_1, \cdots \mathfrak{p}_t \in \mathfrak{F}(_{\mathfrak{i}})$, and so, for each $j = 1, \cdots, t$, there exists $\mathfrak{c}_j \in {}_{\mathfrak{i}}$ such that $\mathfrak{c}_j \subseteq \mathfrak{p}_j$. There exists $h \in \mathbb{N}$ such that

$$(\mathfrak{c}_1 \cdots \mathfrak{c}_t)^{\mathfrak{h}} \subseteq (\mathfrak{p}_1, \cdots \mathfrak{p}_t)^{\mathfrak{h}} \subseteq (\mathfrak{r}_{\mathfrak{A}}(\mathfrak{b}))^{\mathfrak{h}} \subseteq \mathfrak{b};$$

Since Φ_i is a system of ideals of A, there exists $\mathfrak{a}_i \in {}_{\mathfrak{i}}$ with $\mathfrak{a}_i \subseteq (\mathfrak{c}_1 \cdots \mathfrak{c}_t)^{\mathfrak{h}}$. Hence there exists $\mathfrak{a}_0 + \mathfrak{a}_1 + \cdots + \mathfrak{a}_n \in \mathfrak{f}_n$ such that $\mathfrak{a}_0 + \cdots + \mathfrak{a}_n \subseteq \mathfrak{b}$.

(iv) This is now immediate from part (iii), the uniqueness aspect of 3.1(i), and 5.7: Remark 5.7 shows that there is a unique isomorphism of complexes $C(\mathcal{F}, M) \xrightarrow{\cong} \mathcal{H}(\mathcal{S}', M)$ over Id_M.

We have thus answered the question raised immediately before the statement of Lemma 5.14. Some additional comments about that result are appropriate.

5.16 REMARK. Let the situation and notation be as in 5.14 and 5.15. It is easy to deduce from the proof of 5.15 (iii) that, for each $n \in \mathbb{N}_{\nvdash}$,

$$\Phi'_n := \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A and } \mathfrak{b} \supseteq \mathfrak{a} \text{ for some } \mathfrak{a} \in \mathfrak{f}_n\}.$$

5.17 REMARK. Let the situation and notation be as in proposition 5.15. Then 5.15 shows that, for study of the generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$, we can, if we wish, replace $\mathcal{S}$ by $\mathcal{S}'$ (without effecting $\mathcal{H}(\mathcal{S}, M)$ up to isomorphic over Id_M): the family $\mathcal{S}' = (\Phi'_i)_{i\in\mathbb{N}}$ is such that, for each $i \in \mathbb{N}$,

(a) for each $\mathfrak{a} \in {}'_{\mathfrak{i}}$, every ideal $\mathfrak{b}$ of A with $\mathfrak{b} \supseteq \mathfrak{a}$ also belongs to Φ'_i; and

(b) $\Phi'_i \supseteq \Phi'_{i+1}$ and $V(\mathfrak{a}) \subseteq$ Supp (M) for all $\mathfrak{a} \in {}'_{\mathfrak{i}}$.

5.18 REMARK. Let the situation and notation be as in 4.3, and let $F(\Phi_n) := \bigcup_{\mathfrak{a} \in {}_{\mathfrak{n}}} V(\mathfrak{a})$ and $F_n = \bigcap_{i=1}^{n} F(\Phi_i)$ for all $n \in \mathbb{N}$. Set $G_0 := \mathrm{Supp}_B(M \otimes_A B)$, and, for all

$n \in \mathbb{N}$,

$$G_n = \{\mathfrak{q} \in \text{ Supp}_{\mathfrak{B}}(\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \mathfrak{F}_{\mathfrak{n}}\},$$

and

$$\Phi'_n = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of B such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{G}_{\mathfrak{n}}\},$$

a system of ideals of B. Let $\mathcal{S}' = (\Phi'_i)_{i \in \mathbb{N}}$. Then it is easy to see that if $G'_0 :=$ $\text{Supp}_B(M \otimes_A B)$ and $G'_n := \bigcup_{b \in \Phi_n B} V(\mathfrak{b})$ for all $n \in \mathbb{N}$, then $G_n = \bigcap_{i=0}^{n} G'_i$ for all $n \in \mathbb{N}_{\not\vdash}$. Hence, by 4.3 and 5.15, there is a unique morphism of complexes of B-modules and B-homomorphisms from $\mathcal{H}(\mathcal{S}, M) \otimes_A B$ to $\mathcal{H}(\mathcal{S}', M \otimes_A B)$ over $Id_{M \otimes_A B}$. Furthermore, this morphism of complexes is an isomorphism of complexes if and only if the morphism of complexes Ψ of theorem 4.3 is an isomorphism (or equivalently,

$$\eta_{\Phi_{n+1}B} : K^n \otimes_A B \longrightarrow D_{\Phi_{n+1}B}(K^n \otimes_A B)$$

is an isomorphism for every $n \in \mathbb{N}_{\not\vdash}$.)

Note that, if $\mathcal{G} = (G_i)_{i \in \mathbb{N}_{\not\vdash}}$ is a filtration of Spec (B) which admits $M \otimes_A B$, then, by 5.7, there is a unique morphism of complexes of B-modules and B-homomorphisms from $\mathcal{H}(\mathcal{S}, M) \otimes_A B$ to the Cousin complex $C(\mathcal{G}, M \otimes_A B)$ of $M \otimes_A B$ with respect $\mathcal{G}$ over $Id_{M \otimes_A B}$. Furthermore, this morphism of complexes is an isomorphism of complexes if and only if the morphism of complexes Ψ of Theorem 4.3 is an isomorphism (or equivalently,

$$\eta_{\Phi_{n+1}B} : K^n \otimes_A B \longrightarrow D_{\Phi_{n+1}B}(K^n \otimes_A B)$$

is an isomorphism for every $n \in \mathbb{N}_{\not\vdash}$.)

Next, we shall treat an important case in which the morphism of complexes Ψ of 4.3 is actually an isomorphism; namely a case where $\mathcal{H}(\mathcal{S}, M)$ is isomorphic (over Id_M) to a Cousion complex as in 5.7.

5.19 THEOREM. Let $\mathcal{F}' = (F'_i)_{i\in\mathbb{N}_{\not 0}}$ be a filtration of Spec (A) which admits M. For each $n \in \mathbb{N}$, let

$$\Phi_n = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{F}'_\mathfrak{n}\}.$$

It follows from 5.7 that $\mathcal{S} := (\Phi_n)_{n\in\mathbb{N}}$ is a family of system of ideals of A and there is a unique isomorphism of complexes (over Id_M)

$$C(\mathcal{F}', M) \xrightarrow{\cong} \mathcal{H}(\mathcal{S}, M).$$

Let $f : A \longrightarrow B$ be a homomorphism of commutative Noetherian rings, and let $\mathcal{S}B := (\Phi_i B)_{i\in\mathbb{N}}$.

(i) There is a unique isomorphism of complexes of B-modules and B-homomorphisms

$$C(\mathcal{F}', M) \otimes_A B \xrightarrow{\cong} \mathcal{H}(\mathcal{S}B, M \otimes_A B)$$

over $Id_{M\otimes_A B}$.

(ii) For each $n \in \mathbb{N}_{\not 0}$, set

$$G_n := \{\mathfrak{q} \in \text{ Supp } (\mathfrak{M} \otimes_\mathfrak{A} \mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \mathfrak{F}'_\mathfrak{n}\}$$

and

$$\Phi'_n := \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of B suchthat } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{G}_\mathfrak{n}\},$$

a system of ideals of B. Let $\mathcal{S}' := (\Phi'_n)_{n\in\mathbb{N}}$. Then there is a unique isomorphism of complexes of B-modules and B-homomorphisms

$$C(\mathcal{F}', M) \otimes_A B \longrightarrow \mathcal{H}(\mathcal{S}', M \otimes_A B)$$

over $Id_{M\otimes_A B}$.

(iii) Suppose, in addition, that whenever $\mathfrak{q}, \mathfrak{q}' \in \text{Supp}_B(M \otimes_A B)$ with $\mathfrak{q} \subseteq \mathfrak{q}'$ and $f^{-1}(\mathfrak{q}) = \mathfrak{f}^{-1}(\mathfrak{q}')$, then $\mathfrak{q} = \mathfrak{q}'$ (this condition is satisfied if B were integral over its subring $f(A)$, or if f were surjective, or if $\text{ht}_A f^{-1}(\mathfrak{q}) = \text{ht}_B \mathfrak{q}$ for all $\mathfrak{q} \in$ Spec (B).) Then $\mathcal{G} := (G_i)_{i \in \mathbb{N}_{\nvdash}}$ is a filtration of Spec (B) which admits $M \otimes_A B$, and there is a unique isomorphism of complexes of B-modules and B-homomorphisms

$$C(\mathcal{F}', M) \otimes_A B \xrightarrow{\cong} C(\mathcal{G}, M \otimes_A B)$$

over $Id_{M \otimes_A B}$.

Proof. Write the Cousin complex $C(\mathcal{F}', M)$ for M with respect to $\mathcal{F}'$ as

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots.$$

(i) By 4.3 and 1.14(i), it is enough for us to show that

$$\Gamma_{\Phi_{n+1}B}(M^n \otimes_A B) = H^1_{\Phi_{n+1}B}(M^n \otimes_A B) = 0 \text{ for all } n \in \mathbb{N}_{\nvdash}. \qquad (\nVdash)$$

Let $n \in \mathbb{N}_{\nvdash}$. Now $M^n = \bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_n} (\text{Coker } d^{n-2})_{\mathfrak{p}}$. Since, for every $\mathfrak{p} \in$ Spec (A),

$$\begin{aligned} (\text{Coker } d^{n-2})_{\mathfrak{p}} \otimes_A B &\cong (\text{Coker } d^{n-2} \otimes_A A_{\mathfrak{p}}) \otimes_A B \\ &\cong (\text{Coker } d^{n-2} \otimes_A B) \otimes_A A_{\mathfrak{p}} \\ &\cong (\text{Coker } d^{n-2} \otimes_A B)_{\mathfrak{p}}, \end{aligned}$$

it follows that if $\mathfrak{p} \in \text{Supp}_A$ (Coker $d^{n-2} \otimes_A B$), then $\mathfrak{p} \in \text{Supp}_A$ (Coker d^{n-2}). Hence, by 5.2,

$$\text{Supp}_A(\text{Coker } d^{n-2} \otimes_A B) \subseteq F'_n.$$

We can now deduce, with the aid of 5.10 (ii) and the fact that tensor product commutes with direct sum,

$$\begin{aligned} (\bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_n} ((\text{Coker } d^{n-2})_{\mathfrak{p}}) \otimes_A B &\cong \bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_n} \big((\text{Coker } d^{n-2})_{\mathfrak{p}} \otimes_A B\big) \\ &\cong \bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_n} ((\text{Coker } d^{n-2} \otimes_A B)_{\mathfrak{p}}) \\ &\cong D_{\Phi_{n+1}B}(\text{Coker } d^{n-2} \otimes_A B). \end{aligned}$$

Hence, by the part (4) of 1.14 (iii), the proof is complete.

Note that, also we can deduce (1) from 1.10 (iii).

(ii) Set $F(\Phi_n) = \bigcup_{\mathfrak{a}\in_\mathfrak{n}} V(\mathfrak{a})$ and $F_n = \bigcap_{i=0}^{n} F(\Phi_i)$ for all $n \in \mathbb{N}$. By (i) and 5.18, it is enough for us to show that $G_0 = \mathrm{Supp}_B(M \otimes_A B)$ and

$$G_n = \{\mathfrak{q} \in \ \mathrm{Supp}_{\mathfrak{B}}(\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \mathfrak{F}_\mathfrak{n}\}.$$

It follows from 4.1 and the fact that $\mathrm{Supp}_A(M) \subseteq F_0$ that $G_0 = \mathrm{Supp}_B(M \otimes_A B)$. Since $\Phi_n \supseteq \Phi_{n+1}$ for every $n \in \mathbb{N}, \mathbb{F}_\ltimes = \mathbb{F}(\lneqq_\ltimes)$ for all $n \in \mathbb{N}$. Hence, by 4.1 and 5.6 (iii),

$$\begin{aligned} G_n &= \{\mathfrak{q} \in \ \mathrm{Supp}_{\mathfrak{B}}(\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \mathfrak{F}'_\mathfrak{n} \cap \ \mathrm{Supp}\ (\mathfrak{M})\} \\ &= \{\mathfrak{q} \in \ \mathrm{Supp}_{\mathfrak{B}}(\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \mathfrak{F}(_\mathfrak{n}) \cap \ \mathrm{Supp}\ (\mathfrak{M})\} \\ &= \{\mathfrak{q} \in \ \mathrm{Supp}_{\mathfrak{B}}(\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \mathfrak{F}_\mathfrak{n} \cap \ \mathrm{Supp}\ (\mathfrak{M})\} \\ &= \{\mathfrak{q} \in \ \mathrm{Supp}_{\mathfrak{B}}(\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \mathfrak{F}_\mathfrak{n}\}, \end{aligned}$$

for each $n \in \mathbb{N}$.

(iii) It follows from the proof of (ii) and the hypothesis that $\mathcal{G} := (G_i)_{i\in\mathbb{N}_\nvdash}$ is a filtration of Spec (B) which admits $M \otimes_A B$. Hence the claim follows from 5.7 and (ii).

5.20 REMARK. Let the situation and notation be as in 15.19. We present another proof for 5.19 (i) and 5.19 (ii). We need to the following facts:

(1) (See [27, 2.2]) Let $n \in \mathbb{N}_\nvdash$. Then there exists a natrual B-isomorphism

$$\theta_n : \ \mathrm{Coker}\ d^{n-2} \otimes_A B \longrightarrow \ \mathrm{Coker}\ (d^{n-2} \otimes Id_B)$$

for which the diagram

$$\begin{array}{ccc} M^{n-1} \otimes B & \xrightarrow{\pi_{n-1 \otimes} Id_B} & \text{Coker } d^{n-2} \otimes_A B \\ & \searrow^{\sigma_{n-1}} & \downarrow \theta_n \\ & & \text{Coker } (d^{n-2} \otimes Id_B) \end{array}$$

commutes. π_{n-1} and σ_{n-1} are the cononical epimorphisms.

(2) Let $n \in \mathbb{N}_{\nvdash}$. Then, by the proof of 15.19 (i), Supp_A (Coker $d^{n-2} \otimes_A B) \subseteq F'_n$. Hence, by 5.9, there is an A-homomorphism

$$\alpha(\text{Coker } d^{n-2} \otimes_A B): \ (\text{Coker } d^{n-2} \otimes_A B) \longrightarrow \bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_\mathfrak{n}} (\text{Coker } d^{n-2} \otimes_A B)_\mathfrak{p}$$

for which, for $x \in$ Coker $d^{n-2} \otimes_A B$ and $\mathfrak{p} \in \partial \mathfrak{F}'_\mathfrak{n}$, the component of α (Coker $d^{n-2} \otimes_A$ B) in the direct summand (Coker $d^{n-2} \otimes_A B)_\mathfrak{p}$ is $x/1$. It is easy to see that α(Coker $d^{n-2} \otimes_A B$) is a B-homomorphism. By 5.10 (ii), there exists a B-isomorphism

$$\gamma_n : \bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_\mathfrak{n}} (\text{Coker } d^{n-2} \otimes_A B)_\mathfrak{p} \longrightarrow D_{\Phi_n B}(\text{Coker } d^{n-2} \otimes_A B)$$

such that the diagram

$$\begin{array}{ccc} \text{Coker } d^{n-2} \otimes_A B & \xrightarrow{\alpha \ (\text{Coker } d^{n-2} \otimes_A B)} & \bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_\mathfrak{n}} (\text{Coker } d^{n-2} \otimes_A B)_\mathfrak{p} \\ & \searrow^{\eta_{\Phi_{n+1} B} \ (\text{Coker } \mathrm{d}^{\mathrm{n}-2} \otimes_\mathrm{A} \mathrm{B})} & \downarrow \gamma_n \\ & & D_{\Phi B} \ (\text{Coker } d^{n-2} \otimes_A B) \end{array}$$

commutes.

(3) Let $n \in \mathbb{N}_{\nvdash}$. Then there exists a natural B-isomorphism

$$\lambda_n : (\bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_n} \text{Coker } d^{n-2}) \otimes_A B \longrightarrow \bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_n} (\text{Coker } d^{n-2} \otimes_A B)_{\mathfrak{p}}$$

such that the diagram

$$\begin{array}{ccc}
\text{Coker } d^{n-2} \otimes_A B & \xrightarrow{\overline{d^{n-1}} \otimes Id_B} & \left(\bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_n} (\text{Coker } d^{n-2})_{\mathfrak{p}} \right) \otimes_A B \\
& \searrow^{\alpha\,(\text{Coker } d^{n-2} \otimes_A B)} & \Big\downarrow \lambda_n \\
& & \bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_n} (\text{Coker } d^{n-2} \otimes_A B)_{\mathfrak{p}}
\end{array}$$

commutes. Here we have $\overline{d^{n-1}}{}_0 \pi_{n-1} = d^{n-1}$.

(4) Let $n \in \mathbb{N}_{\nvdash}$. $\Phi_{n+1} B$ is a cofinal subset of Φ'_{n+1}.

Now, we can prove 5.19 (i). Write the generalized Hughes complex $\mathcal{H}(\mathcal{S}B, M \otimes_A B)$ as

$$0 \longrightarrow M \otimes_A B \xrightarrow{w^{-1}} L^0 \xrightarrow{w^0} L^1 \longrightarrow \cdots \longrightarrow L^i \xrightarrow{w^i} L^{i+1} \longrightarrow \cdots.$$

It will be convenient to write $L^{-2} = 0, L^{-1} = M \otimes_A B$. Define ψ^{-2} to be the zero homomorphism and $\psi^{-1} : M \otimes_A B \longrightarrow M \otimes_A B$ to be the identity mapping. We shall construct a family $(\psi^i)_{i \geq -2}$ of B-isomorphisms with the required properties by induction on n : a basis for this induction is provied by ψ^{-2} and ψ^{-1}.

So suppose, inductively, that $n \geq 0$ and we have constructed B-isomorphisms $\psi^{-2}, \psi^{-1}, \cdots, \psi^{n-1}$ such that the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M\otimes_A B & \xrightarrow{d^{-1}\otimes Id_B} & M^0\otimes_A B & \longrightarrow & \cdots & \longrightarrow & M^{n-2}\otimes_A B & \xrightarrow{d^{n-2}\otimes Id_B} & M^{n-1}\otimes_A B \\
\downarrow{\scriptstyle \psi^{-2}} & & \downarrow{\scriptstyle \psi^{-1}} & & \downarrow{\scriptstyle \psi^{0}} & & & & \downarrow{\scriptstyle \psi^{n-2}} & & \downarrow{\scriptstyle \psi^{n-1}} \\
0 & \longrightarrow & M\otimes_A B & \xrightarrow{w^{-1}} & L^0 & \longrightarrow & \cdots & \longrightarrow & L^{n-2} & \xrightarrow{w^{n-2}} & L^{n-1}
\end{array}$$

commutes.

From our iductive assumptions we obtain a commutative diagram

$$\begin{array}{ccc}
M^{n-1}\otimes_A B & \xrightarrow{\sigma_{n-1}} & \text{Coker } (d^{n-2}\otimes Id_B) \\
\downarrow{\scriptstyle \psi^{n-1}} & & \downarrow{\scriptstyle \overline{\psi^{n-1}}} \\
L^{n-1} & \xrightarrow{\sigma'_{n-1}} & \text{Coker } w^{n-2}
\end{array}$$

in which σ_{n-1} and σ'_{n-1} are the cononical epimorphisms and $\overline{\psi^{n-1}}$ is the induced B-isomorphism. Hence, by (1), the diagram below is commutative:

$$\begin{array}{ccc}
M^{n-1}\otimes_A B & \xrightarrow{\pi_{n-1}\otimes Id_B} & \text{Coker } d^{n-2}\otimes_A B \\
\downarrow{\scriptstyle \psi^{n-1}} & & \downarrow{\scriptstyle \overline{\psi^{n-1}}_0\theta_n} \\
L^{n-1} & \xrightarrow{\sigma'_{n-1}} & \text{Coker } w^{n-2}
\end{array}$$

Hence $\pi_{n-1} : M^{n-1} \longrightarrow \text{Coker } d^{n-2}$ is the cononical epimorphism and $\overline{\psi^{n-1}}_0\theta_n$ is a B-isomorphism. We now use the morphism of functors $\eta_{\Phi_{n+1}B} : Id \longrightarrow D_{\Phi_{n+1}B}(\bullet)$ of 1.3, we obtain a diagram

$$\begin{array}{ccccc} M^{n-1}\otimes_A B & \xrightarrow{\pi_{n-1}\otimes Id_B} & \text{Coker } d^{n-2}\otimes_A B & \xrightarrow{\eta_{\Phi_{n+1}B}(\text{Coker } d^{n-2}\otimes_A B)} & D_{\Phi_{n+1}B}(\text{Coker } d^{n-2}\otimes_A B) \\ \psi^{n-1}\downarrow & & \downarrow \overline{\psi^{n-1}}{}_0\theta_n & & \downarrow D_{\Phi_{n+1}B}(\overline{\psi^{n-1}}{}_0\theta_n) \\ L^{n-1} & \xrightarrow{\sigma'_{n-1}} & \text{Coker } w^{n-2} & \xrightarrow{\eta_{\Phi_{n+1}B}\ (\text{Coker } w^{n-2})} & D_{\Phi_{n+1}B}\ (\text{Coker } w^{n-2}). \end{array}$$

which commutes. Hence, by (2) and (3), the diagram below is commutative:

$$\begin{array}{ccccc} M^{n-1}\otimes_A B & \xrightarrow{\pi_{n-1}\otimes Id_B} & \text{Coker } d^{n-2}\otimes_A B & \xrightarrow{\overline{d^{n-1}}\otimes Id_B} & \left(\bigoplus_{\mathfrak{p}\in\partial\mathfrak{F}'_n}(\text{Coker } d^{n-2})_{\mathfrak{p}}\right)\otimes_A B \\ \psi^{n-1}\downarrow & & \downarrow \overline{\psi^{n-1}}{}_0\theta_n & & \downarrow D_{\Phi_{n+1}B}(\overline{\psi^{n-1}}{}_0\theta_n)_0\gamma_n 0\lambda_n \\ L^{n-1} & \xrightarrow{\sigma'_{n-1}} & \text{Coker } w^{n-2} & \xrightarrow{\eta_{\Phi_{n+1}B}\ (\text{Coker } w^{n-2})} & D_{\Phi_{n+1}B}\ (\text{Coker } w^{n-2}). \end{array}$$

Because $\overline{\psi^{n-1}}{}_0\theta_n$ is a B-isomorphism, we have $D_{\Phi_{n+1}B}(\overline{\psi^{n-1}}{}_0\theta_n)$ is a B-isomorphism. Now define $\psi^n : M^n\otimes_A B \longrightarrow L^n$ to be $D_{\Phi_{n+1}B}(\overline{\psi^{n-1}}{}_0\theta_n)_0\gamma_n 0\lambda_n$. Then, in view of definition 2.1, ψ^n is a B-isomorphism and the diagram

$$\begin{array}{ccc} M^{n-1}\otimes_A B & \xrightarrow{d^{n-1}\otimes Id_B} & M^n\otimes_A B \\ \psi^{n-1}\downarrow & & \downarrow\psi^n \\ L^{n-1} & \xrightarrow{w^{n-1}} & L^n \end{array}$$

commutes, thus completing the inductive step. The theorem follows by induction. Similar to the above proof, on use of (4), we can prove 5.19 (ii).

We show next how to use theorem 5.19 to obtain a generalization of [27, theorem (2.6)].

5.21 EXAMPLE. Let $f : A \longrightarrow B$ be a homomorphism of commutative Noetherian rings. Let $C(\mathcal{H}(M), M)$ denote the basic Cousin complex for M, where $\mathcal{H}(M) = (H_i)_{i \in \mathbb{N}_{\nvdash}}$ and

$$H_i = \{\mathfrak{p} \in \ \mathrm{Supp}_{\mathfrak{A}}(\mathfrak{M}) : \mathrm{ht}_{\mathrm{M}}\mathfrak{p} \geq \mathfrak{i}\} \text{ for all } \mathfrak{i} \in \mathbb{N}_{\nvdash}.$$

Assume that $\mathrm{ht}_M f^{-1}(\mathfrak{q}) = \mathrm{ht}_{M \otimes_A B}\mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Supp}_B(M \otimes_A B)$. Note that, by 4.1, if $\mathfrak{q} \in \mathrm{Supp}_B(M \otimes_A B)$, then $f^{-1}(\mathfrak{q}) \in \mathrm{Supp}_A(M)$.

The hypothesis of 5.19 (iii) are satisfied, and so we set, for each $n \in \mathbb{N}_{\nvdash}$,

$$\begin{aligned} G_n \ &= \{\mathfrak{q} \in \ \mathrm{Supp}_{\mathfrak{B}}(\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) \in \mathfrak{H}_{\mathfrak{n}}\} \\ &= \{\mathfrak{q} \in \ \mathrm{Supp}_{\mathfrak{B}}(\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{B}) : \mathrm{ht}_{\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}}\mathfrak{q} \geq \mathfrak{n}\}. \end{aligned}$$

The filtration $\mathcal{G} := (G_i)_{i \in \mathbb{N}}$ of Spec (B) is therefore just the $M \otimes_A B$-height filtration, and we can immediately deduce from 5.19 (iii) that the complex $C(\mathcal{H}(M), M) \otimes_A B$ is isomorphic over $Id_{M \otimes_A B}$ to the basic Cousin complex for the B-module $M \otimes_A B$.

In the special case in which $M = A$, such an isomorphism was produced in [27, Theorem (2.6)] under the additional assumption that B is integral over its subring $f(A)$. We present an example of a natural ring homomorphism from a (commutative Noetherian) ring to one of its rings of fractions for which this last condition is not satisfied.

5.22 EXAMPLE. Let A be a unique factorization domain (UFD), and let there exist $c, \pi \in A$ such that c, π are prime elements of A and $c \nmid \pi$ (for example $\mathbb{Z}, \mathbb{Z}[\mathbb{X}_{\nVdash}, \cdots, \mathbb{X}_{\ltimes}], \daleth[\mathbb{X}_{\nVdash}, \cdots, \mathbb{X}_{\ltimes}]$ (k is a field and $\mathbb{Z}$ is the set of integers)). Since Ac is a prime ideal of A,

$$S := \{\alpha \in A : c \nmid \alpha\}$$

is a multiplicatively closed subset of A. Suppose that $\varphi : A \longrightarrow S^{-1}A$ is the natural ring homomorphism. $A\pi$ is a prime ideal of A and $A\pi \cap S \neq \emptyset$. Hence $\varphi(A\pi)$ is a prime ideal of $\varphi(A)$ and there is not $\mathfrak{q} \in$ Spec $(S^{-1}A)$ such that $\mathfrak{q} \cap \varphi(\mathfrak{A}) = \varphi(\mathfrak{A}\pi)$. Hence, by the Lying-over theorem (See [1,5.10]), $S^{-1}A$ is not integral over its subring $\varphi(A)$. On the other hand, we know that $\mathrm{ht}_{S^{-1}A}\mathfrak{q} = \mathrm{ht}_A \varphi^{-1}(\mathfrak{q})$, for every $\mathfrak{q} \in$ Spec $S^{-1}A)$.

Theorem 3.2 can be viewed as a generalization of 5.5 (ii) and 5.5 (iii) in which two Cousin complexes are compared. We explain this.

5.23 REMARK. We introduce some further notation.

Let $\mathcal{F} = (F_i)_{i \in \mathbb{N}_{\not{\circ}}}$ be a filtration of Spec (A) which admits M, and denote the Cousin complex $C(\mathcal{F}, M)$ for M with respect to $\mathcal{F}$ by

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots .$$

Let $\mathcal{G} = (G_i)_{i \in \mathbb{N}_{\not{\circ}}}$ denote a second filtration of Spec (A) which admits M for which $F_n \subseteq G_n$ for all $n \in \mathbb{N}_{\not{\circ}}$, and let $\mathcal{H}(M) = (H_i)_{i \in \mathbb{N}_{\not{\circ}}}$ be the M-hight filtration of Spec (A) given by $H_i = \{\mathfrak{p} \in \text{Supp } (M) : ht_M \mathfrak{p} \geq \mathfrak{i}\}$ for all $i \in \mathbb{N}_{\not{\circ}}$.

By 5.7, there is a unique isomorphism of complexes $\pi : C(\mathcal{F}, M) \longrightarrow \mathcal{H}(\mathcal{S}(\mathcal{F},$ Spec $(A)), M)$ over Id_M, where $\mathcal{S}(\mathcal{F},$ Spec $(A))$ is the family of sysems of ideals $(\Phi_n(\mathcal{F},$ Spec $(A)))_{n \in \mathbb{N}}$ for which

$$\Phi_n(\mathcal{F}, \text{ Spec (A)}) = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{F}_\mathfrak{n}\}$$

for all $n \in \mathbb{N}$. Of course, we can do the same for $\mathcal{G}$: there is a unique isomorphism of complexes $\Omega : C(\mathcal{G}, M) \longrightarrow \mathcal{H}(\mathcal{S}(\mathcal{G},$ Spec $(A)), M)$ over Id_M. However, since $F_n \subseteq G_n$ for all $n \in \mathbb{N}_{\not{\circ}}$, we have

$$\Phi_n(\mathcal{F}, \text{ Spec (A)}) \subseteq \Phi_\mathrm{n}(\mathcal{G}, \text{ Spec (A)}) \text{ for all n} \in \mathbb{N}.$$

We are thus in a situation where 3.1 can be applied; in fact, as we now show, we can apply Theorem 3.2. It will be convenient to denote $\Phi_n(\mathcal{G}, \text{Spec }(A))$ by $\Phi_n(\mathcal{G})$ (for $n \in \mathbb{N}$) during the remainder of this remark.

Let $n \in \mathbb{N}$. By 5.5 (i),

$$M^n = \bigoplus_{\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{n} \cap \partial\mathfrak{H}_\mathfrak{n}} (\text{Coker d}^{\text{n}-2})_\mathfrak{p}.$$

It follows from 1.10 (i) that

$$\Gamma_{\Phi_{n+1}(\mathcal{G})}(M^n) = \bigoplus_{\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{n} \cap \partial\mathfrak{H}_\mathfrak{n}} \Gamma_{\Phi_{n+1}(\mathcal{G})}((\text{Coker d}^{\text{n}-2})_\mathfrak{p}) = \bigoplus_{\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{n} \cap \partial\mathfrak{H}_\mathfrak{n} \cap \mathfrak{G}_{\mathfrak{n}+1}} \Gamma_{\Phi_{\text{n}+1}(\mathcal{G})}((\text{Coker d}^{\text{n}-2})_\mathfrak{p}).$$

Also, for a $\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{n} \cap \partial\mathfrak{H}_\mathfrak{n}$, each element of (Coker $d^{n-2})_\mathfrak{p}$ is annihilated by some power of $\mathfrak{p}$ (by [35, 2.6], for example), and, if $\mathfrak{p} \in \mathfrak{G}_{\mathfrak{n}+1} \cap$ supp (M), then $\mathfrak{p} \in {}_{\mathfrak{n}+1}(\mathcal{G})$ (by 5.6 (iii)). It is easy to deduce from these comments

$$\Gamma_{\Phi_{n+1}(\mathcal{G})}(M^n) = \bigoplus_{\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{n} \cap \partial\mathfrak{H}_\mathfrak{n} \cap \mathfrak{G}_{\mathfrak{n}+1}} (\text{Coker} d^{n-2})_\mathfrak{p}.$$

It is easy to see that

$$\Gamma_{\Phi_{n+1}(\mathcal{G})}(M^n) = \bigoplus_{\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{n} \cap \partial\mathfrak{H}_\mathfrak{n} \setminus \partial\mathfrak{G}_\mathfrak{n}} (\text{Coker d}^{\text{n}-2})_\mathfrak{p} =: \text{S}^\text{n}.$$

It was shown in 5.5 (ii) that

$$d^n(\Gamma_{\Phi_{n+1}(\mathcal{G})}(M^n)) \subseteq \Gamma_{\phi_{n+2}(\mathcal{G})}(M^{n+1}).$$

On use of 1.5 (9) and 1.10 (i), we see that

$$\begin{aligned} H^1_{\Phi_{n+1}(\mathcal{G})}(M^n) &\cong H^1_{\Phi_{n+1}(\mathcal{G})}(M^n/S^n) \\ &\cong H^1_{\Phi_{n+1}(\mathcal{G})}\left(\bigoplus_{\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{n} \cap \partial\mathfrak{H}_\mathfrak{n} \setminus \mathfrak{G}_{\mathfrak{n}+1}} (\text{Coker d}^{\text{n}-2})_\mathfrak{p} \right) = 0. \end{aligned}$$

We can now apply Theorem 3.2. Let $S^\bullet = (S^i)_{i \geq -2}$ denote the subcomplex

$$0 \longrightarrow 0 \longrightarrow S^0 \longrightarrow S^1 \longrightarrow \cdots \longrightarrow S^i \longrightarrow S^{i+1} \longrightarrow \cdots$$

of $C(\mathcal{F}, M)$. Theorem 3.2 shows that the quotient complex $C(\mathcal{F}, M)/S^\bullet$ is isomorphic (over Id_M) to the Cousin complex $C(\mathcal{G}, M)$. This result was established by a different method in 5.5 (ii).

Thus Theorem 3.2 can be viewed as a generalization of 5.5 (ii) and 5.5 (iii).

We proceeded on anologue of 5.5 (iv) for the situation of Theorem 3.2 in 3.3.

§6 Comparison of complexes of modules of generalized fractions and generalized Hughes complexes.

Throughout this section, for any positive integer $n, D_n(R)$ denotes the set of $n \times n$ lower triangular matrices over R. For $H \in D_n(R)$, the determinant of H is denoted by $|H|$, and we use ^{T} to denote matrix transpose. Given $H \in D_n(R)$ with $n > 1, H^*$ will denote the $(n-1) \times (n-1)$ submatrix of H obtained by deletion of the n-th row and n-th column of H.

Let us briefly describe the main ingredients in the construction of modules of generalized fraction. Let M be an R-module. A non empty subset U of R^n is called triangular if (i) whenever $(x_1, \ldots, x_n) \in U$, then $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \in U$ for all choices of $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, and (ii) whenever $x, y \in U$, then there exist $z \in U$ and $H, K \in D_n(R)$ such that $Hx^T = z^T = Ky^T$. Given such a U and an R-module M, one can construct the module of generalized fractions $U^{-n}M = \left\{\frac{a}{x} : a \in M, x \in U\right\}$, where $\frac{a}{x}$ denotes the equivalence class of the pair $(a, x) \in M \times U$ under the equivalence relation $\sim$ on $M \times U$ defined as follows.

For $a, b \in M$ and $x, y \in U$, write $(a, x) \sim (b, y)$ precisely when there exist $(z_1, \cdots, z_n) := z \in U$ and $H, K \in D_n(R)$ such that

$$Hx^T = z^T = Ky^T \text{ and } |\mathrm{H}|\mathrm{a} - |\mathrm{K}|\mathrm{b} \in \left(\sum_{\mathrm{i}=1}^{\mathrm{n}-1} \mathrm{Rz_i}\right) \mathrm{M}.$$

Now $U^{-n}M$ is an R-module under the operations

$$\frac{a}{x} + \frac{b}{y} = \frac{|H|a + |K|b}{z} \text{ and } \mathrm{r}\frac{\mathrm{a}}{\mathrm{x}} = \frac{\mathrm{ra}}{\mathrm{x}}$$

for $r \in R, a, b \in M, x, y \in U$ and any choice of $z \in U$ and $H, K \in D_n(R)$ such that $Hx^T = z^T = Ky^T$. The reader is referred to [38, section 2] for more details of the construction.

The above concept is indeed a generalization of the familiar concept of an ordinary module of fractions: See [38,3.1].

6.1 DEFINITION AND NOTATION. (See [20, p.420]) By a *chain of triangular subsets on* R, we mean a family $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ of sets such that:

(i) U_n is a triangular subset of R^n for every $n \in \mathbb{N}$;

(ii) $(1) \in U_1$;

(iii) for each $n \in \mathbb{N}$ and each $(u_1, \cdots, u_n) \in U_n$, we have $(u_1 \cdots, u_n, 1) \in U_{n+1}$; and

(iv) for each $n \in \mathbb{N}$ with $n > 1$, and each $(u_1 \cdots, u_n) \in U_n$, we have $(u_1 \cdots, u_{n-1}) \in U_{n-1}$.

Such a chain $\mathcal{U}$ determines, for each R-module M, a complex

$$0 \longrightarrow M \xrightarrow{c^0} U_1^{-1}M \xrightarrow{e^1} \cdots \longrightarrow U_n^{-n}M \xrightarrow{e^n} U_{n+1}^{-n-1}M \longrightarrow \cdots .$$

in which $e^0(m) = m/(1)$ for all $m \in M$ and

$$e^n\left(\frac{m}{(u_1, \cdots, u_n)}\right) = \left(\frac{m}{(u_1, \cdots, u_n, 1)}\right)$$

for all $n \in \mathbb{N}, \gtrdot \in \mathbb{M}$ and $(u_1, \cdots, u_n) \in U_n$; we shall denote this complex by $C(\mathcal{U}, M)$.

6.2 REMARK. Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a chain of triangular subsets on R. By [3], for each $n \in \mathbb{N}$, the set

$$\Phi(U_n) = \left\{\sum_{i=1}^{n} Ru_i : (u_1, \cdots, u_n) \in U_n\right\}$$

is a system of ideals of R. Thus $\mathcal{S}(\mathcal{U}) = (\Phi(U_n))_{n\in\mathbb{N}}$ is a family of systems of ideals of R, and one can form the generalized Hughes complex $\mathcal{H}(\mathcal{S}(\mathcal{U}), M)$. Our propose in this section is to compare the complex $\mathcal{H}(\mathcal{S}(\mathcal{U}), M)$ with the complex of modules of generalized fractions $C(\mathcal{U}, M)$ described in 6.1.

Before giving some of the basic properties of generalized fractions, we introduce the important notions of *expansion* and *restriction* (See [38, (3.2) and (3.6)].

6.3 DEFINITION. (i) The triangular subset V of R^n is said to be *expanded* if, whenever $(v_1, \ldots, v_n) \in V$ and i is an integer such that $0 \leq i < n$, then $(v_1, \ldots, v_i, 1, \ldots, 1) \in V$ also.

(ii) If U is a triangular subset of R^n and if m is a positive integer such that $1 \leq m < n$, then the set

$$\{(u_1, \ldots u_m) : (u_1, \ldots, u_m) \in U \text{ for some } \mathrm{u_{m+1}, \ldots, u_n \in R}\}$$

is a triangular subset of R^m called the *restriction* of U to R^m.

6.4 REMARK. Let $n \in \mathbb{N}$. Let M be an R-module and assume that U is triangular subset of R^n. Let $m \in M$ and $(u_1, \ldots, u_n) \in U$. Then

(i) (See [21, 3.3]) if $m \in \left(\sum_{i=1}^{n-1} u_i M \right)$, then $\frac{m}{(u_1, \cdots, u_n)} = 0$ in $U^{-n}M$.

(ii) (See [39,2.1]) if $\frac{u_n m}{(u_1, \cdots, u_n)} = 0$ in $U^{-n}M$, then $\frac{m}{(u_1, \cdots, u_n)} = 0$ in $U^n M$.

(iii) if $v \in U$ is such that $Hu^T = v^T$ for some $H \in D_n(R)$, then $m/u = |H| m/v$.

The next result is a technical lemma which will be of assistance in the 6.15. It can be proved by routine arguments using generalized fractions, based on [38, (2.2) and (2.3)]; alternatively, the reader might like to consider the argument given by Gibson and O'Corroll in [4, 3.3].

6.5 LEEMMA. (See [4, 3.3]) Let M be an R-module.

(i) Let U be an expanded triangular subset of R^1. Then $U \times \{1\}$ is a triangular subset of R^2, and there is an exact sequence

$$M \xrightarrow{e} U^{-1}M \xrightarrow{w} (U \times \{1\})^{-2}M \longrightarrow 0,$$

in which e is the natural homomorphism and $w(m/(u)) = m/(u,1)$ for each $m \in M$ and $(u) \in U$.

(ii) Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a chain of triangular sets on R. Choose $n \in \mathbb{N}$. Then $U_{n+1} \times \{1\}$ is a triangular subset of R^{n+2}, and there is an exact sequence

$$U_n^{-n}M \xrightarrow{e^n} U_{n+1}^{-n-1}M \xrightarrow{w_{n+1}} (U_{n+1} \times \{1\})^{-n-2}M \longrightarrow 0,$$

in which e^n is as defined in 6.1 and $w_{n+1}(m/v) = m/(v,1)$ (with an obvious notation) for each $m \in M$ and $v \in U_{n+1}$.

6.6 DEFINITION. (See [4, P.255]) Let M be an R-module. Suppose that $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a chain of triangular subsets on R. Put F_0=Supp (M) and, for $i \in \mathbb{N}$, define

$$F_i = \{\mathfrak{p} \in \text{ Supp (M) : there exists } (\mathrm{u_1, \cdots, u_i}) \in \mathrm{U_i} \text{ with } \sum_{\mathrm{j=1}}^{\mathrm{i}} \mathrm{Au_j} \subseteq \mathfrak{p}\}.$$

The family $\mathcal{F} = (F_i)_{i\in\mathbb{N}_0}$ of sets of primes of R is called *the sequence of sets of primes induced* by $\mathcal{U}$ and M.

6.7 REMARK. Let M be an R-module, and let $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ be a chain of triangular subsets on R. We can form the complex

$$C(\mathcal{U}, M) : 0 \longrightarrow M \xrightarrow{e^0} U_1^{-1}M \xrightarrow{e^1} U_2^{-2}M \longrightarrow \cdots \longrightarrow U_i^{-i}M \xrightarrow{e^i} U_{i+1}^{-i-1}M \longrightarrow \cdots .$$

It follows from the proof of [4, Theorem 3.6] that, for each $n \in \mathbb{N}_0$,

$$\mathrm{Supp(Coker\ e^{n-1}) \subseteq F_n \text{ and } Supp\ (Ker\ e^n/Im e^{n-1}) \subseteq F_{n+1}}.$$

The special case of theorem 6.8 below provides a short proof of the main part of Thereom 3.5 of [36].

6.8 THEOREM. Let $\mathcal{S} = (\Phi_i)_{i \in \mathbb{N}}$ be a family of systems of ideals of A. Suppose that there exists a chain $\mathcal{U} = (U_i)_{i \in \mathbb{N}}$ of triangular subsets on A such that, for each $i \in \mathbb{N}$, $\sum_{ℷ=\not\vDash}^{ℶ} \mathbb{A}\approxeq_ℷ \in \lneqq_ℶ$ for all $(u_1, \cdots, u_i) \in U_i$, and that, for each ideal $\mathfrak{b} \in {}_{\text{i}}$, there exists $(v_1, \cdots, v_i) \in U_i$ such that $\sum_{j=1}^{i} Av_j \subseteq \mathfrak{b}$. Let M be an A-module.

Then there is a unique isomorphism of complexes $C(\mathcal{U}, M) \xrightarrow{\cong} \mathcal{H}(\mathcal{S}, M)$ over Id_M.

Proof. Write the complex $C(\mathcal{U}, M)$ as

$$0 \xrightarrow{e^{-1}} M \xrightarrow{e^{0}} U_1^{-1}M \xrightarrow{e^{1}} \cdots \longrightarrow U_n^{-n}M \xrightarrow{e^{n}} U_{n+1}^{-n-1}M \longrightarrow \cdots .$$

Let

$$\Phi_0 := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of} \mathfrak{A} \text{ such that } \mathfrak{V}(\mathfrak{a}) \subseteq \text{ Supp}(\mathfrak{M})\}.$$

By 2.6 and 2.7, it is enough for us to show that

(a) Supp (Coker e^{n-1}) $\subseteq \bigcup_{\mathfrak{a} \in_{\mathfrak{n}}} V(\mathfrak{a})$ for all $n \in \mathbb{N}_{\not\vDash}$,

(b) Supp (Ker e^n/Ime^{n-1}) $\subseteq \bigcup_{\mathfrak{a} \in_{\mathfrak{n}+\mathfrak{1}}} V(\mathfrak{a})$ for all $n \in \mathbb{N}_{\not\vDash}$, and

(c) The homomorphism

$$\eta_{\Phi_{n+1}}(U_{n+1}^{-n-1}M) : U_{n+1}^{-n-1}M \longrightarrow D_{\Phi_{n+1}}(U_{n+1}^{-n-1}M)$$

is an isomorphism for all $n \in \mathbb{N}_{\not\vDash}$.

It follows from 6.7 and the hypothesis that (a) and (b) are satisfied.

Let $m \in M$ and $u = (u_1, \cdots, u_{n+1}) \in U_{n+1}$ such that $m/u \in \Gamma_{\Phi_{n+1}}(U_{n+1}^{-n-1}M)$. Then there exists $v = (v_1, \cdots, v_{n+1}) \in U_{n+1}$ such that $\left(\sum_{i=1}^{n+1} Av_i\right)(m/u) = 0$. By the definition of triangular subset, there exists $w = (w_1, \cdots, w_{n+1}) \in U_{n+1}$ and $H, K \in D_{n+1}(A)$ such that $Hu^T = w^T = Kv^T$. Hence, in $U_{n+1}^{-n-1}M$,

$$\frac{|H|w_{n+1}m}{(w_1, \cdots, w_{n+1})} = \frac{w_{n+1}m}{(u_1, \cdots, u_{n+1})} = 0.$$

Hence, by 6.4 (ii), $|H|m/w = 0$, so $m/u = 0$. Therefore we can deduce $\Gamma_{\Phi_{n+1}}(U_{n+1}^{-n-1}M) = 0$. Consequently, by 1.14 (i), $\eta_{\Phi_{n+1}}(U_{n+1}^{-n-1}M)$ is injective. Now, we show that $\eta_{\Phi_{n+1}}(U_{n+1}^{-n-1}M)$ is surjective.

Let $\mathfrak{a} \in {}_{\mathfrak{n}+1}$, and let $f \in Hom_A(\mathfrak{a}, \mathfrak{U}_{\mathfrak{n}+1}^{-\mathfrak{n}-1}\mathfrak{M})$. Then there exists $u = (u_1, \cdots, u_{n+1}) \in U_{n+1}$ such that $\sum_{i=1}^{n+1} Au_i \subseteq \mathfrak{a}$. Since $\mathfrak{a}$ is a finitely generated ideal of A, and finitely many generalized fractions can be put on a common denominator, there exists $t = (t_1, \cdots, t_{n+1}) \in U_{n+1}$ such that

$$Imf \subseteq \left\{ \frac{m}{(t_1, \cdots, t_{n+1})} \in U_{n+1}^{-n-1}M : m \in M \right\}.$$

There exist $w = (w_1, \cdots, w_{n+1}) \in U$ and $H, K \in D_{n+1}(A)$ such that $Hu^T = w^T = Kt^T$. Hence, by 6.4 (iii),

$$Imf \subseteq \left\{ \frac{m}{(w_1, \cdots, w_{n+1})} \in U_{n+1}^{-n-1}M : m \in M \right\}.$$

For each $\mathfrak{b} \in {}_{\mathfrak{n}+1}$, let

$$[\ \ ] : Hom_A(\mathfrak{b}, \mathfrak{U}_{\mathfrak{n}+1}^{-\mathfrak{n}-1}\mathfrak{M}) \longrightarrow \mathfrak{D}_{{}_{\mathfrak{n}+1}}(\mathfrak{U}_{\mathfrak{n}+1}^{-\mathfrak{n}-1}\mathfrak{M})$$

be the comonical homomorphism. Hence $[f] = [f|_{\sum_{i=1}^{n} Aw_i^2 + Aw_{n+1}}]$. Let $f(w_{n+1}) = g/w$. Then, by 6.4 (i),

$$[f] = [f|_{\sum_{i=1}^{n} Aw_i^2 + Aw_{n+1}}] = \eta_{\Phi_{n+1}}(U_{n+1}^{-n-1}M)\left(\frac{g}{(w_1, \cdots, w_n, w_{n+1}^2)}\right).$$

Hence

$$[f] = \eta_{\Phi_{n+1}}(U_{n+1}^{-n-1}M)\left(\frac{g}{(w_1,\cdots,w_n,w_{n+1}^2)}\right).$$

The following lemma plays an important role in the construction of the morphism of complexes $\Theta = (\theta^i)_{i\geq -2}$ in 6.15.

6.9 LEMMA. Let M be an R-module. Let $n \in \mathbb{N}$ with $n > 1$, let U be an expanded triangular subset of R^{n+1}, and let $\overline{U}$ be the restriction of U to R^n. Let $u = (u_1, \cdots, u_{n+1}) \in U$. Let $f \in Hom_R\left(\sum_{i=1}^{n+1} Ru_i, (\overline{U}\times\{1\})^{-n-1}M\right)$. Then there exists $w = (w_1, \cdots, w_{n+1}) \in U$ and $H \in D_{n+1}(R)$ such that

$$Imf \subseteq \left\{\frac{m}{(w_1,\cdots,w_n,1)} \in (\overline{U}\times\{1\})^{-n-1}M : m \in M\right\}$$

and $Hu^T = w^T$. Also there is an R-homomorphism

$$\delta_u : Hom_R\left(\sum_{i=1}^{n+1} Ru_i, (\overline{U}\times\{1\})^{-n-1}M\right) \longrightarrow U^{-n-1}M$$

which is such that, for f and w as above, so that

$$f(w_{n+1}) = \frac{g}{(w_1,\cdots,w_n,1)}$$

for some $g \in M$, we have $\delta_u(f) = g/(w_1, \cdots, w_n, w_{n+1})$.

Proof. Since $\sum_{i=1}^{n+1} Ru_i$ is a finitely generated ideal of R and finitely many generalized fractions can be put on a common denominator, there exists $t = (t_1, \cdots, t_n) \in \overline{U}$ such that

$$Imf \subseteq \left\{\frac{m}{(t_1,\cdots,t_n,1)} \in (\overline{U}\times\{1\})^{-n-1}M : m \in M\right\}.$$

Since $\overline{U}$ is the restriction of U to R^n and U is expanded, $(t_1, \cdots, t_n, 1) \in U$. Hence there exists $w = (w_1, \cdots, w_{n+1}) \in U$ and $H, K \in D_{n+1}(R)$ such that $Hu^T = w^T =$

$K(t,1)^T$, and, since $K^*t^T = (w_1, \cdots, w_n)^T$, in view of 6.4 (iii), it is clear that w meets the reqiurements.

To define a map δ_u as described in the statement of the lemma, suppose that $w' = (w'_1, \cdots, w'_{n+1}) \in U$ and $H' \in D_{n+1}\ (R)$ are such that $H'u^T = w^T$ and

$$Imf \subseteq \left\{ \frac{m}{(w'_1, \cdots, w'_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M \right\}.$$

suppose that

$$f(w'_{n+1}) = \frac{g'}{(w'_1, \cdots, w'_n, 1)},$$

for some $g' \in M$. We must show that $g'/w' = g/w$ in $U^{-n-1}M$.

There are $P, P' \in D_{n+1}(R)$ and $z = (z_1, \cdots, z_{n+1}) \in U$ such that $Pw^T = z^T = P'w'^T$. By 6.4 (iii), there exists $g'' \in M$ such that $f(z_{n+1}) = g''/(z_1, \cdots, z_n, 1)$. We show that $g/w = g''/z$ in $U^{-n-1}M$.

Let $P = (p_{ij})$; then $z_{n+1} = \sum_{i=1}^{n+1} p_{n+1i}w_i$. Hence

$$z_{n+1}^2 = \sum_{i=1}^{n} a_i w_i + p_{n+1n+1}^2 w_{n+1}^2,$$

for some $a_1, \cdots, a_n \in \sum_{i=1}^{n+1} Rw_i$. Since

$$Imf \subseteq \left\{ \frac{m}{(w_1, \cdots, w_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M \right\},$$

it follows form 6.4 (i) that $f(\sum_{i=1}^{n} a_i w_i) = 0$, and so $f(z_{n+1}^2) = f(p_{n+1n+1}^2 w_{n+1}^2)$.

Hence, in $(\overline{U} \times \{1\})^{-n-1}M$,

$$\frac{z_{n+1}g''}{(z_1, \cdots, z_n, 1)} = \frac{p_{n+1n+1}^2 w_{n+1} g}{(w_1, \cdots, w_n, 1)}.$$

Hence by 6.4 (iii),

$$\frac{z_{n+1}g''}{(z_1, \cdots, z_n, 1)} = \frac{|P^*| p_{n+1n+1}^2 w_{n+1} g}{(z_1, \cdots, z_n, 1)}.$$

Since $\overline{U} \times \{1\} \subseteq U$, it follows that, in $U^{-n-1}M$,

$$\frac{z_{n+1}g''}{(z_1, \cdots, z_n, 1)} = \frac{|P^*|p_{n+1n+1}^2 w_{n+1}g}{(z_1, \cdots, z_n, 1)}.$$

that is, by 6.4 (iii),

$$\frac{z_{n+1}^3 g''}{(z_1, \cdots, z_n, z_{n+1}^2)} = \frac{z_{n+1}^2|P^*|p_{n+1n+1}^2 w_{n+1}g}{(z_1, \cdots, z_n, z_{n+1}^2)}.$$

Therefore, by 6.4 (ii),

$$\frac{z_{n+1}g''}{(z_1, \cdots, z_n, z_{n+1}^2)} = \frac{|P^*|p_{n+1n+1}^2 w_{n+1}g}{(z_1, \cdots, z_n, z_{n+1}^2)}.$$

in $U^{-n-1}M$. Let $L = (l_{ij}) \in D_{n+1}(R)$ be such that $L^* = P^*, l_{n+1i} = a_i (1 \leq i \leq n)$ and $l_{n+1n+1} = p_{n+1n+1}^2$. Then $L[w_1, \cdots, w_n, w_{n+1}^2]^T = [z_1, \cdots, z_n, z_{n+1}^2]^T$ and $|L| = |P^*|p_{n+1n+1}^2$. Hence, by 6.4 (iii),

$$\frac{g''}{(z_1, \cdots, z_n, z_{n+1})} = \frac{g}{(w_1, \cdots, w_n, w_{n+1})}$$

in $U^{-n-1}M$.

Similarly, we can prove that $g'/w' = g''/g$ in $U^{-n-1}M$. Hence $g/w = g'/w'$ in $U^{-n-1}M$. It follows that there is indeed a mapping

$$\delta_u : Hom_R(\sum_{i=1}^{n+1} Ru_i, (\overline{U} \times \{1\})^{-n-1}M) \longrightarrow U^{-n-1}M,$$

as described in the statement of the lemma; Now we show that δ_u is an R-homomorphism.

Let $f_1, f_2 \in Hom_R(\sum_{i=1}^{n+1} Ru_i, (\overline{U} \times \{1\})^{-n-1}M$ and let $r \in R$. Then there exist $t = (t_1, \cdots, t_n)$ and $t' = (t_1', \cdots, t_n') \in \overline{U}$ such that

$$Imf_1 \subseteq \{\frac{m}{(t_1, \cdots, t_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\}$$

and

$$Imf_2 \subseteq \{\frac{m}{(t_1', \cdots, t_n', 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\}.$$

There exists $t'' = (t''_1, \cdots, t''_n)$ and $L,\ L' \in D_n(R)$ such, that $Lt^T = t''^T = L't'^T$. Since $\overline{U}$ is the restriction of U to R^n and U is expanded, $(t''_1, \cdots, t''_n, 1) \in U$. Hence there exists $w = (w_1, \cdots, w_{n+1}) \in U$ and $H, K \in D_{n+1}(R)$ such that $Hu^T = w^T = K(t'', 1)^T$. Hence, by 6.4 (iii),

$$Imf_1 \subseteq \{\frac{m}{(w_1, \cdots, w_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\}$$

and

$$Imf_2 \subseteq \{\frac{m}{(w_1, \cdots, w_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\}.$$

Let

$$f_1(w_{n+1}) = \frac{g_1}{(w_1, \cdots, w_n, 1)} \text{ and } \mathrm{f_2(w_{n+1})} = \frac{\mathrm{g_2}}{\mathrm{(w_1, \cdots, w_n, 1)}}$$

for some $g_1, g_2 \in M$. Then

$$\delta_u(f_1) + \delta_u(f_2) = \frac{g_1 + g_2}{(w_1, \cdots, w_n, w_{n+1})} \text{ and } \delta_\mathrm{u}(\mathrm{f_1}) = \frac{\mathrm{g_1}}{\mathrm{(w_1, \cdots, w_n, w_{n+1})}}. \quad (1)$$

On the other hand, we have

$$Im(f_1 + f_2) \subseteq \{\frac{m}{(w_1, \cdots, w_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\}$$

and

$$Im(rf_1) \subseteq \{\frac{m}{(w_1, \cdots, w_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\}.$$

Hence

$$\delta_u(f_1 + f_2) = \frac{g_1 + g_2}{(w_1, \cdots, w_n, w_{n+1})} \text{ and } \delta_\mathrm{u}(\mathrm{rf_1}) = \frac{\mathrm{rg_1}}{\mathrm{(w_1, \cdots, w_n, w_{n+1})}}. \quad (2)$$

It follows from (1) and (2) that δ_u is an R-homomorphism.

6.10 PROPOSITION. Let the situation be as in 6.9. We denote by $\Phi(U)$ the system of ideals of R determined by U (See 6.2). For each $\mathfrak{b} \in (\mathfrak{U})$, let

$$[\ \ ] : Hom_R(\mathfrak{b}, (\overline{\mathfrak{U}} \times \{\mathfrak{1}\})^{-\mathfrak{n}-1}\mathfrak{M}) \longrightarrow \mathfrak{D}_{(\mathfrak{U})}((\overline{\mathfrak{U}} \times \{\mathfrak{1}\})^{-\mathfrak{n}-1}\mathfrak{M})$$

be the cononical homomorphism.

There is an R-monomorphism

$$\delta : D_{\Phi(U)}((\overline{U} \times \{1\})^{-n-1}M) \longrightarrow U^{-n-1}M$$

which is such that, for each $u = (u_1, \cdots, u_{n+1}) \in U$ and each

$$f \in Hom_R(\sum_{i=1}^{n+1} Ru_i, (\overline{U} \times \{1\})^{-n-1}M),$$

we have $\delta([f]) = \delta_u(f)$, where δ_u is the homomorphism defined in lemma 6.9.

Proof. Let $u = (u_1, \cdots, u_{n+1}), u' = (u'_1, \cdots, u'_{n+1}) \in U$ with $\sum_{i=1}^{n+1} Ru'_i \subseteq \sum_{i=1}^{n+1} Ru_i$. We show that the diagram

$$\begin{array}{ccc} Hom_R(\sum_{i=1}^{n+1} Ru_i, (\overline{U} \times \{1\})^{-n-1}M) & & \\ \downarrow & \searrow^{\delta_u} & \\ Hom_R(\sum_{i=1}^{n+1} Ru'_i, (\overline{U} \times \{1\})^{-n-1}M) & \xrightarrow[\delta_{u'}]{} & U^{-n-1}M \end{array}$$

in which the vertical map is the restriction homomorphism, is commutative.

Let $f \in Hom_R(\sum_{i=1}^{n+1} Ru_i, (\overline{U} \times \{1\})^{-n-1}M)$. Then there exists $w' = (w'_1, \cdots, w'_{n+1}) \in U$ and $L \in D_{n+1}(R)$ such that

$$Imf \subseteq \{\frac{m}{(w'_1, \cdots, w'_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\}$$

and $Lu^T = w'^T$. There exists $w = (w_1, \cdots, w_{n+1}) \in U$ and $K', K \in D_{n+1}(R)$ such that $K'w'^T = w^T = Ku'^T$. Let $K'L = H$. Then, by 6.4 (iii),

$$Imf \subseteq \{\frac{m}{(w_1, \cdots, w_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\}$$

and $Hu^T = w^T = Ku'^T$. Let $f(w_{n+1}) = g/(w_1, \cdots, w_n, 1)$. Then

$$f|_{\sum_{i=1}^{n+1} Ru'_i}(w_{n+1}) = \frac{g}{(w_1, \cdots, w_n, 1)}.$$

Also, $Im(f|_{\sum_{i=1}^{n+1} Ru'_i}) \subseteq Imf$. It follows from the definition that

$$\delta_u(f) = g/w = \delta_{u'}(f|_{\sum_{i=1}^{n+1} Ru'_i}).$$

Hence there is an R-homomorphism δ as described in the statement of the proposition. We show that δ is injective.

Let $u = (u_1, \cdots, u_{n+1}) \in U$ and $f \in Hom_R(\sum_{i=1}^{n+1} Ru_i, (\overline{U} \times \{1\})^{-n-1}M)$ be such that $\delta([f]) = \delta_u(f) = 0$. There exist $H \in D_{n+1}(R)$ and $w = (w_1, \cdots, w_{n+1}) \in U$ such that $Hu^T = w^T$ and

$$Imf \subseteq \{\frac{m}{(w_1, \cdots, w_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\}.$$

Let $f(w_{n+1}) = g/(w_1, \cdots, w_{n+1}, 1)$. Then $g/w = 0$ in $U^{-n-1}M$. Therefore there exist $Q \in D_{n+1}(R)$ and $z = (z_1, \cdots, z_{n+1}) \in U$ such that $Qw^T = z^T$ and $|Q|g \in \sum_{i=1}^{n} z_i M$. Let $Q = (q_{ij})$. Then $z_{n+1}^2 = \sum_{i=1}^{n} b_i w_i + q_{n+1n+1}^2 w_{n+1}^2$, where $b_1, \cdots, b_n \in \sum_{i=1}^{n+1} Rw_i$. It follows from 6.4 (i), and the fact that

$$Imf \subseteq \{\frac{m}{(w_1, \cdots, w_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\},$$

that $f(z_i^2) = 0 (1 \leq i \leq n)$ and $f(z_{n+1}^2) = q_{n+1n+1}^2 f(w_{n+1}^2)$. Since

$$Q^*[w_1, \cdots, w_n]^T = [z_1, \cdots, z_n]^T$$

and $|Q| = |Q^*| q_{n+1n+1}$, it follows from 6.4 (iii) that

$$\frac{q_{n+1n+1} g}{(w_1, \cdots, w_n, 1)} = \frac{q_{n+1n+1} |Q^*| g}{(z_1, \cdots, z_n, 1)} = \frac{|Q| g}{(z_1, \cdots, z_n, 1)} = 0$$

in $(\overline{U} \times \{1\})^{-n-1} M$. Hence

$$\frac{q_{n+1n+1}^2 w_{n+1} g}{(w_1, \cdots, w_n, 1)} = 0$$

in $(\overline{U} \times \{1\})^{-n-1} M$. Hence $q_{n+1n+1}^2 f(w_{n+1}^2) = 0$, that is $f(z_{n+1}^2) = 0$. Since $f(z_i^2) = 0 (1 \leq i \leq n)$ and $f(z_{n+1}^2) = 0$, the restriction of f to $\sum_{i=1}^{n+1} R z_i^2$ is zero, and so $[f] = 0$. Therefore δ is injective.

6.11 DEFINITION. (See [8, P.115]) The ring R is called an *N-ring* if, for every ideal $\mathfrak{a}$ of R, there is a commutative Notherian ring extension T of R (having the same identity as R) such that $\mathfrak{a}$ is contracted from T, that is , $\mathfrak{a}\mathfrak{T} \cap \mathfrak{R} = \mathfrak{a}$.

Of course, if R is Noetherian, then it is an N-ring, but an N-ring need not be Noetherian (See [8, P.122]).

The following theorem of Heinzer and Lantz provides a characterization of N-rings which is very useful for our purpose.

6.12 THEOREM (W. Heinzer and D. Lantz [8, Theorem 2.3]). The ring R is an N-ring if and only if; for every ideal $\mathfrak{b}$ of R, the set $\{(\mathfrak{b} : \mathfrak{c}) : \mathfrak{c}$ is an ideal of $R\}$ (partaily ordered by inclusion) satisfies the maximal condition.

6.13 THEOREM. Let the situation be as in 6.10.

If R is an N-ring (See 6.11) (and so, in particular, if R is Noetherican), then the R-monomorphism δ of 6.10 is an isomorphism.

Proof. It is enough to show that δ is surjective. Let $m/(u_1, \cdots, u_{n+1}) \in U^{-n-1}M$. It follows from 6.12 that the following increasing chain

$$\left(\sum_{i=1}^{n} Ru_i : u_{n+1}\right) \subseteq \left(\sum_{i=1}^{n} Ru_i : u_{n+1}^2\right) \subseteq \cdots \subseteq \left(\sum_{i=1}^{n} Ru_i : u_{n+1}^k\right) \subseteq \cdots$$

stabilizes. Hence there exists $t \in \mathbb{N}$ such that $\left(\sum_{i=1}^{n} Ru_i : u_{n+1}^t\right) = \left(\sum_{i=1}^{n} Ru_i : u_{n+1}^{t+1}\right)$. Therefore there exists an R-homomorphism

$$f : \sum_{i=1}^{n} Ru_i + Ru_{n+1}^{t+1} \longrightarrow (\overline{U}) \times \{1\}^{-n-1}M$$

for which

$$f(\sum_{i=1}^{n} a_i u_i + a_{n+1}u_{n+1}^{t+1}) = \frac{a_{n+1}u_{n+1}^t m}{(u_1, \cdots, u_n, 1)}$$

for all $a_1, \cdots, a_{n+1} \in R$. Hence

$$Imf \subseteq \{\frac{m}{(u_1, \cdots, u_n, 1)} \in (\overline{U} \times \{1\})^{-n-1}M : m \in M\}$$

and

$$f(u_{n+1}^{t+1}) = \frac{u_{n+1}^t m}{(u_1, \cdots, u_n, 1)}.$$

Hence by 6.9, we have

$$\delta([f]) = \delta_{(u_1, \cdots, u_n, u_{n+1}^{t+1})}(f) = \frac{u_{n+1}^t m}{(u_1, \cdots, u_n, u_{n+1}^{t+1})} = \frac{m}{(u_1, \cdots, u_n, u_{n+1})}.$$

Therefore $m/(u_1, \cdots, u_{n+1}) \in Im\delta$.

A similar result is available for triangular subsets of R, its proof is similar, but simpler than, the proofs of 6.10 and 6.13.

6.14 PROPOSITION. Let U be an expanded triangular subset of R. We denote by $\Phi(U)$ the system of ideals of R determined by U. For each $\mathfrak{b} \in (\mathfrak{U})$, let [ ]: $Hom_R(\mathfrak{b}, \mathfrak{M}) \longrightarrow \mathfrak{D}_{(\mathfrak{U})}(\mathfrak{M})$ be the canonical homomorphism.

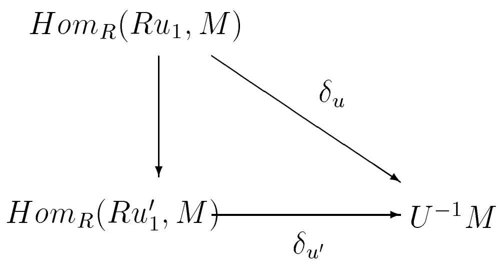
There is a monomorphism $\delta : D_{\Phi(U)}(M) \longrightarrow U^{-1}M$ which is such that $\delta([f]) = f(u_1)/(u_1)$ for each $f \in Hom_R(Ru_1, M)$ where $(u_1) \in U$. Moreover, if R is an N-ring (and, in particular, if R is Noetherian), δ is an isomorphism.

Proof. Let $u = (u_1) \in U$. Then there is an R-homomorphism

$$\delta_u : Hom_R(Ru_1, M) \longrightarrow U^{-1}M$$

which is such that, for $f \in Hom_R(Ru_1, M)$, we have $\delta_u(f) = f(u_1)/(u_1)$.

Let $u = (u_1), u' = (u'_1) \in U$ with $Ru'_1 \subseteq Ru_1$. We show that the diagram



in which the vertical map is the natural homomorphism, is commutative.

Let $f \in Hom_R(Ru_1, M)$ and let $u'_1 = a_1u_1$ where $a_1 \in R$. Then

$$\delta_{u'}(f|_{Ru'_1}) = \frac{f(u'_1)}{(u'_1)} = \frac{a_1 f(u_1)}{(a_1u_1)} = \frac{f(u_1)}{(u_1)} = \delta_u(f).$$

Hence there is an R-homomorphism δ as described in the statement of the proposition. We show that δ is injective.

Let $u = (u_1) \in U$ and $f \in Hom_R(Ru_1, M)$ be such that $\delta([f]) = \delta_u(f) = 0$. Then $f(u_1)/(u_1) = 0$ in $U^{-1}M$. Therefore there exists $a_1 \in R$ and $(v_1) \in U$ such that $a_1u_1 = v_1$ and $a_1 f(u_1) = 0$. Hence $f(a_1u_1) = f(v_1) = 0$, and so $f|_{Rv_1} = 0$. Hence $[f] = 0$. Therefore δ is injective.

Let R be an N-ring. We show that δ is surjective. Let $m/(u_1) \in U^{-1}M$. It follows from 6.12 that the following increasing chain

$$(0 : u_1) \subseteq (0 : u_1^2) \subseteq \cdots \subseteq (0 : u_1^k) \subseteq \cdots$$

stabilizes. Hence there exists $t \in \mathbb{N}$ such that $(0 : u_1^t) = (0 : u_1^{t+1})$. Therefore there exists an R-homomorphism $f : Ru_1^{t+1} \longrightarrow U^{-1}M$ for which $f(a_1u_1^{t+1}) = a_1u_1^t m$. Hence

$$\delta_{(u_1^{t+1})}(f) = \frac{f(u_1^{t+1})}{(u_1^{t+1})} = \frac{u_1^t m}{(u_1^{t+1})} = \frac{m}{(u_1)}.$$

Therefore $\frac{m}{(u_1)} \in Im\delta$.

6.15 THEOREM. Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a chain of triangular sets on R. Denote the complex $C(\mathcal{U}, M)$ of modules of generalized fractions by

$$0 \longrightarrow M \xrightarrow{f^{-1}} F^0 \xrightarrow{f^0} F^1 \longrightarrow \cdots \longrightarrow F^n \xrightarrow{f^n} F^{n+1} \longrightarrow \ldots .$$

(so that $F^n = U_{n+1}^{-n-1}M$ and $f^{n-1} = e^n$ for all $n \in \mathbb{N}_{\not{0}}$), and set $F^{-1} = M$.

Let $\mathcal{S}(\mathcal{U}) = (\Phi(U_n))_{n\in\mathbb{N}}$ be the family of systems of ideals of R determined by $\mathcal{U}$. Denote the generalized Hughes complex $\mathcal{H}(\mathcal{S}(\mathcal{U}), M)$ for M with respect to $\mathcal{S}(\mathcal{U})$ by

$$0 \longrightarrow M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} K^1 \longrightarrow \cdots \longrightarrow K^n \xrightarrow{h^n} K^{n+1} \longrightarrow \ldots .$$

and set $K^{-1} = M$.

Then there is a homomorphism of complexes

$$\Theta = (\theta^i)_{i\geq -1} : \mathcal{H}(\mathcal{S}(\mathcal{U}), M) \longrightarrow C(\mathcal{U}, M)$$

over Id_M. Moreover, Θ is an isomorphism if R is an N-ring (and, in paricular, when R is Noetherian).

Proof. The homomorphism $\Theta = (\theta^i)_{i\geq -1}$ is constructed by a straighforward inductive process.

Let $\theta^{-1} : K^{-1} \longrightarrow F^{-1}$ be the identity mapping on M and use 6.14 to define θ^0. Suppose, inductively, that $n \geq 1$ and we have constructed R-homomorphism $\theta^{-1}, \theta^0, \cdots, \theta^{n-1}$ so that diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \xrightarrow{h^{-1}} & K^0 & \longrightarrow & \cdots & \longrightarrow & K^{n-2} & \xrightarrow{h^{n-2}} & K^{n-1} \\
 & & \theta^{-1}\big\downarrow & & \theta^0\big\downarrow & & & & \theta^{n-2}\big\downarrow & & \theta^{n-1}\big\downarrow \\
0 & \longrightarrow & M & \xrightarrow{f^{-1}} & F^0 & \longrightarrow & \cdots & \longrightarrow & & \xrightarrow{f^{n-2}} & F^{n-1}
\end{array}$$

commutes, and suppose we have shown that $\theta^{-1}, \theta^0, \cdots, \theta^{n-1}$ are all isomorphisms when R is an N-ring. The above diagram induces a homomorphism $\overline{\theta^{n-1}}$: Coker $h^{n-2} \longrightarrow$ Coker f^{n-2} such that the diagram

$$\begin{array}{ccc}
K^{n-1} & \xrightarrow{\pi_{n-1}} & \text{Coker } h^{n-2} \\
\theta^{n-1}\big\downarrow & & \overline{\theta^{n-1}}\big\downarrow \\
F^{n-1} & \xrightarrow{\sigma_{n-1}} & \text{Coker } f^{n-2}
\end{array}$$

(in which σ_{n-1} and π_{n-1} are the canonical epimorphisms) commutes, and $\overline{\theta^{n-1}}$ is an isomorphism if $\theta^{-1}, \cdots, \theta^{n-1}$ are. By 6.5 (ii), there is an R-isomorphism

$$\overline{w_n} : \ \mathrm{Coker\ f^{n-2}} \longrightarrow (\mathrm{U_n} \times \{1\})^{-\mathrm{n}-1}\mathrm{M}.$$

Note that, $\overline{w_n}{}_0\sigma_{n-1}$ is equal to w_n of 6.5 (ii). It will be convenient to abbreviate $\Phi(U_{n+1})$ by Φ_{n+1} (for $n \in \mathbb{N}$) in the remainder of the proof. We now use the morphism of functors $\eta_{\Phi_{n+1}} : Id \longrightarrow D_{\Phi_{n+1}}$ of 1.13 and we obtain a diagram

$$\begin{array}{ccccc}
K^{n-1} & \xrightarrow{\pi_{n-1}} & \text{Coker } h^{n-2} & \xrightarrow{\eta_{\Phi_{n+1}}\ (\text{Coker } h^{n-2})} & D_{\Phi_{n+1}}\ (\text{Coker } h^{n-2}) = K^n \\
\downarrow \theta^{n-1} & & \downarrow \overline{\theta^{n-1}} & & \downarrow D_{\Phi_{n+1}}(\overline{\theta^{n-1}}) \\
F^{n-1} & \xrightarrow{\sigma_{n-1}} & \text{Coker } f^{n-2} & \xrightarrow{\eta_{\Phi_{n+1}}\ (\text{Coker } f^{n-2})} & D_{\Phi_{n+1}}\ (\text{Coker } f^{n-2}) \\
 & & \downarrow \overline{w_n} & & \downarrow D_{\Phi_{n+1}}(\overline{w_n}) \\
 & & (U_n \times \{1\})^{-n-1}M & \xrightarrow{\eta_{\Phi_{n+1}}((U_n \times \{1\})^{-n-1}M)} & D_{\Phi_{n+1}}((U_n \times \{1\})^{-n-1}M)
\end{array}$$

which commutes. By 6.10 and 6.13, there is an R-homomorphism

$$\delta : D_{\Phi_{n+1}}((U_n \times \{1\})^{-n-1}M) \longrightarrow U_{n+1}^{-n-1}M := F^n,$$

and δ is an isomorphism if R is an N-ring. It is routine to check that

$$f^{n-1} = \delta_0 \eta_{\Phi_{n+1}}((U_n \times \{1\})^{-n-1}M)_0 \overline{w_n}_0 \sigma_{n-1}.$$

Let $\theta^n = \delta_0 D_{\Phi_{n+1}}(\overline{w_n})_0 D_{\Phi_{n+1}}(\overline{\theta^{n-1}})$. Then the diagram

$$\begin{array}{ccc}
K^{n-1} & \xrightarrow{h^{n-1}} & K^n \\
\theta^{n-1} \downarrow & & \theta^n \downarrow \\
F^{n-1} & \xrightarrow{f^{n-1}} & F^n
\end{array}$$

commutes, and, also, θ^n must be an isomorphism if R is an N-ring. We are therefore able to complete the inductive step, and the proof.

6.16 REMARK. It is easy to check that, when R is an N-ring, the isomorphism of 6.13

$$\delta : D_{\Phi(U)}((\overline{U} \times \{1\})^{-n-1}M) \longrightarrow U^{-n-1}M$$

is the inverse of the isomorphism

$$\tau_U(M) : U^{n-1}M \longrightarrow D_{\Phi(U)}((\overline{U} \times \{1\})^{-n-1}M)$$

provided by [36, 3.3]. Hence we can deduce, on use of induction, that if R is an N-ring, then the isomorphism of complexes of 6.15 is the inverse of the isomorphism of [36,3.5].

Let the situation and notation be as in 6.8 and suppose that our commutative ring is not necessarily Noetherian. Then we show that there is a morphism of complexes $\mathcal{H}(\mathcal{S}, M) \longrightarrow C(\mathcal{U}, M)$ over Id_M.

6.17 PROPOSITION. Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ and $\mathcal{S}' = (\Phi'_i)_{i\in\mathbb{N}}$ be two families of systems of ideals of R such that, for each $n \in \mathbb{N}$, the system Φ'_n is a cofinal subset of Φ_n. Let M be an R-module. Then there is an isomorphism of complexes

$$\Psi = (\psi^i)_{i\geq -2} : \mathcal{H}(\mathcal{S}', M) \longrightarrow \mathcal{H}(\mathcal{S}, M)$$

over Id_M.

Proof. Write the generalized Hughes complex $\mathcal{H}(\mathcal{S}', M)$ as

$$0 \longrightarrow M \xrightarrow{w^{-1}} L^0 \xrightarrow{w^0} L^1 \longrightarrow \cdots \longrightarrow L^i \xrightarrow{w^i} L^{i+1} \longrightarrow \ldots .$$

and write the generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$ as

$$0 \longrightarrow M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} K^1 \longrightarrow \cdots \longrightarrow K^i \xrightarrow{h^i} K^{i+1} \longrightarrow \cdots .$$

It will be convenient to write $L^{-2} = K^{-2} = 0, L^{-1} = K^{-1} = M$, and to use $h^{-2} : K^{-2} \longrightarrow K^{-1}$ and $w^{-2} : L^{-2} \longrightarrow L^{-1}$ to denote the zero homomorphism.

We shall construct an isomorphism of complexes Ψ by an inductive process. Let $\psi^{-2} : L^{-2} \longrightarrow K^{-2}$ be the zero map and $\psi^{-1} : L^{-1} \longrightarrow K^{-1}$ be the identity mapping on M. These provide a basis for the following induction.

Let $n \in \mathbb{N}_0$ and suppose that we have already constructed R-isomorphisms $\psi^i : L^i \longrightarrow K^i$ for $i = -2, -1, 0, \cdots, n-1$ such that the diagram

$$\begin{array}{ccccccccc}
L^{-2} & \xrightarrow{w^{-2}} & L^{-1} & \longrightarrow & \cdots & \longrightarrow & L^{n-2} & \xrightarrow{w^{n-2}} & L^{n-2} \\
\downarrow{\scriptstyle \psi^{-2}} & & \downarrow{\scriptstyle \psi^{-1}} & & & & \downarrow{\scriptstyle \psi^{n-2}} & & \downarrow{\scriptstyle \psi^{-1}} \\
K^{-2} & \xrightarrow{h^{-2}} & K^{-1} & \longrightarrow & \cdots & \longrightarrow & K^{n-2} & \xrightarrow{h^{n-2}} & K^{n-1}
\end{array}$$

commutes. From our inductive assumptions we obtain a commutative diagram

$$\begin{array}{ccc}
L^{n-1} & \xrightarrow{\pi_{n-1}} & \text{Coker } w^{n-2} \\
\downarrow{\scriptstyle \psi^{n-1}} & & \downarrow{\scriptstyle \overline{\psi^{n-1}}} \\
K^{n-1} & \xrightarrow{\sigma_{n-1}} & \text{Coker } h^{n-2}
\end{array}$$

in which π_{n-1} and σ_{n-1} are the canonical epimorphisms and $\overline{\psi^{n-1}}$ is the induced isomorphism. We now use the morphism of functors $\eta_{\Phi'_{n+1}} : Id \longrightarrow D_{\Phi'_{n+1}}$ of 1.13 and we obtain a diagram

$$\begin{array}{ccccc}
L^{n-1} & \xrightarrow{\pi_{n-1}} & \text{Coker } w^{n-2} & \xrightarrow{\eta_{\Phi'_{n+1}}(\text{Coker } w^{n-2})} & D_{\Phi'_{n+1}}(\text{Coker } w^{n-2}) = L^n \\
\Big\downarrow {\scriptstyle \psi^{n-1}} & & \Big\downarrow {\scriptstyle \overline{\psi^{n-1}}} & & \Big\downarrow {\scriptstyle D_{\Phi'_{n+1}}(\overline{\psi^{n-1}})} \\
K^{n-1} & \xrightarrow{\sigma_{n-1}} & \text{Coker } h^{n-2} & \xrightarrow{\eta_{\Phi'_{n+1}}(\text{Coker } h^{n-2})} & D_{\Phi'_{n+1}}(\text{coker } h^{n-2})
\end{array}$$

which commutes. Since Φ'_{n+1} is cofinal subset of Φ_{n+1}, there exists an R-isomorphism $\lambda_n : D_{\Phi'_{n+1}}(\text{Coker } h^{n-2}) \longrightarrow D_{\Phi_{n+1}}(\text{Coker } h^{n-2})$ such that the diagram

$$\begin{array}{ccc}
\text{Coker } h^{n-2} & \xrightarrow{\eta_{\Phi'_{n+1}}(\text{Coker } h^{n-2})} & D_{\Phi'_{n+1}}(\text{Coker } h^{n-2}) \\
& \searrow {\scriptstyle \eta_{\Phi_{n+1}}(\text{Coker } h^{n-2})} & \Big\downarrow {\scriptstyle \lambda_n} \\
& & D_{\Phi_{n+1}}(\text{Coker } h^{n-2})
\end{array}$$

commutes. Set $\psi^n = \sigma_n o D_{\Phi'_{n+1}}(\overline{\psi^{n-1}})$. Then ψ^n is an R-isomorphism and the diagram

$$\begin{array}{ccc}
L^{n-1} & \xrightarrow{w^{n-1}} & L^n \\
\Big\downarrow {\scriptstyle \psi^{n-1}} & & \Big\downarrow {\scriptstyle \psi^n} \\
K^{n-1} & \xrightarrow{h^{n-1}} & K^n
\end{array}$$

commutes. We are therefore able to complete the inductive step, and the proof.

6.18 COROLLARY. Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of R. Suppose that there exists a chain $\mathcal{U} = (U_i)_{i\in\mathbb{N}}$ of triangular subsets of R such that, for each $i \in \mathbb{N}$, we have $\sum_{j=1}^{i} Ru_j \in \Phi_i$ for all $(u_1, \cdots, u_i) \in U_i$, and, for each ideal $\mathfrak{b} \in {}_i$, there exists $(v_1, \cdots, v_i) \in U_i$ such that $\sum_{j=1}^{i} Rv_j \subseteq \mathfrak{b}$. Let M be an R-module. Then there is a morphism of complexes $\mathcal{H}(\mathcal{S}, M) \longrightarrow C(\mathcal{U}, M)$ over Id_M.

Proof. For each $n \in \mathbb{N}$, let

$$\Phi(U_n) := \left\{ \sum_{j=1}^{n} Ru_j : (u_1, \cdots, u_n) \in U_n \right\}.$$

Then, by the hyppothesis, $\Phi(U_n)$ is cofinal subset of Φ_n for all $n \in \mathbb{N}$. Hence the claim follows from 6.15 and 6.17.

6.19 A COUNTEREXAMPLE. A multiplicatively closed subset of R is a triangular subset of R. We give an example of a commutative ring R and a multiplicatively closed subset S of R for which the natural map

$$\delta : \varinjlim_{sR \in \Phi(S)} Hom_R(sR, R) = D_{\Phi(S)}(R) \longrightarrow S^{-1}R$$

of 6.14 is not surjective. Since S can be incorporated into the chain of triangular subsets $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ on R, where $U_1 = S$ and $U_n = S \times \{1\} \times \cdots \times \{1\} \subseteq R^n$ for all $n \in \mathbb{N}$ with $n > 1$, this example is enough to show that the morphism of complexes of 6.15 in not always an isomorphism.

Consider $R = k[X_1, X_2, \cdots, X_n, \cdots]/I$ where k is a field and

$$I := (X_1X_2, X_1^2X_3, \cdots, X_1^{n-1}X_n, \cdots).$$

Let x_i denote the natural image of X_i in R. We show that $(0 :_R x_1^{n-1}) \subset (0 :_R x_1^n)$, for each $n \in \mathbb{N}$.

Since $x_1^n x_{n+1} = 0$, we have $x_{n+1} \in (0 :_R x_1^n)$. It is enough to show that $x_{n+1} \notin (0 :_R x_1^{n-1})$. Suppose that $x_{n+1} \in (0 :_R x_1^{n-1})$, so that $X_1^{n-1}X_{n+1} \in I$. Hence there are $t \in \mathbb{N}$ and $f_1(X_1, \cdots, X_t), \cdots, f_t(X_1, \cdots, X_t) \in k[X_1, \cdots, X_t]$ such that $t > n+1$ and

$$X_1^{n-1}X_{n+1} = \sum_{i=1}^{t} X_1^i X_{i+1} f_i(X_1, \cdots, X_t)$$

in $k[X_1, \cdots, X_{t+1}]$. Evaluate at $X_2 = \cdots = X_n = X_{n+2} = \cdots = X_{t+1} = 0$ in $k[X_1, \cdots, X_{t+1}]$. We obtain that

$$X_1^{n-1}X_{n+1} = X_1^n X_{n+1} f_n(X_1, 0, \cdots, 0, X_{n+1}, 0, \cdots, 0),$$

and this contradiction shows that $x_{n+1} \notin (0 :_R x_1^{n-1})$.

We note in passing that the strictly ascending chain

$$(0 :_R x_1) \subset (0 :_R x_1^2) \subset \cdots (0 :_R x_1^n) \subset (0 :_R x_1^{n+1}) \subset \cdots$$

shows that R is not an N-ring.

Take $S = \{x_1^i : i \in \mathbb{N}_{\nvDash}\}$, we show that $1/x_1 \notin Im\delta$. Suppose that $1/x_1 \in Im\delta$. Then there are $l \in \mathbb{N}$ and $f \in Hom_R(x_1^l R, R)$ such that $1/x_1 = f(x_1^l)/x_1^l$ in $S^{-1}R$. Note that $(0 :_R x_1^l) \subseteq (0 :_R f(x_1^l))$.

We can assume that

$$f(x_1^l) = \sum_{i=1}^{m} a_i x_1^{\alpha_{i1}} \cdots x_u^{\alpha_{iu}},$$

Where $u, m \in \mathbb{N}, \eth_{\nvDash}, \cdots, \eth_{\gtrdot} \in \urcorner$ and $(\alpha_{i1}, \cdots, \alpha_{iu}) \in \mathbb{N}_{\nvDash}^{\approxeq}(\nvDash \leq \sqsupset \leq \gtrdot)$. If, for any $1 \leq i \leq m$, there exists $2 \leq j \leq u$ such that $1 \leq \alpha_{ij}$, then $x_1^{j-1}x_j^{\alpha_{ij}} = 0$ in R, and hence

$$\frac{a_i x_1^{\alpha_{i1}\cdots} x_n^{\alpha_{iu}}}{x_1^l} = 0$$

in $S^{-1}R$. Hence, in $S^{-1}R$,

$$\frac{1}{x_1} = \frac{f(x_1^l)}{x_1^l} = \sum_{i=1}^{h} \frac{b_i x_1^{\beta_i}}{x_1^l},$$

Where $h \in \mathbb{N}, \nVdash, \cdots, \eqsim \in \daleth$ and $(\beta_1, \cdots, \beta_h) \in \mathbb{N}_{\nVdash}^{\eqsim}$. Then there exists $x_1^q \in S$ such that $x_1^{q+l} = \sum_{i=1}^{h} b_i x_1^{\beta_i+q+1}$ in R. It follows from the definition of I that

$$X_1^{q+l} = \sum_{i=1}^{l} b_i X_1^{\beta_i+q+1}$$

in $k[X_1]$. Hence there exists $1 \leq p \leq h$ such that $b_i = 0$ for all $1 \leq i \leq h$ and $i \neq p$, $b_p = 1$ and $\beta_p = l - 1$. Thus

$$f(x_1^l) = x_1^{l-1} + \sum_{i=1}^{s} c_i x_1^{\gamma_{i1}} \cdots x_u^{\gamma_{iu}},$$

where $s \in \mathbb{N}, \nVdash, \cdots, \sim \in \daleth$ and $(\gamma_{i1}, \cdots, \gamma_{iu}) \in (\mathbb{N}_{\nVdash} \times \mathbb{N} \times \cdots \times \mathbb{N})(\nVdash \leq \beth \leq \sim)$. Now

$$x_1^{u-1}(\sum_{i=1}^{s} c_i x_1^{\gamma_{i1}} \cdots x_u^{\gamma_{iu}}) = 0$$

in R. Hence $x_1^{u-1} f(x_1^l) = x_1^{u+l-2}$ and $x_{u+l} x_1^{u-1} f(x_1^l) = x_{u+l} x_1^{u+l-2} \neq 0$, since $x_{u+l} \notin (0 :_R x_1^{u+l-2})$. However $x_{u+l} x_1^{u-1}(x_1^l) = x_{u+l} x_1^{u+l-1} = 0$. we have thus show that

$$x_{u+l} x_1^{u-1} \in (0 :_R x_1^l) \backslash (0 :_R f(x_1^l)),$$

and this contradiction shows that $1/x_1 \notin Im\delta$.

§7 Generalized Hughes complexes and kersken's denominator systems.

In [14], M. Kersken defined the concept of *denominator system over R*. He then constructed, for an R-module M and a denominator system $\mathcal{L}$, a complex $\widetilde{C^{\bullet}}$ $(\mathcal{L}; M)$ (See [14, (2.4)]) which he called the *Cousin complex of M with respect to $\mathcal{L}$*. In [6,2.3], M.A. Hamieh and H. Zakeri showed that every Kersken's Cousin complexes is isomorphic to a complex of modules of generalized fractions. In this section, we show that the methods of § 2 can be used to show very quickly that every kersken's Cousin complexes is isomorphic to a generalized Hughes Complex.

We now describe the theory of denominator systems developed by kersken in [14].

7.1 DEFINITION. A *denominator system over R* is a set $\mathcal{L} \subseteq \bigcup_{i\in\mathbb{N}_{\nvdash}} R^i$. (where R^0 is interpreted as $\{\emptyset\}$) for which the following are satisfied:

$(a) \mathcal{L} \neq \emptyset$;

(b) whenever $i \in \mathbb{N}$ and $(u_1, \cdots u_i) \in \mathcal{L}^i = \mathcal{L} \cap R^i$, then $(u_1, \cdots, u_j) \in \mathcal{L}$ for all $j = 0, \cdots, i$;

(c) whenever $i \in \mathbb{N}$ and $v = (v_1, \cdots, v_i) \in \mathcal{L}^i$, then

$$S(v) := \{v_{i+1} \in R : (v_1, \cdots, v_i, v_{i+1}) \in \mathcal{L}\}$$

is a multiplicatively closed subset of R; and

(d) whenever $i \in \mathbb{N}$ and $u = (u_1, \cdots, u_i), v = (v_1, \cdots, v_i) \in \mathcal{L}^i$ with $\sum_{j=1}^{i} v_j R \subseteq \sum_{j=1}^{i} u_j R$, then $S(v) \subseteq S(u) \subseteq \overline{(S(v) + \sum_{j=1}^{i} u_j R)}$, where $\overline{T}$, for a multiplicatively closed subset T of R, denotes the saturation of T.

7.2 LEMMA (See [14,1.1]) Let $\mathcal{L}$ be a denominator system over R. Let $(u_1, \cdots, u_p), (v_1, \cdots, v_p) \in \mathcal{L}^p$ where $p \in \mathbb{N}_{\not\succ}$. Then there exists $(w_1, \cdots, w_p) \in \mathcal{L}^p$ such that $w_j \in (\sum_{i=1}^{j} u_i R).(\sum_{i=1}^{j} v_i R)$ for all $j = 0, \cdots, p$.

7.3 DEFINITION. For an ideal $\mathfrak{a}$ of R, the $\mathcal{L}$ hleight of $\mathfrak{a}$ [14, page 390], denoted $\mathcal{L}$-ht $\mathfrak{a}$, is defined to be

$$\mathrm{Sup}\ \{\mathrm{i} \in \mathbb{N}_{\not\succ} : \text{there exists } (\mathrm{u}_1, \cdots, \mathrm{u}_\mathrm{i}) \in \mathcal{L}^\mathrm{i} \text{ such that } \sum_{\mathrm{j}=1}^{\mathrm{i}} \mathrm{u}_\mathrm{j}\mathrm{R} \subseteq \mathfrak{a}\}.$$

Let M be an R-module. The $\mathcal{L}$-height of M [14, page 390], denoted $\mathcal{L}$-ht (M), is defined to be

$$inf\{\mathcal{L} - ht\ (0 :_R x) : x \in M\}.$$

7.4 LEMMA. (See [14, (2.1)]) Let M be an R-module. Assume that $\mathcal{L}$-ht $(Ann_R(M)) \geq p$, where $p \in \mathbb{N}_{\not\succ}$.

Let $u = (u_1, \cdots, u_p), v = (v_1, \cdots, v_p) \in \mathcal{L}^p$ be in $Ann_R(M)$. Then $(S(u))^{-1}M$ and $(S(v))^{-1}M$ are isomorphic.

By the above isomorphisms we obtain the desired result.

7.5 DEFINITION. (See [14, page 392]) Let H be a finitely generated R-module and let $x_1, \cdots, x_n$ generate H. Assume that $\mathcal{L}$- ht $(H) \geq p$, where $p \in \mathbb{N}_{\not\succ}$. Then $\mathcal{L} - ht(0 :_R x_i) \geq p$ for all $i = 1, \cdots, n$. consequently, by 7.2, there exists $u = (u_1, \cdots, u_p) \in \mathcal{L}^p$ such that $\sum_{i=1}^{p} u_i R \subseteq Ann_R(H).C(\mathcal{L}^p; H)$ is defined to the module $(S(u))^{-1}H$ (this is sensible by 7.4) and $\epsilon_H : H \longrightarrow C(\mathcal{L}^P; H)$ denotes the natural homomorphism.

7.6 DEFINITION. (See [14, (2.2)]) Let M be an R-module. Assume that $\mathcal{L} - ht(M) \geq p$, where $p \in \mathbb{N}_{\not{0}}$. Let

$$K = \{H : H \text{ is a finitely generated submodule of} \text{M}\}.$$

We define a partial order on K by letting $H \leq H_1$ if $H \subseteq H_1$; this makes K into a direct set. Let $H, H_1 \in K$ and $u = (u_1, \cdots, u_{\mathfrak{p}}) \in \mathcal{L}^{\mathfrak{p}}$ be such that $H \subseteq H_1$ and $\sum_{i=1}^{p} u_i R \subseteq Ann_R(H_1)$. Let $\epsilon_{H,H_1} : C(\mathcal{L}^p; H) \longrightarrow C(\mathcal{L}^p; H_1)$ be the homomorphism in which $x/t \in C(\mathcal{L}^p; H)$ maps into x/t. Clearly the homomorphisms ϵ_{H,H_1} turn the family $\{C(\mathcal{L}^p; H)\}_{H \in K}$ into a direct system over K. $C(\mathcal{L}^p; M)$ is defined to be the direct limit $\varinjlim_{H \in K} C(\mathcal{L}^p; H)$.

7.7 REMARK. Given an R-module M, write K for the collection of all finitely generated submodules of M. We define a partial order on K by letting $H, H_1 \in K$ and $H \leq H_1$ if $H \subseteq H_1$; this makes K into a directed set, and we write $\mu_{HH_1} : H \longrightarrow H_1$ for the natural indusion. Then $\{H, \mu_{HH_1}\}$ is a direct system of R-modules, the limit of which is the original M, that is $M = \varinjlim_{H \in K} H$.

7.8 LEMMA. (See [14, (2.2)]) Let the situation and notation be as in 7.5, 7.6 and 7.7. Then the system of homomorphisms $\{\epsilon_H : H \longrightarrow C(\mathcal{L}^p; H)\}_{H \in K}$ is a morphism from direct system of 7.7 to direct system of 7.6 Hence it induces an R-homomorphism $\epsilon_M : M \longrightarrow C(\mathcal{L}^p; M)$. Also, we have $\mathcal{L} - ht\ (Ker \epsilon_M) \geq p + 1$ and $\mathcal{L} - ht$ (Coker ϵ_M) $\geq p + 1$.

7.9 THEOREM. (See [14, (2.4)]) Let M be an R-module, and let $\mathcal{L}$ be a denominator system over R. Then there exists a unique (up to an isomorphism) complex

$$\tilde{C}^\bullet(\mathcal{L}; M) : \cdots \longrightarrow \tilde{C}^i(\mathcal{L}; M) \xrightarrow{\delta^i} \tilde{C}^{i+1}(\mathcal{L}; M) \longrightarrow \cdots$$

such that:

(1) $\tilde{C}^i(\mathcal{L}; M) = 0$ and $\delta^i = 0$ for all $i < -1$

(2) $\tilde{C}^{-1}(\mathcal{L}; M) = M$;

(3) for each $p \in \mathbb{N}_{\not 0}$, $\tilde{\mathbb{C}}^{\shortmid}(\mathcal{L}; \mathbb{M}) = \mathbb{C}(\mathcal{L}^{\shortmid}; \text{Coker } \delta^{p-2})$ and δ^{p-1} is the composition of the natural epimorphism from $\tilde{C}^{p-1}(\mathcal{L}; M)$ to Coker $\delta^{\mathfrak{p}-2}$ and the homomorphism $\epsilon_{\text{Coker } \delta^{\text{p}-2}}$: Coker $\delta^{\text{p}-2} \longrightarrow \tilde{\text{C}}^{\text{p}}(\mathcal{L}; \text{M})$.

Furthermore, $\mathcal{L} - ht(H^i(\tilde{C}^\bullet(\mathcal{L}; M))) \geq i + 2$ for all $i \in \mathbb{Z}$.

7.10 REMARK. Let the situation and notation be as in 7.9. Then it follows from 7.8, on use of induction, that $\mathcal{L} - ht$ (Coker $\delta^{i-2}) \geq i$ for all $i \in \mathbb{N}_{\not 0}$.

7.11 THEOREM. Let $\mathcal{L}$ be a denominator system over A, as described in 7.1. Let M be an A-module. For each $i \in \mathbb{N}$, set

$$\Phi_i = \{\sum_{j=1}^{i} u_j A : (u_1, \cdots, u_i) \in \mathcal{L}^i = \mathcal{L} \cap A^i\}.$$

Then $\mathcal{S} := (\Phi_i)_{i \in \mathbb{N}}$ is a family of systems of ideals of A, and there is a unique isomorphism of complexes (over Id_M) between kersken's Cousin complex $\tilde{C}^\bullet(\mathcal{L}; M)$ and the generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$.

Proof. It is immediate from 7.2 that Φ_i is a system of ideals of A (for all $i \in \mathbb{N}$).

Write $\tilde{C}^\bullet(\mathcal{L}; M)$ as

$$0 \xrightarrow{\delta^{-2}} M \xrightarrow{\delta^{-1}} \tilde{C}^0(\mathcal{L}; M) \xrightarrow{\delta^0} \longrightarrow \cdots \longrightarrow \tilde{C}^n(\mathcal{L}; M) \xrightarrow{\delta^n} \tilde{C}^{n+1}(\mathcal{L}; M) \longrightarrow \cdots.$$

Let

$$\Phi_0 = \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{a}) \subseteq \text{ Supp } (\mathfrak{M})\}.$$

By 2.6, 2.7 and 1.14 (i), it is enough for us to show that

(a) Coker δ^{n-2} is Φ_n-torsion for all $n \in \mathbb{N}_{\not{\scriptscriptstyle 0}}$,

(b) $H^{n-1}(\widetilde{C^{\bullet}}\ (\mathcal{L}; M))$ is Φ_{n+1}-torsion for all $n \in \mathbb{N}_{\not{\scriptscriptstyle 0}}$, and

(c) $\Gamma_{\Phi_{n+1}}(\widetilde{C^n}\ (\mathcal{L}; M)) = H^1_{\Phi_{n+1}}(\widetilde{C^n}\ (\mathcal{L}; M)) = 0$ for all $n \in \mathbb{N}_{\not{\scriptscriptstyle 0}}$.

Let $n \in \mathbb{N}_{\not{\scriptscriptstyle 0}}$. By 7.9, 7.10 and the definition 7.3, each element of Coker δ^{n-2} is annihilated by an ideal in Φ_n and each element of $H^{n-1}(\widetilde{C^{\bullet}}\ (\mathcal{L}; M))$ is annihilated by an ideal in Φ_{n+1}. Hence points (a) and (b) have been verified.

Let $i \in \mathbb{N}_{\not{\scriptscriptstyle 0}}$. Since, by 1.5(5), the functor $H^i_{\Phi_{n+1}}$ commutes with direct limits, it follows from 7.6, 7.9 and 7.10 that it is enough for us to show that, for an orbitrary finitely generated submodule D of Coker δ^{n-2} and a sequence $u := (u_1, \cdots, u_n) \in \mathcal{L}^n$ for which $(\sum_{j=1}^{n} u_j A)D = 0$, we have

$$H^i_{\Phi_{n+1}}((S(u))^{-1}D) = 0.$$

To see this, let $(v_1, \cdots, v_{n+1}) \in \mathcal{L}^{n+1}$. By 7.2, there exists $(w_1, \cdots, w_{n+1}) \in \mathcal{L}^{n+1}$. such that

$$\sum_{j=1}^{n} w_j A \subseteq (\sum_{j=1}^{n} u_j A) \cap (\sum_{j=1}^{n} v_j A)$$

and $w_{n+1} \in \sum_{j=1}^{n+1} u_j A$. Thus, by 7.1 (d), $w_{n+1} \in S(u)$. Thus $\mathfrak{a} \cap \mathfrak{S}(\mathfrak{u}) \neq \emptyset$, for each $\mathfrak{a} \in \Phi_{n+1}$. Hence $\Phi_{n+1}(S(u))^{-1}A = \{(S(u))^{-1}A\}$. Hence, by 1.5 (6),

$$H^i_{\Phi_{n+1}}((S(u))^{-1}D) \cong H^i_{\Phi_{n+1(S(u))^{-1}A}}((S(u))^{-1}D) = 0.$$

Thus point (c) has been verified, and the proof is complete.

CHAPTER (II)

ACTION OF CERTAIN GROUPS ON CERTAIN MODULES AND COMPLEXES

Throughout this chapter, for a subgroup G of automorphisms of R, we use R^G to denote the fixed subring

$$\{a \in R : \sigma(a) = a \text{ for all } \sigma \in \mathrm{G}\},$$

and $\mathcal{C}(R^G)$ to denote the category of all R^G-modules and R^G-homomorphisms. Also, for R-module M and for a group H of R^G-automorphisms of M, we use M^H to denote the fixed submodule

$$\{x \in M : \sigma(x) = x \text{ for all } \sigma \in \mathrm{H}\}.$$

§ 8 Group action and functors.

Throughout this section, G is a finite group of automorphisms of R and M is an R-module. We shall assume that there exists a finite group H of R^G-automorphisms of M such that $|H|$, the order of H, is invertible in R. Also, we shall let R' be a commutative ring with non-zero identity such that $|H|$ is invertible in it.

The next two results 8.1 and 8.2 is needed in this section.

8.1 PROPOSITION. (See [19, P.36, Proposition 4]). Let T be an additive functor from $\mathcal{C}(R)$ to $\mathcal{C}(R')$. Then if R-homomorphism f is null (a homomorphism is null if its domain and kernel coincide), then $T(f) = 0$.

8.2 THEOREM (See [19, P.37, Theorem 2]). Let T be an additive functor from $\mathcal{C}(R)$ to $\mathcal{C}(R')$, and let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a split exact sequence of R-modules. If T is covariant, then the sequence

$$0 \longrightarrow T(M') \longrightarrow T(M) \longrightarrow T(M'') \longrightarrow 0$$

is a split exact sequence, while if T is contravariant this is the case for sequence

$$0 \longrightarrow T(M'') \longrightarrow T(M) \longrightarrow T(M') \longrightarrow 0.$$

There is an R^G-homomorphism $\eta : M \longrightarrow M^H$ which is such that

$$\eta(x) = \frac{1}{|H|}\sum_{\sigma\in H}\sigma(x)$$

for all $x \in M$. We shall need to use the exact sequences

$$0 \longrightarrow M^H \xrightarrow{\rho} M \xrightarrow{\pi} \frac{M}{M^H} \longrightarrow 0$$

and

$$0 \longrightarrow Ker\eta \xrightarrow{\mu} M \xrightarrow{\eta} M^H \longrightarrow 0$$

in which ρ, π and μ are the obvious natural homomorphisms. It is clear that $\eta_0\rho = Id_{M^H}$. Therefore we have the following.

8.3 LEMMA. The exact sequence of R^G-modules and R^G-homomorphisms

$$0 \longrightarrow M^H \xrightarrow{\rho} M \xrightarrow{\pi} \frac{M}{M^H} \longrightarrow 0$$

and

$$0 \longrightarrow Ker\eta \xrightarrow{\mu} M \xrightarrow{\eta} M^H \longrightarrow 0$$

are split exact sequences.

8.4 DEFINITION. Let T be a functor from $\mathcal{C}(R^G)$ to $\mathcal{C}(R')$. We define

$$(T(M))^H := \{x \in T(M) : T(\sigma)(x) = x \text{ for all } \sigma \in \mathrm{H}\}.$$

8.5 THEOREM. Let T be an additive covariant functor from $\mathcal{C}(R^G)$ to $\mathcal{C}(R')$. Then the R'-homomorphism $T(\rho) : T(M^H) \longrightarrow (T(M))^H$ is an isomorphism.

Proof. It follows from 8.2 and 8.3 that the sequence

$$0 \longrightarrow T(M^H) \xrightarrow{T(\rho)} T(M) \xrightarrow{T(\pi)} T(\frac{M}{M^H}) \longrightarrow 0 \quad (1)$$

is a split exact sequence.

Eeach $\sigma \in H$ induces an R^G-isomorphism $\overline{\sigma} : M/M^H \longrightarrow M/M^H$ which is such that $\overline{\sigma}(y + M^H) := \sigma(y) + M^H$ for all $y \in M$. We define

$$(T(\frac{M}{M^H}))^H := \{x \in T(\frac{M}{M^H}) : T(\overline{\sigma})(x) = x \text{ for all } \sigma \in \mathrm{H}\}.$$

For each $\sigma \in H$, we have a commutative diagram

$$\begin{array}{ccccc}
M^H & \xrightarrow{\rho} & M & \xrightarrow{\pi} & \frac{M}{M^H} \\
{\scriptstyle Id_{M^H}} \downarrow & & \sigma \downarrow & & \overline{\sigma} \downarrow \\
M^H & \xrightarrow{\rho} & M & \xrightarrow{\pi} & \frac{M}{M^H}
\end{array}$$

We now use the functor T, we obtain a commutative diagram

$$\begin{array}{ccccc} T(M^H) & \xrightarrow{T(\rho)} & T(M) & \xrightarrow{T(\pi)} & T(\frac{M}{M^H}) \\ {\scriptstyle Id_{T(M^H)}}\downarrow & & \downarrow{\scriptstyle T(\sigma)} & & \downarrow{\scriptstyle T(\overline{\sigma})} \\ T(M^H) & \xrightarrow{T(\rho)} & T(M) & \xrightarrow{T(\pi)} & T(\frac{M}{M^H}) \end{array}$$

for each $\sigma \in H$. It follows from the above commutative diagrams that

$$Im(T(\rho)) \subseteq (T(M))^H \text{ and } \mathrm{Im}(\mathrm{T}(\pi)|_{(\mathrm{T(M)})^{\mathrm{H}}}) \subseteq (\mathrm{T}(\frac{\mathrm{M}}{\mathrm{M^H}}))^{\mathrm{H}}. \quad (2)$$

Set $\pi' = T(\pi)|_{(T(M))^H}$. By (1) and (2), the sequence of R'-modules and R'-homomorphisms

$$0 \longrightarrow T(M^H) \xrightarrow{T(\rho)} T(M)^H \xrightarrow{\pi'} (T(\frac{M}{M^H}))^H$$

in an exact sequence. Hence it is enough for us to show that $\left(T(\frac{M}{M^H})\right)^H = 0$.

Let $x \in (T(M/M^H))^H$. Then

$$|H|x = \left(\sum_{\sigma \in H} T(\overline{\sigma})\right)(x) = \left(T(\sum_{\sigma \in H} \overline{\sigma})\right)(x).$$

Since $\sum_{\sigma \in H} \sigma(y) \in M^H$ for all $y \in M$, $\sum_{\sigma \in H} \overline{\sigma} = 0$. Hence, by 8.1, $|H|x = 0$. We have $x = 0$ because $\frac{1}{|H|} \in R'$.

8.6 THEOREM. Let U be an additive contravariant functor from $\mathcal{C}(R^G)$ to $\mathcal{C}(R')$. Then the R'-homomorphism $U(\eta) : U(M^H) \longrightarrow (U(M))^H$ is an isomorphism.

Proof. It follows from 8.2 and 8.3 that the sequence

$$0 \longrightarrow U(M^H) \xrightarrow{U(\eta)} U(M) \xrightarrow{U(\mu)} U(Ker\eta) \longrightarrow 0 \quad (1)$$

is a split sequence.

For each $\sigma \in H$, we have commutative a diagram

$$\begin{array}{ccccc}
Ker\eta & \xrightarrow{\mu} & M & \xrightarrow{\eta} & M^H \\
\sigma|_{Ker\eta} \downarrow & & \sigma \downarrow & & Id_{M^H} \downarrow \\
Ker\eta & \xrightarrow{\mu} & M & \xrightarrow{\eta} & M^H.
\end{array}$$

Using the functor U, we obtain a commutative digram

$$\begin{array}{ccccc}
U(M^H) & \xrightarrow{U(\eta)} & U(M) & \xrightarrow{U(\mu)} & U(Ker\eta) \\
Id_{U(M^H)} \downarrow & & U(\sigma) \downarrow & U(\sigma|_{Ker\eta}) & \downarrow \\
U(M^H) & \xrightarrow{U(\eta)} & U(M) & \xrightarrow{U(\mu)} & U(Ker\eta)
\end{array}$$

for each $\sigma \in H$. It follows from the above commutative diagrams that

$$Im(U(\eta)) \subseteq (U(M))^H \text{ and } \mathrm{Im}(\mathrm{U}(\mu)|_{(\mathrm{U(M)})^{\mathrm{H}}}) \subseteq (\mathrm{U(Ker}\eta))^{\mathrm{H}} \quad (2).$$

Set $\mu' = U(\mu)|_{(U(M))^H}$. By (1) and (2), the sequence of R'-modules and R'-homomorphisms

$$0 \longrightarrow U(M^H) \xrightarrow{U(\eta)} U(M)^H \xrightarrow{\mu'} (U(Ker))^H$$

is an exact sequence. Hence it is enough for us to show that $(U(Ker))^H = 0$.

Let $x \in (U(Ker\eta))^H$. Then

$$|H|x = \left(\sum_{\sigma\in H} U(\sigma|_{Ker\eta})\right)(x) = \left(U(\sum_{\sigma\in H} \sigma|_{Ker\eta})\right)(x).$$

Since $\frac{1}{|H|}\sum_{\sigma\in H}\sigma(y) = 0$ for all $y \in Ker\ \eta$, $\sum_{\sigma\in H}\sigma|_{Ker\eta} = 0$. Hence, by 8.1, $|H|x = 0$. We have $x = 0$ because $\frac{1}{|H|} \in R'$.

The following corollaries of 8.5 and 8.6 are quite useful. Note that, $\frac{1}{|H|} \in R^G$ because $|H|\sigma(\frac{1}{|H|}) = 1_R$ for all $\sigma \in G$.

8.7 COROLLARY. Let N be an R^G-module, and let

$$(N \otimes_{R^G} M)^H := \{x \in N \otimes_{R^G} M : (Id_N \otimes \sigma)(x) = x \text{ for all } \sigma \in \mathrm{H}\}.$$

Then

$$Id_N \otimes \rho : N \otimes_{R^G} M^H \longrightarrow (N \otimes_{R^G} M)^H$$

is an R^G-isomorphism.

8.8 COROLLARY. Let $\mathfrak{b}$ be an ideal of R, and let $i \in \mathbb{N}_0$. Let N be an R^G-module, and let

$$(H^i_{\mathfrak{b}\cap\mathfrak{R}^{\mathfrak{G}}}(N \otimes_{R^G} M))^H := \{x \in H^i_{\mathfrak{b}\cap\mathfrak{R}^{\mathfrak{G}}}(N \otimes_{R^G} M) : H^i_{\mathfrak{b}\cap\mathfrak{R}^{\mathfrak{G}}}(Id_N \otimes \sigma)(x) = x \text{ for all } \sigma \in \mathrm{H}\}.$$

Then

$$(H^i_{\mathfrak{b}\cap\mathfrak{R}^{\mathfrak{G}}}(Id_N \otimes \rho)) : H^i_{\mathfrak{b}\cap\mathfrak{R}^{\mathfrak{G}}}(N \otimes_{R^G} M^H) \longrightarrow (H^i_{\mathfrak{b}\cap\mathfrak{R}^{\mathfrak{G}}}(N \otimes_{R^G} M))^H$$

is an R^G-isomorphism.

8.9 COROLLARY. Let N be an R^G-module, and let

$$(Hom_{R^G}(M, N))^H := \{x \in Hom_{R^G}(M, N) : Hom_{R^G}(\sigma, Id_N)(x) = x \text{ for all } \sigma \in H\}.$$

Then

$$Hom_{R^G}(\eta, Id_N) : Hom_{R^G}(M^H, N) \longrightarrow (Hom_{R^G}(M, N))^H$$

is an R^G-isomorphism.

8.10 REMARK. Let F be a free R-module with basis B. Then each $\sigma \in G$ induces an R^G-isomorphism $\sigma^* : F \longrightarrow F$ which is such that

$$\sigma^*(a_1 b_1 + \cdots + a_n b_n) = \sigma(a_1) b_1 + \cdots + \sigma(a_n) b_n$$

for all $a_i \in A (1 \leq i \leq n)$, for all $b_i \in B (1 \leq i \leq n)$ and for all $n \in \mathbb{N}$. It is clear that

$$G^* := \{\sigma^* : \sigma \in G\}$$

is a group of R^G-isomorphisms from F to F.

Let the situation and notation be as in 8.5 and 8.6, and let G be a finite group such that $|G|$, the order of G, is invertible in R. Then, by 8.5 and 8.6, $T(\rho) : T(F^{G^*}) \longrightarrow (T(F))^{G^*}$ and $U(\eta) : U(F^{G^*}) \longrightarrow (U(F))^{G^*}$ are R'-isomorphisms.

§ 9 Group action on local cohomology modules.

Throughout this section, G is a finite group of automorphisms of A and M is an A-module; N is an A^G-module. We shall assume that there exists H, a finite group of A^G-module automorphisms M, such that $|H|$, the order of H, is invertible in A.

9.1 DEFINITION AND REMARK. Let $i \in \mathbb{N}_{\not{\scriptscriptstyle 0}}$, and let $\mathfrak{b}$ be an ideal of A. If L is an A-module, then we use $L\lceil_{A^G}$ to indicate that we are regarding L as an A^G-module. It is well known that, there is an A^G-isomorphism

$$\theta : H^i_{\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}}}(N \otimes_{A^G} M) \longrightarrow (H^i_{(\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}})\mathfrak{A}}(N \otimes_{A^G} M))\lceil_{A^G}$$

(See 1.2 and 1.5(6)). We define

$(H^i_{(\mathfrak{b}\cap\mathfrak{R}^{\mathfrak{G}})\mathfrak{A}}(N\otimes_{A^G} M))^H := \{x \in H^i_{(\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}})\mathfrak{A}}(N\otimes_{A^G} M) : (\theta_0 H^i_{\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}}}(Id_N \otimes \sigma)_0\theta^{-1})(x) = x$ for all $\sigma \in \mathrm{H}\}$.

9.2 THEOREM. Let $i \in \mathbb{N}_{\not{\scriptscriptstyle 0}}$, and let $\mathfrak{b}$ be an ideal of A. Let $\mathfrak{q}_1, \cdots, \mathfrak{q}_r$ be the minimal prime ideals associated to $\mathfrak{b}$. If

$$\{\mathfrak{q}_1, \cdots, \mathfrak{q}_r\} = \{\sigma(\mathfrak{q}_1), \cdots, \sigma(\mathfrak{q}_r)\}$$

for all $\sigma \in G$, then

$$H^i_{(\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}})\mathfrak{A}}(N \otimes_{A^G} M) = H^i_{\mathfrak{b}}(N \otimes_{A^G} M);$$

and consequently $H^i_{\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}}}(N \otimes_{A^G} M^H)$ and $(H^i_{\mathfrak{b}}(N \otimes_{A^G} M))^H$ are A^G-isomorphic.

Proof. By 8.8 and 9.1, it is enough to show that

$$H^i_{(\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}})\mathfrak{A}}(N \otimes_{A^G} M) = H^i_{\mathfrak{b}}(N \otimes_{A^G} M).$$

Hence, by 1.5(1) and 1.2, it is enough, in order to complete the proof, to show that $r_A((\mathfrak{b} \cap \mathfrak{A}^{\mathfrak{G}})\mathfrak{A}) = \mathfrak{r}_{\mathfrak{A}}(\mathfrak{b})$. It is clear that $r_A((\mathfrak{b} \cap \mathfrak{A}^{\mathfrak{G}})\mathfrak{A}) \subseteq \mathfrak{r}_{\mathfrak{A}}(\mathfrak{b})$. We complete the proof by showing that if $\mathfrak{q} \in \text{Spec }(A)$ and $(\mathfrak{b} \cap \mathfrak{A}^{\mathfrak{G}})\mathfrak{A} \subseteq \mathfrak{q}$, then $\mathfrak{b} \subseteq \mathfrak{q}$.

Let $\mathfrak{q} \in \text{Spec }(A)$ and $(\mathfrak{b} \cap \mathfrak{A}^{\mathfrak{G}})\mathfrak{A} \subseteq \mathfrak{q}$. Firstly, we show that $\mathfrak{b} \subseteq \bigcup_{\sigma\in\mathfrak{G}} \sigma(\mathfrak{q})$. Let $\alpha \in \mathfrak{b}$. Then $\Pi_{\sigma\in G}\sigma(\alpha) \in \mathfrak{b} \cap \mathfrak{A}^{\mathfrak{G}}$. Hence $\Pi_{\sigma\in G}\sigma(\alpha) \in \mathfrak{q}$. consequently, there exists $\lambda \in G$ such that $\lambda(\alpha) \in \mathfrak{q}$. So we can deduce that $\mathfrak{b} \subseteq \bigcup_{\sigma\in\mathfrak{G}} \sigma(\mathfrak{q})$. Therefore there exists $\theta \in G$ such that $\mathfrak{b} \subseteq \theta(\mathfrak{q})$. Hence there exists $i(1 \leq i \leq r)$ such that $\mathfrak{q}_{\mathfrak{i}} \subseteq \theta(\mathfrak{q})$. Hence $\theta^{-1}(\mathfrak{q}_{\mathfrak{i}}) \subseteq \mathfrak{q}$. By the hypothesis, $\mathfrak{b} \subseteq \theta^{-1}(\mathfrak{q}_{\mathfrak{i}}) \subseteq \mathfrak{q}$.

We can deduce from 9.2 the following.

9.3 COROLLARY. Let $i \in \mathbb{N}_{\not{\!\circ}}$, and let $\mathfrak{b}$ be an ideal of A. Let $\mathfrak{q}_1, \cdots, \mathfrak{q}_{\mathfrak{r}}$ be the minimal prime ideals associated to $\mathfrak{b}$. Let $|G|$, the order of G, be invertible in A. Then if

$$\{\mathfrak{q}_1, \cdots, \mathfrak{q}_{\mathfrak{r}}\} = \{\sigma(\mathfrak{q}_1), \cdots, \sigma(\mathfrak{q}_{\mathfrak{r}})\}$$

for all $\sigma \in G$, then

$$H^i_{(\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}})\mathfrak{A}}(N \otimes_{A^G} A) = H^i_{\mathfrak{b}}(N \otimes_{A^G} A)$$

and furtermore $H^i_{\mathfrak{b}\cap\mathfrak{A}^{\mathfrak{G}}}(N)$ and $(H^i_{\mathfrak{b}}(N \otimes_{A^G} A))^G$ are A^G-isomorphic.

Now, we present some examples about ideals which satisfy the condition of the 9.2.

9.4 EXAMPLES. (i) Let $\sigma(\mathfrak{b}) \subseteq \mathfrak{b}$ for all $\sigma \in G$. Then $\mathfrak{b}$ satisfy the condition of 9.2.

(ii) Let $\mathfrak{b}$ be an ideal of A and Suppose that there exists an ideal $\mathfrak{a}$ of A^G such that $\mathfrak{b} = \mathfrak{a}\mathfrak{A}$. Then $\sigma(\mathfrak{b}) = \mathfrak{b}$ for all $\sigma \in G$.

(iii) Let A be a semi-local ring and $\mathfrak{m}_1, \cdots, \mathfrak{m}_t$ its maximal ideals. Then $\sigma(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t$ for all $\sigma \in G$.

(iv) Let A be a local ring and $\mathfrak{m}$ its maximal ideal. Then $\sigma(\mathfrak{m}) = \mathfrak{m}$ for all $\sigma \in G$.

§ 10 Group action on Cousin complexes.

Throughout this section, G is a finite group of automorphisms of A and M is an A-module; N is an A^G-module. We shall assume that there exists a finite group H of R^G-isomorphisms from M to M such that $|H|$, the order of H, is invertible in A.

We shall use the following facts (10.1, 10.2, 10.4 (iii), 10.6 and 10.7) in this section.

10.1 LEMMA. Let $|G|$, the order of G, be invertible in A. Then

(i) The ring homomorphism $A^G \longrightarrow A$ is pure;

(ii) $(IA) \cap A^G = I$ for any ideal I of A^G;

(iii) A^G is a Noetherian ring.

Proof. (i) Let $\varphi : R \longrightarrow R'$ be a ring homomorphism. φ is said to be pure if, for all R-module L, the natural homomorphism $L \longrightarrow L \otimes_R R'$ is injective. It follows that we have purity if R is a direct summand of R' as an R-module (See [17, page 54, Example 2]). Therefore with the aid of 8.3 we have the claim.

(ii) By (i), it is enough to show that if the ring homomorphism $\varphi : R \longrightarrow R'$ is pure, then $\varphi^{-1}(IR') = I$ for any ideal I of R. Let $\varphi : R \longrightarrow R'$ be a pure ring homomorphism, and let I be an ideal of R. Then the natural R-homomorphism $\sigma : R/I \longrightarrow R/I \otimes_R R'$ is injective. Let $\lambda : R/I \otimes_R R' \longrightarrow R'/IR'$ be the natural R-isomorphism. Then the R-homomorphism $\lambda_0\sigma : R/I \longrightarrow R'/IR'$ given by $(\lambda_0\sigma)(r + I) = \varphi(r) + IR'$ is a monomorphism. Hence $\varphi^{-1}(IR') = I$.

(ii) Let $I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be ascending claim ideals in A^G. Then $(I_nA)_{n\in\mathbb{N}}$ stabilized in A, so that $((I_nA) \cap A^G)_{n\in\mathbb{N}} = (I_n)_{n\in\mathbb{N}}$ stabilizes in A^G.

10.2 LEMMA. (i) A is integral over A^G;

(ii) Let $\mathfrak{p} \in$ Spec (A^G), and let $S = A^G - \mathfrak{p}$. Then J is a maximal ideal of $S^{-1}A$ if only if there exists $\mathfrak{q} \in$ Spec (A) such that $S^{-1}\mathfrak{q} = \mathfrak{J}$ and $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{q}$;

(iii) $\mathrm{Supp}_{A^G}(N) = \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathrm{Supp}_A(N \otimes_{A^G} A)\}$;

(iv) If $\mathfrak{q} \in \mathrm{Supp}_A(M)$, then $\sigma(\mathfrak{q}) \in \mathrm{Supp}_A(M)$ for all $\sigma \in G$;

(v) Supp $_A(N \otimes_{A^G} A) = \{\mathfrak{q} \in$ Spec $(A) : \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} \in \mathrm{Supp}_{A^G}(N)\}$;

(vi) dim $A = dimA^G$ and if $\mathfrak{q} \in \mathfrak{Spec}(\mathfrak{A})$, then dim $(A/\mathfrak{q}) = \mathfrak{dim}(\mathfrak{A}^{\mathfrak{G}}/\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}})$;

(vii) Let $\mathfrak{q} \in \mathrm{Supp}_A(N \otimes_{A^G} A)$. Then $ht_{N \otimes_{A^G} A}(\mathfrak{q}) = \mathfrak{ht}_{\mathfrak{N}}(\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}})$.

Proof. (i) This follows from [1, page 68, Exercise 12].

(ii) Let J be a maximal ideal of $S^{-1}A$. Then there is $\mathfrak{q} \in$ Spec (A) such that $S^{-1}\mathfrak{q} = \mathfrak{J}$ and $\mathfrak{q} \cap \mathfrak{S} = \emptyset$. Hence $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} \subseteq \mathfrak{p}$. Thus $S^{-1}(\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}) \subseteq \mathfrak{S}^{-1}\mathfrak{p}$. It follows from (i) and [1, 5.6 (ii)] that $S^{-1}A$ is integral over $S^{-1}A^G$. Hence, by [1, 5.8], $(S^{-1}\mathfrak{q}) \cap (\mathfrak{S}^{-1}\mathfrak{A}^{\mathfrak{G}}) = \mathfrak{S}^{-1}(\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}})$ is a maximal ideal of $S^{-1}A^G$, so that $S^{-1}(\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}) = \mathfrak{S}^{-1}\mathfrak{p}$. Hence $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$

Let $\mathfrak{q} \in$ Spec (A) and $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$. Then $(S^{-1}\mathfrak{q}) \cap (\mathfrak{S}^{-1}\mathfrak{A}^{\mathfrak{G}})$ is a maximal ideal of $S^{-1}A^G$. Hence, by [1, 5.8], $S^{-1}\mathfrak{q}$ is a maximal ideal of $S^{-1}A$.

(iii) By 4.1, it is enough to show that

$$\mathrm{Supp_{A^G}(N)} \subseteq \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathrm{Supp_A(N \otimes_{A^G} A)}\}.$$

Let $\mathfrak{p} \in \mathrm{Supp}_{A^G}(N)$. Set $S = A^G - \mathfrak{p}$. Then $N \otimes_{A^G} S^{-1}A^G \neq 0$. We shall denote the inclusion map $A^G \longrightarrow A$ by ρ'. By 8.4 and 8.5

$$Id_N \otimes S^{-1}\rho' : N \otimes_{A^G} S^{-1}A^G \longrightarrow N \otimes_{A^G} S^{-1}A$$

is an A^G-monomorphism, because $N \otimes_{A^G} S^{-1}(\bullet)$ is an additive covariant functor from $\mathcal{C}(A^G)$ to itself. Therefore $N \otimes_{A^G} S^{-1}A \neq 0$. Hence there exists a maximal ideal J of $S^{-1}A$ such that $((N \otimes_{A^G} S^{-1}A) \otimes_{S^{-1}A} (S^{-1}A)_J) \neq 0$. By (ii), there exists $\mathfrak{q} \in \text{Spec }(A)$ such that $J = S^{-1}\mathfrak{q}$ and $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$. Consequently,

$$((N \otimes_{A^G} S^{-1}A) \otimes_{S^{-1}A} (S^{-1}A)_{S^{-1}\mathfrak{q}}) \neq 0. \quad (1)$$

We have

$$((N \otimes_{A^G} S^{-1}A) \otimes_{S^{-1}A} (S^{-1}A)_{S^{-1}\mathfrak{q}}) \cong N \otimes_{A^G} (S^{-1}A \otimes_{S^{-1}A} (S^{-1}A)_{S^{-1}\mathfrak{q}}) \cong N \otimes_{A^G} (S^{-1}A)_{S^{-1}\mathfrak{q}}$$

$$\cong N \otimes_{A^G} A_{\mathfrak{q}} \cong N \otimes_{A^G} (A \otimes_A A_{\mathfrak{q}}) \cong (N \otimes_{A^G} A) \otimes_A A_{\mathfrak{q}}. \quad (2)$$

It follows from (1) and (2) that $((N \otimes_{A^G} A) \otimes_A A_{\mathfrak{q}}) \neq 0$. Hence $\mathfrak{q} \in \text{Supp}_A(N \otimes_{A^G} A)$ such that $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$.

(iv) Let $\mathfrak{q} \in \text{Supp}_A(M)$. Then there exists the ring isomorphism $f : A_{\mathfrak{q}} \longrightarrow A_{\sigma(\mathfrak{q})}$ which is such that, for each $a/s \in A_{\mathfrak{q}}$, $f(a/s) = \sigma(a)/\sigma(s)$. Consequenctly, $A_{\mathfrak{q}}$ is an $A_{\sigma(\mathfrak{q})}$-module. Therefore

$$M \otimes_A A_{\mathfrak{q}} \cong M \otimes_A (A_{\sigma(\mathfrak{q})} \otimes_{A_{\sigma(\mathfrak{q})}} A_q) \cong (M \otimes_A A_{\sigma(\mathfrak{q})}) \otimes_{A_{\sigma(\mathfrak{q})}} A_{\mathfrak{q}} \cong M_{\sigma(\mathfrak{q})} \otimes_{A_{\sigma(\mathfrak{q})}} A_{\mathfrak{q}}.$$

Hence $(M_{\sigma(\mathfrak{q})} \otimes_{A_{\sigma(\mathfrak{q})}} A_{\mathfrak{q}}) \neq 0$. Thus $\sigma(\mathfrak{q}) \in \text{Supp}_A(M)$.

(v) By (iii), it is enough to show that

$$\{\mathfrak{q} \in \text{Spec (A)} : \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} \in \text{Supp}_{\text{A}^{\text{G}}}(\text{N})\} \subseteq \text{Supp}_{\text{A}}(\text{N} \otimes_{\text{A}^{\text{G}}} \text{A}).$$

Let $\mathfrak{q} \in \text{Spec }(A)$ such that $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} \in \text{Supp}_{A^G}(N)$. Then, by (iii), there exists $\mathfrak{q}' \in \text{Spec }(A)$ such that $\mathfrak{q}' \in \text{Supp}_A(N \otimes_{A^G} A)$ and $\mathfrak{q}' \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}$. Hence, by [1, page 68, Exercise 13], there exists $\theta \in G$ such that $\mathfrak{q} = \theta(\mathfrak{q}')$. Thus, by (iv), $\mathfrak{q} \in \text{Supp}_A(N \otimes_{A^G} A)$.

(vi) This follows from (i), [1, 5.6 (i)] and [17, Exercise 9.2].

(vii) Let $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_t = \mathfrak{q}$ be a stritly decreasing chain of prime ideals of $\text{Supp}_A(N \otimes_{A^G} A)$. Then, by (i), the Incomparalility theorem [1,5.9] and (iii),

$\mathfrak{q}_o \cap \mathfrak{A}^{\mathfrak{G}} \subset \mathfrak{q}_1 \cap \mathfrak{A}^{\mathfrak{G}} \subset \cdots \subset \mathfrak{q}_t \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}$ is a strictly decreasing chain of prime ideals of $\mathrm{Supp}_{A^G}(N)$. Therfore

$$ht_{N\otimes_{A^G}A}(\mathfrak{q}) \leq \mathfrak{ht}_{\mathfrak{N}}(\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}).$$

Now, suppose that $\mathfrak{p}_o \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_l = \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}$ is a strictly decresing chain of prime ideals of $\mathrm{Supp}_{A^G}(N)$. By, (i) and the Lying-over theorem [1,5.10], there exists $\mathfrak{q}_o \in$ Spec (A) such that $\mathfrak{q}_o \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}_o$. It now follows from (i) and the Going-up theorem [1,5.11] that, there exists $\mathfrak{q}_o \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_l$ a strictly decreasing chain of prime ideals of A such that $\mathfrak{q}_i \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}_i$ for each $0 \leq i \leq l$. By [1, P.68, Exercise 13], there exists $\sigma \in G$ such that $\sigma(\mathfrak{q}_l) = \mathfrak{q}$. It is clear that $\sigma(\mathfrak{q}_o) \subset \sigma(\mathfrak{q}_1) \subset \cdots \subset \sigma(\mathfrak{q}_l) = \mathfrak{q}$ is a strictly decreasing chain of prime ideals of A such that $\sigma(\mathfrak{q}_i) \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{q}_i \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}_i$ for each $0 \leq i \leq l$. By (v), $\sigma(\mathfrak{q}_i) \in \mathrm{Supp}_A(N \otimes_{A^G} A)$ for each $0 \leq i \leq l$. Hence $ht_N(\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}) \leq \mathfrak{ht}_{\mathfrak{N}\otimes_{\mathfrak{A}^{\mathfrak{G}}}\mathfrak{A}}(\mathfrak{q})$.

Using a proof Similar to the proof of 4.3 and 5.19, we can prove theorem 10.3 and theorem 10.4.

10.3 THEOREM. Let $\mathcal{S} = (\Phi_i)_{i\in\mathbb{N}}$ be a family of systems of ideals of A. Write the generalized Hughes complex $\mathcal{H}(\mathcal{S}, M)$ as

$$0 \xrightarrow{h^{-2}} M \xrightarrow{h^{-1}} K^0 \xrightarrow{h^0} \cdots \longrightarrow K^i \xrightarrow{h^i} K^{i+1} \longrightarrow \cdots .$$

Let $f : A \longrightarrow B$ be a homomorphism of commutative Noetherian rings, and let L be a B-module. Let $\mathcal{S}B = (\Phi_i B)_{i\in\mathbb{N}}$ (we are using notation introduced in 1.4), a family of systems ofideals of B. Then there is a unique morphism of complexes of B-modules and B-homomorphisms

$$\Psi = (\psi^n)_{n\geq -2} : \mathcal{H}(\mathcal{S}, M) \otimes_A L \longrightarrow \mathcal{H}(\mathcal{S}B, M \otimes_A L)$$

over $Id_{M\otimes_A L}$.

Furthermore, Ψ is an isomorphism of complexes if and only if

$$\eta_{\Phi_{n+1}B}(K^n \otimes_A L) : K^n \otimes_A L \longrightarrow D_{\Phi_{n+1}B}(K^n \otimes_A L)$$

is an isomorphism for every $n \in \mathbb{N}_{\not\vdash}$.

10.4 THEOREM. Let $\mathcal{F}' = (F'_i)_{i \in \mathbb{N}_{\not\vdash}}$ be a filtration of Spec (A) which admits M. For each $n \in \mathbb{N}$, let

$$\Phi_n = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{F}'_\mathfrak{n}\}.$$

It follows from 5.7 that $\mathcal{S} := (\Phi_n)_{n \in \mathbb{N}}$ is a family of system of ideals of A and there is a unique isomorphism of complexes (over Id_M)

$$C(\mathcal{F}', M) \xrightarrow{\cong} \mathcal{H}(\mathcal{S}, M).$$

Let $f : A \longrightarrow B$ be a homomorphism of commutative Noetherian rings, and let $\mathcal{S}B := (\Phi_i B)_{i \in \mathbb{N}}$. Let L be a B-module.

(i) There is a unique isomorphism of complexes of B-modules and B-homomorphisms

$$C(\mathcal{F}', M) \otimes_A L \xrightarrow{\cong} \mathcal{H}(\mathcal{S}B, M \otimes_A L)$$

over $Id_{M \otimes_A L}$.

(ii) For each $n \in \mathbb{N}_{\not\vdash}$, set

$$G_n := \{\mathfrak{q} \in \mathrm{Supp_B(M \otimes_A L) : f^{-1}}(\mathfrak{q}) \in \mathfrak{F}'_\mathfrak{n}\}$$

and

$$\Phi'_n = \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of B such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{G}_\mathfrak{n}\},$$

a system of ideals of B. Let $\mathcal{S}' := (\Phi'_n)_{n \in \mathbb{N}}$. Then there is a unique isomorphism of complexes of B-modules and B-homomorphisms

$$C(\mathcal{F}', M) \otimes_A L \xrightarrow{\cong} \mathcal{H}(\mathcal{S}', M \otimes_A L)$$

over $Id_{M\otimes_A L}$.

(iii) Suppose, in addition, that whenever $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Supp}_B(M\otimes_A L)$ with $\mathfrak{q} \subset \mathfrak{q}'$ and $f^{-1}(\mathfrak{q}) = \mathfrak{f}^{-1}(\mathfrak{q}')$, then $\mathfrak{q} = \mathfrak{q}'$ (This condition would be satisfied if B were integral over its subring $f(A)$, or if f were surjective, or if $ht_A f^{-1}(\mathfrak{q}) = \mathfrak{ht}_{\mathfrak{B}}\mathfrak{q}$ for all $\mathfrak{q} \in$ Spec (B)). Then $\mathcal{G} := (G_i)_{i\in\mathbb{N}_{\not{\,}}}$ is a filtration of Spec (B) which admits $M \otimes_A L$, and there is a unique isomorphism of complexes of B-modules and B-homomorphisms

$$C(\mathcal{F}', M) \otimes_A L \xrightarrow{\cong} C(\mathcal{G}, M \otimes_A L)$$

over $Id_{M\otimes_A L}$.

In this section, we can use either 10.4 (iii) or the following theorem.

10.5 THEOREM. Let $\mathcal{F}' = (F_i')_{i\in\mathbb{N}_{\not{\,}}}$ be a filtration of Spec (A) which admits M, and let $f : A \longrightarrow B$ be a homomorphism of commutative Noetherian rings. Let L be a B-module. Set

$$G_n := \{\mathfrak{q} \in \text{ Supp (B)} : \mathrm{f}^{-1}(\mathfrak{q}) \in \mathfrak{F}_\mathfrak{n}' \cap \text{ Supp (M)}\}$$

for each $n \in \mathbb{N}_{\not{\,}}$. It is clear that $\mathcal{G} := (G_i)_{i\in\mathbb{N}_{\not{\,}}}$ is a descending sequence of subsets of Spec (B). Let $\mathcal{G} := (G_i)_{i\in\mathbb{N}_{\not{\,}}}$ be a filtration of Spec (B). Then $\mathcal{G}$ admits $M \otimes_A L$ and there exists a unique isomorphism of complexes of B-modules and B-homomorphisms

$$C(\mathcal{G}, M \otimes_A L) \xrightarrow{\cong} C(\mathcal{F}', M) \otimes_A L$$

over $Id_{M\otimes_A L}$.

Proof. Let $\mathfrak{q} \in \mathrm{Supp}_B(M \otimes_A L)$. Then, by 4.1, $f^{-1}(\mathfrak{q}) \in \mathrm{Supp}_A(M) \subseteq F_0'$. Hence $\mathfrak{q} \in \mathfrak{G}_\mathfrak{o}$. Hence $\mathcal{G}$ admits $M \otimes_A L$. Write the Cousin complex $C(\mathcal{F}', M)$ for M with respect to $\mathcal{F}'$ as

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots .$$

By 5.11, it is enough for us to show that for each $n \in \mathbb{N}_{\not\vdash}$,

(a) $\mathrm{Supp}_B(M^n \otimes_A L) \subseteq G_n$;

(b) Supp_B (Coker $(d^{n-2} \otimes Id_L)) \subseteq G_n$;

(c) $\mathrm{Supp}_B(H^{n-1}(C(\mathcal{F}', M) \otimes_A L)) \subseteq G_{n+1}$;

(d) the natural B-homomorphism

$$\alpha(M^n \otimes_A L) : M^n \otimes_A L \longrightarrow \bigoplus_{\mathfrak{q} \in \partial \mathfrak{G}_{\mathfrak{n}}} (M^n \otimes_A L)_{\mathfrak{q}}$$

such that, for $x \in (M^n \otimes_A L)$ and $\mathfrak{q} \in \partial \mathfrak{G}_{\mathfrak{n}}$, the component of $\alpha(M^n \otimes_A L)(x)$ in the summand $(M^n \otimes_A L)_{\mathfrak{q}}$ is $x/1$ (it follows from condition (a) and 5.9 that there is such an B-homomorphism), is an isomorphism.

(a) , (b) and (c) follows from 4.1 , 4.2 and 5.2.

For each $n \in \mathbb{N}_{\not\vdash}$, set

$$\Phi_n := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{a}) \subseteq \mathfrak{F}'_{\mathfrak{n}} \cap \text{ Supp (M)}\}$$

and

$$\Phi'_n := \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of B such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{G}_{\mathfrak{n}}\}.$$

It is easy to see that $\Phi_n B \subseteq \Phi'_n$ for all $n \in \mathbb{N}_{\not\vdash}$. We show that $\Phi_n B$ is a cofinal subset of Φ_n for all $n \in \mathbb{N}_{\not\vdash}$. Let $n \in \mathbb{N}_{\not\vdash}$. Let $\mathfrak{b} \in {}'_{\mathfrak{n}}$, and suppose that $\mathfrak{b}$ is proper. Let $\mathfrak{q}_1, \cdots, \mathfrak{q}_t$ be the minimal primes of $\mathfrak{b}$. Then there exists $h \in \mathbb{N}$ such that

$$(\mathfrak{q}_1 \cdots \mathfrak{q}_t)^{\mathfrak{h}} \subseteq (\mathfrak{r}(\mathfrak{b}))^{\mathfrak{h}} \subseteq \mathfrak{b}.$$

Hence

$$(f^{-1}(\mathfrak{q}_1) \cdots \mathfrak{f}^{-1}(\mathfrak{q}_t))^{\mathfrak{h}} \subseteq \mathfrak{f}^{-1}(\mathfrak{b}).$$

Since $\mathfrak{b} \subseteq \mathfrak{q}_i$ for all $i = 1, \cdots, t$, $f^{-1}(\mathfrak{q}_i) \in \mathfrak{F}'_{\mathfrak{n}} \cap$ Supp (M) for all $i = 1, \cdots, t$. By 5.6 (iii), $F'_n \cap$ Supp $(M) \subseteq \Phi_n$, so that $f^{-1}(\mathfrak{q}_i) \in {}_{\mathfrak{n}}$ for all $i = 1, \cdots, t$. Hence there exists

$\mathfrak{a} \in {}_{\mathfrak{n}}$ such that $\mathfrak{a} \subseteq \mathfrak{f}^{-1}(\mathfrak{b})$, because Φ_n is a system of ideals of A. Thus $\mathfrak{a}\mathfrak{B} \in {}_{\mathfrak{n}}\mathfrak{B}$ and $\mathfrak{a}\mathfrak{B} \subseteq \mathfrak{b}$.

Let $n \in \mathbb{N}_{\not\vdash}$. It follows from 1.10 (iii) that

$$H^i_{\Phi'_{n+1}}(M^n \otimes_A L) \cong H^i_{\Phi_{n+1}B}(M^n \otimes_A L) = H^i_{\Phi_{n+1}B}\left((\bigoplus_{\mathfrak{p} \in \partial \mathfrak{F}'_{\mathfrak{n}}} (\mathrm{Coker\ d^{n-2}})_{\mathfrak{p}}) \otimes_{\mathrm{A}} \mathrm{L} \right) = 0$$

for each $i \in \mathbb{N}_{\not\vdash}$. Hence, by 1.14 (i), $\eta_{\Phi'_{n+1}}(K^n \otimes_A L)$ is a B-isomorphism, so that, by 5.10 (i), $\alpha(M^n \otimes_A L)$ is a B-isomorphism.

10.6 THEOREM. (See [33,1.6]) Let $\mathcal{F}' = (F'_i)_{i \in \mathbb{N}_{\not\vdash}}$ be a filtration of Spec (A) which admits M, and let V be a subset of Spec (A). Suppose that the filtration $\mathcal{F}' \cap V = (F'_i \cap V)_{i \in \mathbb{N}_{\not\vdash}}$ of Spec (A) admits M. Then there is an isomorphism of complexes of A-modules and A-homomorphisms

$$C(\mathcal{F}', M) \xrightarrow{\cong} C(\mathcal{F}' \cap V, M).$$

over Id_M.

10.7 THEOREM. Let $\mathcal{F}' = (F'_i)_{i \in \mathbb{N}_{\not\vdash}}$ be a filtration of Spec (A) which admits M, and let

$$F''_n = \{\mathfrak{q} \in \text{ Spec } (\mathfrak{A}) : \sigma(\mathfrak{q}) \in \mathfrak{F}'_{\mathfrak{n}} \text{ for all } \sigma \in \mathfrak{G}\}$$

for each $n \in \mathbb{N}_{\not\vdash}$. Then $\mathcal{F}'' = (F''_i)_{i \in \mathbb{N}_{\not\vdash}}$ is a filtration of Spec (A) which admits M and there is exactly one isomorphism of complexes

$$C(\mathcal{F}', M) \xrightarrow{\cong} C(\mathcal{F}'', M)$$

over Id_M.

Proof. By 10.2 (iv), it is easy to check that $\mathcal{F}''$ is a filtration of Spec (A) which admits M.

Write the Cousin complex $C(\mathcal{F}'', M)$ for M with respect to $\mathcal{F}''$ as

$$0 \xrightarrow{d'^{-2}} M \xrightarrow{d'^{-1}} M'^{0} \xrightarrow{d'^{0}} M'^{1} \longrightarrow \cdots \longrightarrow M'^{n} \xrightarrow{d'^{n}} M'^{n+1} \longrightarrow \cdots .$$

It follows from 5.2 and hypothesis that for all $n \in \mathbb{N}_{\nvdash}$,

$$\text{Supp } (M'^{n}) \subseteq \text{ Supp } (\text{Coker } d'^{n-2}) \subseteq F''_n \subseteq F'_n$$

and

$$\text{Supp } \big(H^{n-1}(C(\mathcal{F}'', M))\big) \subseteq F''_{n+1} \subseteq F'_{n+1}.$$

Hence, in view of 5.12, it is enough for us to show that the natural A-homomorphism

$$\alpha(M'^{n}) : M'^{n} \longrightarrow \bigoplus_{\mathfrak{q} \in \partial \mathfrak{F}'_{\mathfrak{n}}} (M'^{n})_{\mathfrak{q}}$$

such that, for $x \in M'^{n}$ and $\mathfrak{q} \in \partial F'_n$, the component of $\alpha(M'^{n})(x)$ in the summand $(M'^{n})_{\mathfrak{q}}$ is $x/1$ (it follows from Supp $(M'^{n}) \subseteq F'_n$ and 5.9 that there is such an A-homomorphism), is an isomorphism for all $n \in \mathbb{N}_{\nvdash}$.

Let

$$\Phi_n := \{\mathfrak{b} : \mathfrak{b} \text{ is an ideal of A such that } \mathfrak{V}(\mathfrak{b}) \subseteq \mathfrak{F}''_{\mathfrak{n}}\}$$

for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}_{\nvdash}$. We use the morphism of functors $\eta_{\Phi_{n+1}} : Id \longrightarrow D_{\Phi_{n+1}}$ and obtain a diagram

$$\begin{array}{ccc}
M'^{n} & \xrightarrow{\alpha(M'^{n})} & \bigoplus_{\mathfrak{q} \in \partial \mathfrak{F}'_{\mathfrak{n}}} (M'^{n})_{\mathfrak{q}} \\
\Big\downarrow \scriptstyle{\eta_{\Phi_{n+1}}(M'^{n})} & & \Big\downarrow \scriptstyle{\eta_{\Phi_{n+1}}(\bigoplus_{\mathfrak{q} \in \partial \mathfrak{F}'_{\mathfrak{n}}} (M'^{n})_{\mathfrak{q}})} \\
D_{\Phi_{n+1}}(M'^{n}) & \xrightarrow{D_{\Phi_{n+1}}(\alpha(M'^{n}))} & D_{\Phi_{n+1}}(\bigoplus_{\mathfrak{q} \in \partial \mathfrak{F}'_{\mathfrak{n}}} (M'^{n})_{\mathfrak{q}})
\end{array}$$

which commutes. By 1.10 (i), for each $i \in \mathbb{N}_{\not{0}}$,

$$H^i_{\Phi_{n+1}}(\bigoplus_{\mathfrak{q} \in \partial \mathfrak{F}'_{\mathfrak{n}}} (M'^n)_{\mathfrak{q}}) = 0$$

and

$$H^i_{\Phi_{n+1}}(M'^n) = H^i_{\Phi_{n+1}}(\bigoplus_{\mathfrak{q} \in \partial \mathfrak{F}''_{\mathfrak{n}}} (\text{Coker } d'^{n-2})_{\mathfrak{q}}) = 0.$$

Hence, by 1.14 (i), $\eta_{\Phi_{n+1}}(\bigoplus_{\mathfrak{q} \in \partial \mathfrak{F}'_{\mathfrak{n}}} (M'^n)_{\mathfrak{q}})$ and $\eta_{\Phi_{n+1}}(M'^n)$ are A-isomorphisms. Therefore it is enough for us to show that $D_{\Phi_{n+1}}(\alpha(M'^n))$ is an A isomorphism. It follows from 5.9 that

$$\text{Supp } (Ker\ \alpha(M'^n)) \subseteq F'_{n+1} \text{ and Supp } (\text{Coker } \alpha(M'^n)) \subseteq F'_{n+1}.$$

hence, by 10.2 (iv),

$$\text{Supp } (Ker\ \alpha(M'^n)) \subseteq F''_{n+1} \text{ and Supp } (\text{Coker } \alpha(M'^n)) \subseteq F''_{n+1}.$$

Thus, by 1.8, $Ker\ \alpha(M'^n)$ and Coker $\alpha(M'^n)$ are Φ_{n+1}-torsion. Therefore, by 1.15, $D_{\Phi_{n+1}}(\alpha(M'^n))$ is an A-isomorphism.

10.8 NOTATION AND DEFINITION. For the remainder of this section, we shall assume that $|G|$, the order of G, is invertible in A, and we shall let $\mathcal{G} = (G_i)_{i \in \mathbb{N}_{\not{0}}}$ is a filtration of Spec (A) which admits $N \otimes_{A^G} A$. Also, we shall let

$$F_n := \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \sigma(\mathfrak{q}) \in \mathfrak{G}_{\mathfrak{n}} \text{ for all } \sigma \in \mathfrak{G}\}$$

for all $n \in \mathbb{N}_{\not{0}}$. It is clear that $\mathcal{F} = (F_i)_{n \in \mathbb{N}_{\not{0}}}$ is a descending sequence of subsets of Spec (A^G).

10.9 PROPOSITION. Let the situation and notation be as in 10.8. Then $\mathcal{F}$ is a filtration of Spec (A^G) which admits N and there is exactly one isomorphism of complexes of A-modules and A-homomorphisms

$$C(\mathcal{G}, N \otimes_{A^G} A) \xrightarrow{\cong} C(\mathcal{F}, N) \otimes_{A^G} A$$

over $Id_{N \otimes_{A^G} A}$.

Proof. Let

$$G'_n = \{\mathfrak{q} \in \text{ Spec}(\mathfrak{A}) : \sigma(\mathfrak{q}) \in \mathfrak{G}_\mathfrak{n} \text{ for all } \sigma \in \mathfrak{G}\}$$

for each $n \in \mathbb{N}_{\not{0}}$. Then, by 10.7, $\mathcal{G}' = (G'_i)_{n \in \mathbb{N}_{\not{0}}}$ is a filtration of Spec (A) which admits $N \otimes_{A^G} A$. Also, we have

$$F_n = \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathfrak{G}'_\mathfrak{n}\}$$

for each $n \in \mathbb{N}_{\not{0}}$.

Firstly, we show that $\mathcal{F}$ is a filtration of Spec (A^G). Let $n \in \mathbb{N}_{\not{0}}$. Let $\mathfrak{p} \in \partial\mathfrak{F}_\mathfrak{n}$ and $\mathfrak{p}' \in \mathfrak{F}_\mathfrak{n}$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$. Then there exist $\mathfrak{q} \in \mathfrak{G}'_\mathfrak{n}$ and $\mathfrak{q}' \in \mathfrak{G}'_\mathfrak{n}$ such that $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}, \mathfrak{q}' \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}', \mathfrak{q} \notin \mathfrak{G}'_{\mathfrak{n}+1}$ and $\mathfrak{q}' \cap \mathfrak{A}^{\mathfrak{G}} \subseteq \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}$. If $\alpha \in \mathfrak{q}'$, then $\Pi_{\sigma \in G}\sigma(\alpha) \in \mathfrak{q}' \cap \mathfrak{A}^{\mathfrak{G}} \subseteq \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}$, hence $\theta(\alpha) \in \mathfrak{q}$ for some $\theta \in G$. Hence $\mathfrak{q}' \subseteq \bigcup_{\sigma \in \mathfrak{G}} \sigma(\mathfrak{q})$. Thus there exists $\mu \in G$ such that $\mathfrak{q}' \subseteq \mu(\mathfrak{q})$. Since $\mu(\mathfrak{q}) \in \partial\mathfrak{G}'_\mathfrak{n}$ and $\mathfrak{q}' \in \mathfrak{G}'_\mathfrak{n}, \mathfrak{q}' = \mu(\mathfrak{q})$. Therefore $\mathfrak{q}' \cap \mathfrak{A}^{\mathfrak{G}} = \mu(\mathfrak{q}) \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}$.

Now, we show that $\mathcal{F}$ admits N. Let $\mathfrak{p} \in$ Supp $_{A^G}(N)$. Then, by 10.2 (iii), there exists $\mathfrak{q} \in \text{Supp}_A(N \otimes_{A^G} A) \subseteq G'_0$ such that $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$. Hence $\mathfrak{p} \in \mathfrak{F}_\mathfrak{o}$.

Set

$$G''_n = \{\mathfrak{q} \in \text{ Spec}_{\mathfrak{A}}(\mathfrak{N} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{A}) : \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} \in \mathfrak{F}_\mathfrak{n}\} \quad (1)$$

for all $n \in \mathbb{N}_{\not{0}}$. It is easy to see that $G'_n \cap \text{Supp}_A(N \otimes_{A^G} A) \subseteq G''_n$, for each $n \in \mathbb{N}_{\not{0}}$. Now, we show that $G''_n \subseteq G'_n \cap \text{Supp}_A(N \otimes_{A^G} A)$, For each $n \in \mathbb{N}_{\not{0}}$. Let $n \in \mathbb{N}_{\not{0}}$ and

$\mathfrak{q} \in \mathfrak{G}''_{\mathfrak{n}}$. Then there exists $\mathfrak{q}' \in \mathfrak{G}'_{\mathfrak{n}}$ such that $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{q}' \cap \mathfrak{A}^{\mathfrak{G}}$. Hence it follows from [1, P.68, Exersise 13] that there exists $\sigma \in G$ such that $\mathfrak{q} = \sigma(\mathfrak{q}')$. Hence, by the definition of $G'_n, \mathfrak{q} \in \mathfrak{G}'_{\mathfrak{n}}$. Hence $\mathfrak{q} \in \mathfrak{G}'_{\mathfrak{n}} \cap \operatorname{Supp}_{A^G}(N \otimes_{A^G} A)$. Consequently, for each $n \in \mathbb{N}_{\not\vdash}, \mathbb{G}''_{\ltimes} = \mathbb{G}'_{\ltimes} \cap \operatorname{Supp}_A(N \otimes_{A^G} A)$ (2).

By 10.7, there is a unique isomorphism of complexes

$$C(\mathcal{G}, N \otimes_{A^G} A) \xrightarrow{\cong} C(\mathcal{G}', N \otimes_{A^G} A) \quad (3)$$

over $Id_{N \otimes_{A^G} A}$.

By (2), $\mathcal{G}'' = (G''_i)_{i \in \mathbb{N}_{\not\vdash}}$ is a filtration of Spec (A) which admits $N \otimes_{A^G} A$. It follows from 5.2 and 5.11 that there is a unique morphism of complexes

$$\Psi = (\psi^i)_{i \geq -2} : C(\mathcal{G}'', N \otimes_{A^G} A) \longrightarrow C(\mathcal{G}', N \otimes_{A^G} A) \quad (4)$$

over $Id_{N \otimes_{A^G} A}$. By 10.6, Ψ is an isomorphism of complexes.

By (1), 10.4 (iii) and the fact that $\mathcal{F}$ is a filtration of Spec (A^G) which admits M, there is a unique isomorphism of complexes

$$C(\mathcal{G}'', N \otimes_{A^G} A) \xrightarrow{\cong} C(\mathcal{F}, N) \otimes_{A^G} A \quad (5).$$

over $Id_{N \otimes_{A^G} A}$.

The claim follows form (3) , (4) and (5).

10.10 DEFINITION. Let $C^\bullet$ be a complex of R-modules R-homomorphisms, write $C^\bullet$ as

$$\cdots \longrightarrow C^{n-1} \xrightarrow{e^{n-1}} C^n \xrightarrow{e^n} C^{n+1} \longrightarrow \cdots ,$$

and let K be a group of isomorphism of complexes from $C^\bullet$ to $C^\bullet$. For each integer n, set

$$(C^n)^K := \{x \in C^n : h^n(x) = x \text{ for all } h = (h^n) \in K\}.$$

Then $(C^n)^K$ is an R-module and $e^n((C^n)^K) \subseteq (C^{n+1})^K$, for each integer n. If u^n denotes the restriction of e^n to $(C^n)^K$ (for each integer n), then

$$\cdots \longrightarrow (C^{n-1})^K \xrightarrow{u^{n-1}} (C^n)^K \xrightarrow{u^n} (C^{n+1})^K \longrightarrow \cdots$$

is a subcomplex of $C^\bullet$; we denote this subcomplex by $(C^\bullet)^K$.

10.11 NOTATION AND DEFINITION. Let the situation and notation be as in 10.8.

(i) Write $C(\mathcal{F}, N)$ as

$$0 \longrightarrow N \xrightarrow{e^{-1}} N^0 \xrightarrow{e^0} N^1 \xrightarrow{e^1} \cdots \longrightarrow N^n \xrightarrow{e^n} N^{n+1} \longrightarrow \cdots .$$

Each $\sigma \in G$ induces an isomorphism of complexes of A^G-modules and A^G-homomorphism

$$\sigma_* = (\sigma_*^i)_{i \geq -2} : C(\mathcal{F}, N) \otimes_{A^G} A \longrightarrow C(\mathcal{F}, N) \otimes_{A^G} A$$

Which is such that $\sigma_*^i = Id_{N^i} \otimes \sigma$ for each $i \geq -2$. Set $G_* = \{\sigma_* : \sigma \in G\}$. Then it is clear that G_* is a group of isomorphisms of complexes from $C(\mathcal{F}, N) \otimes_{A^G} A$ to $C(\mathcal{F}, N) \otimes_{A^G} A$. We shall denote the $(C(\mathcal{F}, N) \otimes_{A^G} A)^{G_*}$ by $(C(\mathcal{F}, N) \otimes_{A^G} A)^G$.

(ii) By 10.9, there is exactly one isomorphism of complexes

$$\Psi = (\psi^i)_{i \geq -2} : C(\mathcal{G}, N \otimes_{A^G} A) \longrightarrow C(\mathcal{F}, N) \otimes_{A^G} A$$

over $Id_{N \otimes_{A^G} A}$. Each $\sigma \in G$ induces an isomorphism of complexes of A^G-modules and A^G-homomorphisms

$$\sigma_{**} = (\sigma_{**}^i)_{i \geq -2} : C(\mathcal{G}, N \otimes_{A^G} A) \longrightarrow C(\mathcal{G}, N \otimes_{A^G} A)$$

which is such that $\sigma_{**}^i = (\psi^i)_0^{-1} \sigma_*^i o \psi^i$ for each $i \geq -2$. Set $G_{**} = \{\sigma_{**} : \sigma \in G\}$. It is clear that G_{**} is a group of isomorphism of complexes from $C(\mathcal{G}, N \otimes_{A^G} A)$ to

$C(\mathcal{G}, N \otimes_{A^G} A)$. We shall denote $(C(\mathcal{G}, N \otimes_{A^G} A))^{G_{**}}$ by $(C(\mathcal{G}, N \otimes_{A^G} A))^G$. Write $C(\mathcal{G}, N \otimes_{A^G} A)$ as

$$0 \longrightarrow N \otimes_{A^G} A \xrightarrow{w^{-1}} L^0 \xrightarrow{w^0} L^1 \xrightarrow{w^1} \cdots \longrightarrow L^n \xrightarrow{w^n} L^{n+1} \longrightarrow \cdots.$$

We shall denote $(L^i)^{G_{**}}$ (for each $i \geq -2$) by $(L^i)^G$. We use u^i to denote the restriction of w^i to $(L^i)^G$, for each $i \geq -2$.

(iii) For every $i \geq -2$, each $\sigma \in G$ induces and A^G-isomorphism

$$\overline{\sigma_{**}^i} : H^i(C(\mathcal{G}, N \otimes_{A^G} A)) \longrightarrow H^i(C(\mathcal{G}, N \otimes_{A^G} A))$$

which is such that $\overline{\sigma_{**}^i}(x + Imw^{i-1}) = \sigma_{**}^i(x) + Imw^{i-1}$ for all $x \in \ Ker\ w^i$. We define

$$\left(H^i(C(\mathcal{G}, N \otimes_{A^G} A))\right)^G = \{y \in H^i(C(\mathcal{G}, N \otimes_{A^G} A)) : \overline{\sigma_{**}^i}(y) = y \text{ for all } \sigma \in G\}$$

for all $i \geq -2$.

10.12 THEOREM. Let the situation and notaion be as in 10.8 and 10.11. Then

(i) there is an isomorphism of complexes of A^G-modules and A^G-homomorphisms

$$(C(\mathcal{G}, N \otimes_{A^G} A))^G \xrightarrow{\cong} C(\mathcal{F}, N)$$

(ii) $\left(H^i(C(\mathcal{G}, N \otimes_{A^G} A))\right)^G$ and $H^i(C(\mathcal{F}, N))$ are A^G-isomorphic, for each $i \geq -1$.

Proof. (i) It is clear that

$$(\psi^i|_{(L^i)^G})_{i \geq -2} : (C(\mathcal{G}, N \otimes_{A^G} A))^G \longrightarrow (C(\mathcal{F}, N) \otimes_{A^G} A)^G \quad (1)$$

is an isomorphism of complexes of A^G-modules and A^G-homomorphisms. Let ρ' be the inclusion map $A^G \longrightarrow A$. Then, by 8.7, the homomorphism of complexes of A^G-modules and A^G-homomorphism

$$(Id_{N^i} \otimes \rho')_{i\geq -2} : C(\mathcal{F}, N) \otimes_{A^G} A^G \longrightarrow (C(\mathcal{F}, N) \otimes_{A^G} A)^G \quad (2)$$

is an isomorphism. The claim follows from (1) , (2) and the fact that the comploxes $C(\mathcal{F}, N) \otimes_{A^G} A^G$ and $C(\mathcal{F}, N)$ are isomorphic of complexes.

(ii) By (i), it is enough to prove that $\left(H^i(C(\mathcal{G}, N \otimes_{A^G} A))\right)^G$ and $H^i\left((C(\mathcal{G}, N \otimes_{A^G} A))^G\right)$ are A^G-isomorphic, for each $i \geq -1$.

$(C(\mathcal{G}, N \otimes_{A^G} A))^G$ is a subcomplex of $C(\mathcal{G}, N \otimes_{A^G} A)$. Therefore, there is a monomorphism of complexes of A^G-modules and A^G-homomorphisms

$$\Theta = (\theta^i)_{i\geq -2} : (C(\mathcal{G}, N \otimes_{A^G} A))^G \longrightarrow C(\mathcal{G}, N \otimes_{A^G} A)$$

which is such that, for each $i \geq -2, \theta^i(x) = x$ for all $x \in (L^i)^G$.Θ induces a homomorphism of complexes of A^G-modules and A^G-homomorphisms

$$\overline{\Theta} = (\overline{\theta^i})_{i\geq -2} : H^i((C(\mathcal{G}, N \otimes_{A^G} A))^G) \longrightarrow \left(H^i(C(\mathcal{G}, N \otimes_{A^G} A))\right)^G$$

which is such that, for each $i \geq -2$, $\overline{\theta^i}(x + Imu^{i-1}) = x + Imw^{i-1}$ for all $x \in \ Ker\ u^i$. We show that $\overline{\Theta}$ is an isomorphism.

Assume that $i \geq -1$. We show that $\overline{\theta^i}$ is surjective. Let $x + Imw^{i-1} \in \left(H^i(C(\mathcal{G}, N \otimes_{A^G} A))\right)^G$. Then

$$x \in \ Ker\ w^i \text{ and } x + Imw^{i-1} = \left(\frac{1}{|G|}\sum_{\sigma\in G} \sigma^i_{**}(x)\right) + Imw^{i-1}.$$

Let $y = \frac{1}{|G|}\sum_{\sigma\in G} \sigma^i_{**}(x)$. Then $y \in \ Ker\ u^i$ and $\overline{\theta^i}(y + Imu^{i-1}) = y + Imw^{i-1} = x + Imw^{i-1}$.

Now, we show that $\overline{\theta^i}$ is injective. Let $y + Imu^{i-1} \in H^i((C(\mathcal{G}, N \otimes_{A^G} A))^G)$ be such that $\overline{\theta^i}(y + Imu^{i-1}) = Imw^{i-1}$. Then $y \in Imw^{i-1}$. Hence there exists $x \in L^{i-1}$ such that $y = w^{i-1}(x)$. Therefore, since $y \in (L^i)^G$,

$$y = \frac{1}{|G|} \sum_{\sigma \in G} \sigma^i_{**}(y) = \frac{1}{|G|} \sum_{\sigma \in G} \sigma^i_{**}(w^{i-1}(x)) = w^{i-1}\left(\frac{1}{|G|} \sum_{\sigma \in G} \sigma^{i-1}_{**}(x)\right).$$

Thus, since $\frac{1}{|G|} \sum_{\sigma \in G} \sigma^{i-1}_{**}(x) \in (L^{i-1})^G$, we have $y \in Imu^{i-1}$.

10.13 COROLLARY. (i) Let $\mathcal{G} = (G_i)_{i \in \mathbb{N}_{\nvdash}}$ be the dimension filtration of Spec (A). which is difined by

$$G_i = \{\mathfrak{q} \in \text{Spec }(\mathfrak{A}) : \mathfrak{dim}(\mathfrak{A}/\mathfrak{q}) \leq \mathfrak{dim}\mathfrak{A} - \mathfrak{i}\}$$

for each $i \in \mathbb{N}_{\nvdash}$. There is an isomorphism of A^G-modules and A^G-homomorphisms from $(C(\mathcal{G}, N \otimes_{A^G} A))^G$ to $C(\mathcal{D}, N)$, the Cousin complex for N with respect to $\mathcal{D} = (D_i)_{i \in \mathbb{N}_{\nvdash}}$, where $\mathcal{D}$ is the dimension filtration of Spec (A^G). Furthermore, $H^i(C(\mathcal{D}, N))$ and $\left(H^i(C(\mathcal{G}, N \otimes_{A^G} A))\right)^G$ are A^G-isomorphic, for each $i \geq -1$.

(ii) Let $\mathcal{G} = (G_i)_{i \in \mathbb{N}_{\nvdash}}$ be the $N \otimes_{A^G} A$-height filtration of Spec (A), which is defined by

$$G_i = \{\mathfrak{q} \in \text{Supp}_{\mathfrak{A}}(\mathfrak{N} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{A}) : \mathfrak{ht}_{\mathfrak{N} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{A}}(\mathfrak{q}) \geq \mathfrak{i}\}$$

for each $i \in \mathbb{N}_{\nvdash}$. Then there is an isomorphism of complexes of A^G-modules and A^G-homomorphisms from $(C(\mathcal{G}, N \otimes_{A^G} A))^G$ to $C(\mathcal{H}, N)$, the Cousin complex for N with respect to $\mathcal{H} = (H_i)_{i \in \mathbb{N}}$, where $\mathcal{H}$ is the N-hight filtration of Spec (A^G). Furthermore, $H^i(C(\mathcal{H}, N))$ and $\left(H^i(C(\mathcal{G}, N \otimes_{A^G} A))\right)^G$ are A^G-isomorphic, for each $i \geq -1$.

Proof. (i) By 10.12, it is enough for us to show that

$$D_i = \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \sigma(\mathfrak{q}) \in \mathfrak{G}_{\mathfrak{i}} \text{ for all } \sigma \in \mathfrak{G}\}$$

for each $i \in \mathbb{N}_{\nvdash}$.

Let $i \in \mathbb{N}_{\nvdash}$. There exists an isomorphism of rings from $A/\mathfrak{q}$ to $A/\sigma(\mathfrak{q})$ for all $\mathfrak{q} \in$ Spec (A) and $\sigma \in G$. Hence $dim\ (A/\sigma(\mathfrak{q})) = \mathfrak{dim}(\mathfrak{A}/\mathfrak{q})$ for all $\mathfrak{q} \in$ Spec (A) and $\sigma \in G$. Therefore.

$$\{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathfrak{G}_{\mathfrak{i}}\} = \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \sigma(\mathfrak{q}) \in \mathfrak{G}_{\mathfrak{i}} \text{ for all } \sigma \in \mathfrak{G}\}. \quad (1)$$

It follows from 10.2 (vi) that

$$\{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathfrak{G}_{\mathfrak{i}}\} \subseteq \mathfrak{D}_{\mathfrak{i}}.$$

Let $\mathfrak{p} \in$ Spec (A^G) be such that $dim(A^G/\mathfrak{p}) \leq \mathfrak{dim}\mathfrak{A}^{\mathfrak{G}} - \mathfrak{i}$. Then, by 10.2 (i) and the Lying over theorem [1, 5.10], there exists $\mathfrak{q} \in$ Spec (A) such that $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$. Hence, by 10.2 (vi), $\mathfrak{q} \in \mathfrak{G}_{\mathfrak{i}}$. Thus $\mathfrak{p} \in \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathfrak{G}_{\mathfrak{i}}\}$. Therefore

$$\{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathfrak{G}_{\mathfrak{i}}\} = \mathfrak{D}_{\mathfrak{i}}. \quad (2)$$

The claim follows from (1) and (2).

(ii) By 10.12, it is enough for us to show that

$$H_i = \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \sigma(\mathfrak{q}) \in \mathfrak{G}_{\mathfrak{i}} \text{ for all } \sigma \in \mathfrak{G}\}$$

for each $i \in \mathbb{N}_{\nvdash}$.

Let $i \in \mathbb{N}_{\nvdash}$. It is easy to see that $ht_{N \otimes_{A^G} A}(\sigma(\mathfrak{q})) = \mathfrak{ht}_{\mathfrak{N} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{A}}(\mathfrak{q})$ for all $\mathfrak{q} \in$ Supp $(N \otimes_{A^G} A)$. Hence

$$\{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathfrak{G}_{\mathfrak{i}}\} = \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \sigma(\mathfrak{q}) \in \mathfrak{G}_{\mathfrak{i}} \text{ for all } \sigma \in \mathfrak{G}\}. \quad (3)$$

It follows from 10.2 (vii) and 10.2 (v) that

$$\{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathfrak{G}_{\mathfrak{i}}\} \subseteq \mathfrak{H}_{\mathfrak{i}}.$$

Let $\mathfrak{p} \in \mathrm{Supp}_{A^G}(N)$ such that $ht_N(\mathfrak{p}) \geq \mathfrak{i}$. Then, by 10.2 (i) and the Lying over theorem [1, 5.10]. there exists $\mathfrak{q} \in$ Spec (A) such that $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$. Hence, by 10.2 (v) and 10.2 (vii), $\mathfrak{q} \in \mathfrak{G}_{\mathfrak{i}}$. Thus $\mathfrak{p} \in \{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathfrak{G}_{\mathfrak{i}}\}$. Therefore

$$\{\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} : \mathfrak{q} \in \mathfrak{G}_{\mathfrak{i}}\} = \mathfrak{H}_{\mathfrak{i}}. \quad (4)$$

The claim follows from (3) and (4).

§11 Group action on injective envelops.

Throughout this section, G is a finite group of automorphisms of A such that $|G|$, the order of G, is invertible in A and $f : A \longrightarrow B$ is a homomorphism of commutative Noetherian rings. If $\varphi : R \longrightarrow S$ is a homomorphism of commutative rings and $\mathfrak{p} \in$ Spec (R), then we use $F_S(\mathfrak{p})$ to denote the set $\{\mathfrak{q} \in$ Spec $(\mathfrak{S})$: $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}\}$. The hight filtration of Spec (R) with denote by $\mathcal{H}(R)$ (See 5.4). An R-module L is *indecomposable* if $(a) L \neq 0$ and (b) the only direct summands of L are 0 and L itself. Let K be an R-module and $L \subseteq K$ a submodule; we say that K is an *essential extension* of L if $N \cap L \neq (0)$ for every non-zero submodule $N \subseteq K$. Now suppose that L is an R-module and that E is an essential extension of L which is also an injective A-module. Then E is called an *injective envelop* of L, and writen as $E(L)$ or $E_R(L)$.

We shall use the following theorems (11.1, 11.2, 11.3, 11.4, 11.5 and 11.8 (i)) in this seciton.

11.1 THEOREM. (See [17, 18.4]) Let $\mathfrak{p} \in$ Spec (A)

(i) $E(A/\mathfrak{p})$ is indecomposable.

(ii) Any indecomposable injective A-module is of the form $E(A/\mathfrak{p})$ for some $\mathfrak{p} \in$ Spec (A).

(iii) For any $x \in E(A/\mathfrak{p})$ there exists a positive integer t (depending on x) such that $\mathfrak{p}^{\mathfrak{t}}\mathfrak{x} = \mathfrak{o}$

(iv) If $\mathfrak{q} \in$ Spec (A) is such that $\mathfrak{q} \subseteq \mathfrak{p}$, the $E(A/\mathfrak{q})$ is an $A_{\mathfrak{p}}$-module and $E_A(A/\mathfrak{q}) \cong \mathfrak{E}_{\mathfrak{A}_{\mathfrak{p}}}(\mathfrak{A}_{\mathfrak{p}}/\mathfrak{q}\mathfrak{A}_{\mathfrak{p}})$.

(v) if $a \in A - \mathfrak{p}$, The multiplication by a induces an automorphism of $E(A/\mathfrak{p})$.

11.2 THEOREM. (See [17, 18.5]) We consider modules over A.

(i) A direct sum of any number of injective modules is injetive.

(ii) Every injective module is a direct sum of indecomposable injective modules.

(iii) The direct sum decomposition in (ii) is unique, in the sense that if

$$M = \oplus M_i \text{ (with indecomposable } M_i),$$

then for any $\mathfrak{p} \in$ Spec (A), the sum $M(\mathfrak{p})$ of all the M_i isomorphic to $E(A/\mathfrak{p})$ depends only on M and $\mathfrak{p}$, and not on the decomposition $M = \oplus M_i$. Moreover, the number of M_i isomorphic to $E(A/\mathfrak{p})$ is equal to

$$dim_{k(\mathfrak{p})} Hom_{A_\mathfrak{p}}(\mathcal{K}(\mathfrak{p}), \mathfrak{M}_\mathfrak{p}), (\mathfrak{where}\ \mathcal{K}(\mathfrak{p}) = \mathfrak{A}_\mathfrak{p}/\mathfrak{p}\mathfrak{A}_\mathfrak{p}).$$

11.3 DEFINITION AND REMARK. By means of 11.2, for an injective A-module E, one can write

$$E \cong \bigoplus_{\mathfrak{p} \in \text{Spec }(\mathfrak{A})} \mu(\mathfrak{p}, \mathfrak{E})\mathfrak{E}(\mathfrak{A}/\mathfrak{p}),$$

where, for all $\mathfrak{p} \in$ Spec (A), $\mu(\mathfrak{p}, \mathfrak{E})\mathfrak{E}(\mathfrak{A}/\mathfrak{p})$ denotes the direct sum of $\mu(\mathfrak{p}, \mathfrak{E})$ copies of $E(A/\mathfrak{p})$. Note that, by 11.2,

$$dim_{k(\mathfrak{p})} Hom_{A_\mathfrak{p}}(\mathcal{K}(\mathfrak{p}), \mathfrak{E}_\mathfrak{p}) = \mu(\mathfrak{p}, \mathfrak{E})$$

for all $\mathfrak{p} \in$ Spec (A). Also, for a multiplicative set $S \subseteq A$,

$$\mu(\mathfrak{p}, \mathfrak{E}) = \mu(\mathfrak{S}^{-1}\mathfrak{p}, \mathfrak{S}^{-1}\mathfrak{E})$$

for all $\mathfrak{p} \in$ Spec (A) where $\mathfrak{p} \cap \mathfrak{S} = \emptyset$ (see [17, page 150]). Note that, $S^{-1}E$ is an injective S^{-1} A-module (see [17, page 144, lemma 5]).

11.4 THEOREM. (see [17, 18.8]) A necessary and sufficient condition for a rign A to be Gorenstein is that a minimal injective resolution

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \cdots \longrightarrow \cdots I^i \longrightarrow \cdots$$

of A satisfies $I^i \cong \bigoplus_{ht\mathfrak{p}=\mathfrak{i}} E(A/\mathfrak{p})$, for each $i \in \mathbb{N}_{\not{\kappa}}$.

11.5 THEOREM. (see [15, 3.4 (4)]) Let $\mathfrak{p} \in$ Spec (A) and $A_i = \{x \in E(A/\mathfrak{p}) : \mathfrak{p}^{\mathfrak{i}}\mathfrak{x} = \mathfrak{o}\}$ for each $i \in \mathbb{N}_{\not{\kappa}}$. Let K be the quotient field of $A/\mathfrak{p}$. Then A_{i+1}/A_i is a rector space over K, and $A_1 \cong K$.

Next we shall prove theorem [28, 4.3] under weaker conditions. Indeed to do this we shall use the same argument as in the proof of [28, 4.3] in conjanction with the following results 11.6, 11.7 and 11.8.

11.6 REMARK. (see [28, (2.1) and (2.2)]) Let $\mathfrak{p} \in$ Spec (A) and $S = f(A - \mathfrak{p})$. We may form the possibly trivial ring $S^{-1}B$; we shall use $f' : A_{\mathfrak{p}} \longrightarrow S^{-1}B$ to denote, the ring homomorphism for which $f'(a/t) = f(a)/f(t)$ (for $a \in A$ and $t \in A - \mathfrak{p}$). It is easy to see that

$$F_{S^{-1}B}(\mathfrak{p}\mathfrak{A}_{\mathfrak{p}}) = \{\mathfrak{S}^{-1}\mathfrak{q} : \mathfrak{q} \in \mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})\}.$$

Let $f : A \longrightarrow B$ be a ring homomorphism with the property that B, when regarded as an A-module by means of f, is finitely generated. Then $S^{-1}B$ is a finitely generated $A_{\mathfrak{p}}$-module; hence it is integral over $A_{\mathfrak{p}}$ [1, 5.1]. It follows from 5.8 of [1] that $F_{S^{-1}B}(\mathfrak{p}\mathfrak{A}_{\mathfrak{p}})$ is equal to the set of maximal ideals of $S^{-1}B$.

11.7 LEMMA. (See [28, 3.3]) Let E be an injective A-module, and $\mathfrak{p} \in$ Spec (A). Then $Hom_A(B, E)$, when regarded as a B-module in the natural way, is B-

injective.

11.8 LEMMA. (see [28, 3.4]) Let $\mathfrak{p}$ be a prime ideal of A, and suppose $\mathfrak{q} \in$ Spec $(B)$is such that the injective B-module $Hom_A(B, E_A(A/\mathfrak{p}))$ has a direct summand which is B-isomorphic to $E_B(B/\mathfrak{q})$. Then

(i) $f^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$;

(ii) if B, when regarded as an A-module by means of f, is finitely generated, then $f^{-1}(\mathfrak{q}) = \mathfrak{p}$.

11.9 THEOREM. Let $f : A \longrightarrow B$ be a ring homomorphism with the property that B, when regarded as an A-module by means of f, is a finitely generated A-module, and let $\mathfrak{p} \in$ Spec (A). Then

$$Hom_A(B, E_A(A/\mathfrak{p})) \cong \bigoplus_{\mathfrak{q} \in \mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} \mathfrak{E}_{\mathfrak{B}}(\mathfrak{B}/\mathfrak{q})(\text{as } \mathfrak{B} - \text{modules}).$$

Proof. (Note that, we repeat the proof of [28, 4.3]). By 11.7, $Hom_A(B, E_A(A/\mathfrak{p}))$, when regarded as a B-module in the natural way, is B-injective; furthermore, by 11.8, when this injective B-module is expressed as a direct sum of indecomposable injective B-modules, the only prime ideals which can occur ((11.2)) in this decomposition are the members of $F_B(\mathfrak{p})$. Let $\mathfrak{q}$ be a typical member of $F_B(\mathfrak{p})$, and let H denote the B-module $Hom_A(B, E_A(A/\mathfrak{p})$; it is enough to show that $\mu(\mathfrak{q}, \mathfrak{H}) = \mathfrak{1}$.

However, $S = f(A - \mathfrak{p})$ doesnot meet $\mathfrak{q}$, so it is sufficient ((11.3)) to show that $\mu(S^{-1}\mathfrak{q}, \mathfrak{S}^{-1}\mathfrak{H}) = \mathfrak{1}$. Now $S^{-1}H$ can be regarded as an $A_{\mathfrak{p}}$-module by means of $f' : A_{\mathfrak{p}} \longrightarrow S^{-1}B$, and there are obvious natural $A_{\mathfrak{p}}$-isomorphisms

$$S^{-1}H \cong H_{\mathfrak{p}} \cong Hom_{A_{\mathfrak{p}}}(B_{\mathfrak{p}}, [E_A(A/\mathfrak{p})]_{\mathfrak{p}}).$$

However, $B_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-isomorphic to $S^{-1}B$, and by 11.1 (iv), $[E_A(A/\mathfrak{p})]_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-isomorphic to $E_{A_\mathfrak{p}}(\mathcal{K}_A(\mathfrak{p}))$. Hence, we arrive at a natural $A_\mathfrak{p}$-isomorphism

$$S^{-1}H \cong Hom_{A_\mathfrak{p}}(S^{-1}B, E_{A\mathfrak{p}}(\mathcal{K}_A(\mathfrak{p}))),$$

and it is straightforward to check that this is , in fact, an $S^{-1}B$-isomorphism. Hence it is enough for us to show that $\mu(S^{-1}\mathfrak{q}, \mathfrak{Hom}_{\mathfrak{A}_\mathfrak{p}}(\mathfrak{S}^{-1}\mathfrak{B}, \mathfrak{E}_{\mathfrak{A}_\mathfrak{p}}(\mathcal{K}(\mathfrak{p})))) = \mathfrak{1}$. Morever, by 11.6, $S^{-1}\mathfrak{q} \in \mathfrak{F}_{\mathfrak{S}^{-1}\mathfrak{B}}(\mathfrak{p}\mathfrak{A}_\mathfrak{p})$ and $S^{-1}\mathfrak{q}$ is a maximal ideal of $S^{-1}B$. It follows that, in order to show that $\mu(\mathfrak{q}, \mathfrak{H}) = \mathfrak{1}$, we may in addition assume that A is local and $\mathfrak{p}$ is the maximal ideal of A and $\mathfrak{q}$ is a maximal ideal of B. By 11.3 and the fact that

$$Hom_{B_\mathfrak{q}}(\mathcal{K}_B(\mathfrak{q}), [Hom_A(B, E_A(A/\mathfrak{p}))]_\mathfrak{q})$$

is $B_\mathfrak{q}$-isomorphic to $[Hom_B(B/\mathfrak{q}, Hom_A(B, E_A(A/\mathfrak{p})))]_\mathfrak{q}$, it is enough to show that the dimension of $[Hom_B(B/\mathfrak{q},\ Hom_A(B, E_A(A/\mathfrak{p})))]_\mathfrak{q}$ as a vector space over $\mathcal{K}_B(\mathfrak{q})$ is 1. It is easy to see that

$$dim_{B/\mathfrak{q}}[Hom_B(B/\mathfrak{q}, Hom_A(B, E_A(A/\mathfrak{p})))] = dim_{\mathcal{K}_B(\mathfrak{q})}[Hom_B(B/\mathfrak{q}, Hom_A(B, E_A(A/\mathfrak{p})))]_\mathfrak{q} :$$

Use of this fact reduces the problem to the case in which showing that the dimension of $Hom_B\ (B/\mathfrak{q}, Hom_A(B, E_A(A/\mathfrak{p})))$ as a vector space over $B/\mathfrak{q}$ in 1.

Next, there is a B-isomorphism

$$Hom_B(B/\mathfrak{q}, Hom_A(B, E_A(A/\mathfrak{p}))) \cong Hom_A(B/\mathfrak{q}, E_A(A/\mathfrak{p})).$$

But any A-homomorphism $B/\mathfrak{q} \longrightarrow E_A(A/\mathfrak{p})$ has its image comtained in the A-submodule $X = \{z \in E_A(A/\mathfrak{p}) : \mathfrak{p}\mathfrak{z} = \mathfrak{o}\}$ of $E_A(A/\mathfrak{p})$; since ((11.5)) X is A-isomorphic to $A/\mathfrak{p}$, we have reduced the problem to showing that $Hom_A(B/\mathfrak{q}, A/\mathfrak{p})$, as vector space over $B/\mathfrak{q}$, has dimension 1. However, $Hom_A(B/\mathfrak{q}, A/\mathfrak{p}) = Hom_{A/\mathfrak{p}}(B/\mathfrak{q}, A/\mathfrak{p})$, and the result follows from a easy argument on finite dimensional vector spaces.

11.10 COROLLARY. Let $\mathfrak{p} \in \text{Spec } (A^G)$. Then

(i) Each $\sigma \in G$ induces an A^G-automorphism σ^* on

$$E := \bigoplus_{\substack{\mathfrak{q}\in \ Spec \ (A) \\ \mathfrak{q}\cap A^G \subseteq \mathfrak{p}}} \mu\left(\mathfrak{q}, Hom_{A^G}(A, E(A^G/\mathfrak{p}))\right) E(A/\mathfrak{p})$$

such that $G^* = \{\sigma^* : \sigma \in G\}$ is a group and if

$$E^G := \{x \in E : \sigma^*(x) = x \text{ for all } \sigma \in G\},$$

then $E(A^G/\mathfrak{p})$ and E^G are A^G-isomorphic.

(ii) If A, when regarded as an A^G-module, is finitely generated, then each $\sigma \in G$ induces an A^G-automorphism σ^* on $\bigoplus_{\mathfrak{q}\in F_A(\mathfrak{p})} E(A/\mathfrak{q})$ such that $G^* = \{\sigma^* : \sigma \in G\}$ is a group and if

$$(\bigoplus_{\mathfrak{q}\in F_A(\mathfrak{p})} E(A/\mathfrak{q}))^G := \{x \in \bigoplus_{\mathfrak{q}\in F_A(\mathfrak{p})} E(A/\mathfrak{q}) : \sigma^*(x) = x \text{ for all } \sigma \in G\},$$

then $E_{A^G}(A^G/\mathfrak{p})$ and $(\bigoplus_{\mathfrak{q}\in F_A(\mathfrak{p})} E(A/\mathfrak{q}))^G$ are A^G-isomorphic.

Proof. (i) It follows from 11.3, 11.7 and 11.8 that

$$Hom_{A^G}(A, E(A^G/\mathfrak{p})) \cong \bigoplus_{\substack{\mathfrak{q}\in \ Spec \ (A) \\ \mathfrak{q}\cap A^G \subseteq \mathfrak{p}}} \mu\left(\mathfrak{q}, \mathfrak{Hom}_{\mathfrak{A}^\mathfrak{G}}(\mathfrak{A}, \mathfrak{E}(\mathfrak{A}^\mathfrak{G}/\mathfrak{p}))\right) \mathfrak{E}(\mathfrak{A}/\mathfrak{p}) := \mathfrak{E}(\text{as } \mathfrak{A}-\text{modules}).$$

Hence A^G-automorphism $Hom_{A^G}(\sigma, Id_{E(A^G/\mathfrak{p})})$ induces and A^G-automorphism σ^* on E, for each $\sigma \in G$. It is easy to see that $G^* = \{\sigma^* : \sigma \in G\}$ is a group and

$$\left(Hom_{A^G}(A, E(A^G/\mathfrak{p}))\right)^G \cong E^G(\text{as } A^G - \text{modules}),$$

where $\left(Hom_{A^G}(A, E(A^G/\mathfrak{p}))\right)^G = \{x \in Hom_{A^G}(A, E(A^G/\mathfrak{p})) : \mathfrak{Hom}_{\mathfrak{A}^\mathfrak{G}}(\sigma, \mathfrak{Id}_{\mathfrak{E}(\mathfrak{A}^\mathfrak{G}/\mathfrak{p})})(\mathfrak{x}) = \mathfrak{x}\text{for all}\sigma \in \mathfrak{G}\}$. Hence, by 8.9, $E(A^G/\mathfrak{p})$ and E^G are A^G-isomorphic.

(ii) It follows from 11.9 that

$$Hom_{A^G}(A, E(A^G/\mathfrak{p})) \cong \bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{A}}(\mathfrak{p})} \mathfrak{E}(\mathfrak{A}/\mathfrak{q})(\text{as } \mathfrak{A} - \text{modules}).$$

Similar to the proof of (i), we can prove the claim.

11.11 LEMMA. Let A and B be Gorenstein rings, and let $C(\mathcal{H}(B), B)$ the Cousin complex for B with respect to $\mathcal{H}(B)$ and $C(\mathcal{H}(A), A) \otimes_A B$ be isomorphic (as complexes of B-modules and B-homomorphisms). Let $\mathfrak{p}_\circ \in$ Spec (A). Then

$$B \otimes_A E_A(A/\mathfrak{p}_\circ) \cong \bigoplus_{\substack{ht\mathfrak{q}=ht\mathfrak{p}_\circ \\ \mathfrak{q}\in F_B(\mathfrak{p}_0)}} \mathfrak{E}_{\mathfrak{B}}(\mathfrak{B}/\mathfrak{q})(\text{as } \mathfrak{B} - \text{modules}).$$

Proof. Assume that $ht\mathfrak{p}_\circ = \mathfrak{n}$ and $S = A - \mathfrak{p}_\circ$. It follows from 5.4 that $C(\mathcal{H}(A), A)$ (respectively $C(\mathcal{H}(B), B)$) is a minimal injective resolution of A(respectively B). Hence, by 11.4,

$$B \otimes_A (\bigoplus_{ht\mathfrak{p}=\mathfrak{n}} E(A/\mathfrak{p})) \cong \bigoplus_{\mathfrak{ht}\mathfrak{q}=\mathfrak{n}} \mathfrak{E}(\mathfrak{B}/\mathfrak{q})(\text{as } \mathfrak{B} - \text{modules}).$$

Hence

$$\left(B \otimes_A \bigoplus_{ht\mathfrak{p}=\mathfrak{n}} E(A/\mathfrak{p})\right) \otimes_A S^{-1}A \cong (\bigoplus_{ht\mathfrak{q}=\mathfrak{n}} E(B/\mathfrak{q})) \otimes_{\mathfrak{A}} \mathfrak{S}^{-1}\mathfrak{A}(\text{as } \mathfrak{B} - \text{modules}). \quad (1)$$

Since tensor product commute with direct sum, it follows that, as B-modules,

$$\left(B \otimes_A \bigoplus_{ht\mathfrak{p}=\mathfrak{n}} E(A/\mathfrak{p})\right) \otimes_A S^{-1}A \cong B \otimes_A \left(\bigoplus_{ht\mathfrak{p}=\mathfrak{n}} S^{-1}(E(A/\mathfrak{p}))\right) \quad (2)$$

and

$$(\bigoplus_{ht\mathfrak{q}=\mathfrak{n}} E(B/\mathfrak{q})) \otimes_{\mathfrak{B}} \mathfrak{S}^{-1}\mathfrak{A} \cong \mathfrak{B} \otimes_{\mathfrak{A}} \bigoplus_{\mathfrak{ht}\mathfrak{q}=\mathfrak{n}} \mathfrak{S}^{-1}(\mathfrak{E}(\mathfrak{B}/\mathfrak{p})) \cong \bigoplus_{\mathfrak{ht}\mathfrak{q}=\mathfrak{n}} \mathfrak{f}(\mathfrak{S})^{-1}(\mathfrak{E}(\mathfrak{B}/\mathfrak{q})) \quad (3).$$

Hence, by (1), (2) and (3),

$$\left(B \otimes_A \bigoplus_{ht\mathfrak{p}=\mathfrak{n}} S^{-1}(E(A/\mathfrak{p}))\right) \cong \bigoplus_{ht\mathfrak{q}=\mathfrak{n}} f(S)^{-1}(E(B/\mathfrak{p}))(\text{as } \mathfrak{B}-\text{modules}).$$

It follows form 11.1 (iii) that if $\mathfrak{p} \in$ Spec (A) and $\mathfrak{p} \cap \mathfrak{S} \neq \emptyset$ (respectively $\mathfrak{q} \in$ Spec (B) and $f(S) \cap \mathfrak{q} \neq \emptyset$), then $S^{-1}(E(A/\mathfrak{p})) = \mathfrak{o}$ (respectively $f(S)^{-1}(E(B/\mathfrak{q})) = \mathfrak{o}$). Also, by 11.1 (v), if $\mathfrak{p} \in$ Spec (A) and $\mathfrak{p} \cap \mathcal{S} = \emptyset$ (respectively $\mathfrak{q} \in$ Spec (B) and $\mathfrak{q} \cap \mathfrak{f}(\mathfrak{S}) = \emptyset$), then $E(A/\mathfrak{p})$ (respectively $E(B/\mathfrak{q})$) has a natural structere as an $S^{-1}A$-module (respectively $f(S)^{-1}B$-module), so that $S^{-1}(E(A/\mathfrak{p})) \cong \mathfrak{E}(\mathfrak{A}/\mathfrak{p})$ (respectively $S^{-1}(E(B/\mathfrak{q})) \cong \mathfrak{E}(\mathfrak{B}/\mathfrak{q})$) as an $S^{-1}A$-module (respectively $f(S)^{-1} - B$ module). Hence

$$B \otimes_A \bigoplus_{\substack{ht\mathfrak{p}=n \\ \mathfrak{p}\subseteq\mathfrak{p}_0}} E(A/\mathfrak{p})) \cong \bigoplus_{\substack{ht\mathfrak{q}=n \\ f^{-1}(\mathfrak{q})\subseteq\mathfrak{p}_0}} \mathfrak{E}(\mathfrak{B}/\mathfrak{q})(\text{as } \mathfrak{B}-\text{modules}).$$

Since $ht\mathfrak{p}_{\mathfrak{o}} = \mathfrak{n}$,

$$B \otimes_A E(A/\mathfrak{p}_{\mathfrak{o}}) \cong \bigoplus_{\substack{ht\mathfrak{q}=n \\ f^{-1}(\mathfrak{q})\subseteq\mathfrak{p}_0}} \mathfrak{E}(\mathfrak{B}/\mathfrak{q})(\text{as } \mathfrak{B}-\text{modules}). \quad (4)$$

Let $\mathfrak{q} \in$ Spec (B) such that the $B \otimes_A E(A/\mathfrak{p}_{\mathfrak{o}})$ has a direct summand which is B-isomorphic to $E(B/\mathfrak{q})$. It is enough for us, in order to complete the proof, to show that $\mathfrak{p}_0 \subseteq f^{-1}(\mathfrak{q})$. Let $r \in \mathfrak{p}_0$. Then, by 11.1 (iii) and (4), for any $x \in E(B/\mathfrak{q})$ there exists a positive integer t(depending on x) such that $r^t x = 0$. Hence multiplication by r on $E(B/\mathfrak{q})$ does not provide an automorphism. But multivlication by r on $E(B/\mathfrak{q})$ has the some effect as multivlication by $f(r)$. Hence, by 11.1 (v), $f(r) \in \mathfrak{q}$.

11.12 COROLLARY. Let A and A^G be Gorenstion rings and $\mathfrak{p} \in$ Spec (A^G). Then

(i) $A \otimes_{A^G} E(A^G/\mathfrak{p}) \cong \bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{A}}(\mathfrak{p})} E(A/\mathfrak{q})$ (as A-modules).

(ii) Each $\sigma \in G$ induces an A^G-automorphism σ^* on $\bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})} E(A/\mathfrak{q})$ such that $G^* = \{\sigma^* : \sigma \in G\}$ is a group and if

$$(\bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})} E(A/\mathfrak{q}))^{\mathfrak{G}} = \{\mathfrak{x} \in \bigoplus_{\mathfrak{q} \in \mathfrak{F}_{\mathfrak{A}}(\mathfrak{p})} \mathfrak{E}(\mathfrak{A}/\mathfrak{q}) : \sigma^*(\mathfrak{x}) = \mathfrak{x} \text{ for all} \sigma \in \mathfrak{G}\},$$

then $E(A^G/\mathfrak{p})$ and $(\bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})} E(A/\mathfrak{q}))^{\mathfrak{G}}$ are A^G-isomorphic.

Proof. (i) It follows from 10.2 (vii) and 5.21 that there is an isomorphism of complexes of A-modules and A-homomorphisms from $C(\mathcal{H}(A^G), A^G) \otimes_{A^G} A$ to $C(\mathcal{H}(A), A)$ over Id_A, Hence the cliam follows from 11.11.

(ii) By (i), A^G-automorphism $\sigma \otimes Id_{E(A^G/\mathfrak{p})}$ induces an A^G-automorphism σ^* on $\bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})} E(A/\mathfrak{q})$, for each $\sigma \in G$. It is clear that $G^* = \{\sigma^* : \sigma \in G\}$ is a group and

$$(A \otimes_{A^G} E(A^G/\mathfrak{p}))^{\mathfrak{G}} \cong (\bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})} \mathfrak{E}(\mathfrak{A}/\mathfrak{q}))^{\mathfrak{G}} (\text{as } \mathfrak{A}^{\mathfrak{G}} - \text{modules})$$

where

$$(A \otimes_{A^G} E(A^G/\mathfrak{p}))^{\mathfrak{G}} = \{\mathfrak{x} \in \mathfrak{A} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{E}(\mathfrak{A}^{\mathfrak{G}}/\mathfrak{p}) : (\sigma \otimes \mathfrak{Id}_{\mathfrak{E}(\mathfrak{A}^{\mathfrak{G}}/\mathfrak{p})})(\mathfrak{x}) = \mathfrak{x} \text{ for all } \sigma \in \mathfrak{G}\}.$$

Hence the cliam follows from 8.8.

We show next how to use 11.10 (ii) to obtain a generalization of the main result of [43].

11.13 EXAMPLEXE. Let k be a field and A be a Noetherian k-algebra, and let K be a finite, separable, and normal extension field of k for which $B = A \otimes_k K$ is a Noetherian ring. Let $\Gamma := Gal(K : k)$ denote the Galois group of K over k (that is, group of maps which are both field and k-module automorphisms of K). By 12.2, $K^{\Gamma} = k$. Suppose that $f : A \longrightarrow B$ is the natural homomorphism of rings.

By [31, (2.2)], f is faithfully flat. Hence f is injective. Let $\mathfrak{p}$ be a prime ideal of A. By [43 , 3.7], if A is a Gorenstion ring, then $Id_A \otimes \sigma$ induces an A-automorphism σ^* on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q})$ for all $\sigma \in \Gamma$ such that if

$$(\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q}))^{\mathfrak{d}} := \{\mathfrak{x} \in (\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} \mathfrak{E}(\mathfrak{B}/\mathfrak{q})) : \sigma^*(\mathfrak{x}) = \mathfrak{x} \text{ for all } \sigma \in \mathfrak{d}\},$$

then $(\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q}))^{\mathfrak{d}}$ is an injective envelope of the A-module $A/\mathfrak{p}$. In [43], this result was proved by use of modules of generalized fractions. If $|\Gamma|$, the order of Γ, is invertible in A, then we shall obtain a similar result, but without imposing any restriction on A and by use of a method which does not involve modules of generalized fractions.

Let $|\Gamma|$, the order of Γ, be invertible in A. It follows from 8.7 and the fact that $K^{\Gamma} = k$ that $B^{\Gamma} = f(A)$ where

$$B^{\Gamma} = \{x \in B : (Id_A \otimes \sigma)(x) = x \text{ for all } \sigma \in \Gamma\}.$$

Hence the ring B^{Γ} and A are isomorphic. Consequenctly, B, when regarded as B^{Γ}-module, is finitely generated, and so it follows from 11.10 (ii) that $Id_A \otimes \sigma$ induces an A-automorphism σ^* an $\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q})$ for all $\sigma \in \Gamma$ such that if

$$(\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q}))^{\mathfrak{d}} = \{\mathfrak{x} \in \bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} \mathfrak{E}(\mathfrak{B}/\mathfrak{q}) : \sigma^*(\mathfrak{x}) = \mathfrak{x} \text{ for all } \sigma \in \mathfrak{d}\},$$

then $(\bigoplus_{\mathfrak{q}\in\mathfrak{F}_{\mathfrak{B}}(\mathfrak{p})} E(B/\mathfrak{q}))^{\mathfrak{d}}$ and $E(A/\mathfrak{p})$ are A-isomorphic.

Note that, if A is a Gorenstion ring, then, by [41, proposition 2], B is also a Gorenstein ring, and so we can deduce the above result from 11.12 (ii).

§ 12 Generalization of 11.13.

Throughout this section, k is a field, A is a Noetherian k-algebra and K is an algebraic extension field of k for which $B = A \otimes_k K$ is a Noetherian ring (Note that, by [30, 2.1], this would be the case if A were a finitely generated k-algebra, or if K were a finitely generated extension field of k). By [31, (2.2) and (2.4)], the natural homomorphism of rings $f : A \longrightarrow B$ is faithfully flat and B is integral over its subring $f(A)$. Hence f is a k-algebra monomorphism. We shall use $\mathfrak{p}$ to denote a typical prime ideal of A, and S will denote $f(A \backslash \mathfrak{p})$, a multiplicatively closed subset of B. We shall denote the set $\{\mathfrak{q} \in \text{ Spec } (\mathfrak{B}) : \mathfrak{f}^{-1}(\mathfrak{q}) = \mathfrak{p}\}$ by $F(\mathfrak{p})$. By [17, (9.5)] and [17, Exercise (9.8) and (9.9)] and the facts that f is flat and B is integral over its subring $f(A)$, $ht_A\mathfrak{p} = \mathfrak{ht}_{\mathfrak{B}}\mathfrak{q}$ for all $\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})$. We shall let $G :=$ Gal $(K : k)$ denote the Galois group of K over k (that is, group of maps which are both field and k-module automorphism of K). We shall use F to denote the fixed subfield

$$\{a \in K : \sigma(a) = a \text{ for all } \sigma \in G\}.$$

For any R-module L, the injective envelope of L is denoted by $E_R(L)$ or $E(L)$.

In this section, the 12.1 (and Also 12.8) extends the result 11.13.

12.1 PROPOSITION. Let K be a finite extension field of k. and let $|G|$, the order of G, is invertible in A. Let $[F : k]$ be the dimension of F considered as a vector space over k. Then each $\sigma \in G$ induces an A-automorphism σ^* on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B\backslash\mathfrak{q})$ such that $G^* = \{\sigma^* : \sigma \in G\}$ is a group and if

$$(\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B\backslash(\mathfrak{q}))^{\mathfrak{G}} := \{\mathfrak{x} \in (\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} \mathfrak{E}(\mathfrak{B}\backslash\mathfrak{q})) : \sigma^*(\mathfrak{x}) = \mathfrak{x} \text{ for all } \sigma \in \mathfrak{G}\},$$

then $\oplus[F : k]E(A\backslash\mathfrak{p})$ and $(\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B\backslash\mathfrak{q}))^{\mathfrak{G}}$ are A-isomorphic, where $\oplus[F : k]E(A/\mathfrak{p})$

denotes a direct sum of $[F:k]$ copies of $E(A/\mathfrak{p})$.

Proof. B, when regarded as A-module, is finitely generated. Hence, by 11.9,

$$Hom_A(B, E(A/\mathfrak{p})) \cong \bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} \mathfrak{E}(\mathfrak{B}/\mathfrak{q}) \quad (\text{as } \mathfrak{B}-\text{modules}).$$

Hence $Hom_A(Id_A \otimes \sigma, Id_{E(A/\mathfrak{p})})$ induces an A-automorphism σ^* on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q})$ for all $\sigma \in G$. It is easy to see that $G^* = \{\sigma^* : \sigma \in G\}$ is a group and

$$(Hom_A(B, E(A/\mathfrak{p})))^G \cong \left(\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q})\right)^G \quad (\text{as } A-\text{modules}), \ (1)$$

where

$$(Hom_A(B, E(A/\mathfrak{p})))^G = \{x \in Hom_A(B, E(A/\mathfrak{p})) : \mathfrak{Hom}_{\mathfrak{A}}(\mathfrak{Id}_{\mathfrak{A}} \otimes \sigma, \mathfrak{Id}_{\mathfrak{E}(\mathfrak{A}/\mathfrak{p})})(\mathfrak{x}) = \mathfrak{x} \text{for all} \sigma \in \mathfrak{G}\}.$$

By [18, page 29, Exercise 38], G is finite. Also, $Hom_A(A \otimes_k \bullet, E(A/\mathfrak{p}))$ is an additive contravariant functor from the category of all F-modules and F-homomorphisms to $\mathcal{C}(A)$. Hence, by 8.6, $Hom_A(A \otimes_k F, E(A/\mathfrak{p}))$ and $(Hom_A(B, E(A/\mathfrak{p})))^{\mathfrak{G}}$ are A-isomorphic. On the other hand, $A \otimes_k F$ and $A^{[F:k]}$ are A-isomorphic. Hence $Hom_A(A \otimes_k F, E(A/\mathfrak{p}))$ and $\oplus[F:k]E(A/\mathfrak{p})$ are A-isomorphic. Hence

$$\oplus[F:k]E(A/\mathfrak{p}) \cong (\mathfrak{Hom}_{\mathfrak{A}}(\mathfrak{B}, \mathfrak{E}(\mathfrak{A}/\mathfrak{p})))^{\mathfrak{G}} \quad (\text{as } \mathfrak{A}-\text{modules}). \ (2)$$

The claim follows from (1) and (2).

We shall need to use the following facts for 12.7

12.2 THEOREM. (See [18, page 59]) The field K is a Galois extension field of k (that is, $K^G = k$) if and only if K is a separable and normal extension of k.

12.3 THEOREM. (See [18, page 17, Theorem 19]) Let K be a finite extension of k. Then there is a finite normal extension W of k which contains K and which is the smallest such extension in the sense that if L is a normal extension of k which contains K then there is a map $\sigma : W \longrightarrow L$ which is both field and K-module monomorphism.

12.4 LEMMA. (See [40, Lemma 8.7]) Let K be a separable algebraic extension field of k, and let L be a field such that $k \subseteq L \subseteq K$. Then L is a separable extension of k and K is a separable extension field of L.

12.5 PROPOSITION. (See [18, page 59, proposition 2]) Let K be a normal and separable extension field of k, and let L be a field such that $k \subseteq L \subseteq K$. Let $\sigma : L \longrightarrow K$ be a map which is both field and k-module monomorphism. Then there is a map η which is both field and k-module automorphism of K and $\eta(a) = \sigma(a)$ for all $a \in L$.

12.6 LEMMA. (See [19, page 21, corollary]) Let M be a free R-module with base $(y_i)_{i \in I}$, and let N be an R-module. Then each element of $N \otimes_R M$ has a unique representation in the form

$$\sum_i (n_i \otimes y_i)$$

where n_i belongs to N and $n_i = 0$ for almost all i.

12.7 LEMMA. Let N be an A-module, and let $\rho : F \longrightarrow K$ be the inclusion map. Then the A-homomorphism

$$Id_N \otimes \rho : N \otimes_k F \longrightarrow (N \otimes_k K)^G$$

is an isomorphism, where

$$(N \otimes_k K)^G := \{x \in (N \otimes_k K) : (Id_N \otimes \sigma)(x) = x \text{ for all } \sigma \in G\}.$$

Proof. Every k-module is flat over k. Hence

$$Id_N \otimes \rho : N \otimes_k F \longrightarrow N \otimes_k K$$

is a A-monomorphism. It is clear that $Im(Id_N \otimes \rho) \subseteq (N \otimes_k K)^G$. Therefore, it is enough for us to show that $(N \otimes_k K)^G \subseteq Im(Id_N \otimes \rho)$.

Let $x = \sum_{i=1}^{t} n_i \otimes a_i \in (N \otimes_k K)^G$. It is clear that Gal $(K : k) = Gal(K : F)$ and

$$F = \{x \in K : \sigma(x) = x \text{ for all } \sigma \in Gal(K : F)\}.$$

Hence, by 12.2, K is a separable and normal extension field of $F.F(a_1, \cdots, a_t)$ is a finite extension field of F. Hence, by 12.3, there exists L a finite normal entension field of F such that $F(a_1, \cdots, a_t) \subseteq L \subseteq K$. It follows from 12.4 and the fact that K is a separable extension field of F that L is a separable extension field of F. Hence L is a finite, separable and normal extension field F such that $a_1, \cdots a_t \in L$. Note that, by [18, page 29, Exercise 38], Gal $(L : F)$ is finite. By [18, P.68, Exercise 14], if Gal $(L : F) = \{\eta_1, \cdots, \eta_s\}$, then there exists $\gamma \in L$ such that $\sum_{i=1}^{s} \eta_i(\gamma) = 1$ and $\{\eta_i(\gamma) : i = 1, \cdots s\}$ form a basis of L over F. Let $\{\alpha_j\}_{j \in J}$ be a basis of F over k. Then

$$x = \sum_{i=1}^{s} \sum_{j \in J} (n_{ij} \otimes \alpha_j \eta_i(\gamma))$$

where $n_{ij} \in N$ and $n_{ij} = 0$ for almost all j. For each $\theta \in$ Gal $(L : F)$, let $\lambda(\theta)$ be the permutation on the set $\{1, \cdots, s\}$ such that $\theta_0 \eta_i = \eta_{\lambda(\theta)(i)}$ for all $i = 1, \cdots, s$. By 12.5, for each $\theta \in$ Gal $(L : F)$ there exists $\sigma_\theta \in$ Gal $(K : F)$ such that $\sigma_\theta|_L = \theta$.

Hence, for each $\theta \in$ Gal $(L:F)$,

$$\begin{aligned} x = (Id_N \otimes \sigma_\theta)(x) &= \sum_{i=1}^{s} \sum_{j \in J} (n_{ij} \otimes \alpha_j (\theta_0 \eta_i)(\gamma) \\ &= \sum_{i=1}^{s} \sum_{j \in J} (n_{ij} \otimes \alpha_j \eta_{\lambda(\theta)(i)}(\gamma)) \\ &= \sum_{i=1}^{s} \sum_{j \in J} (n_{\lambda(\theta)^{-1}(i)j} \otimes \alpha_j \eta_i(\gamma)). \end{aligned}$$

Since $\lambda(\theta)^{-1} = \lambda(\theta^{-1})$ for all $\theta \in$ Gal $(L:F)$,

$$x = \sum_{i=1}^{s} \sum_{j \in J} (n_{\lambda(\theta^{-1})(i)j} \otimes \alpha_j \eta_i(\gamma))$$

for all $\theta \in$ Gal $(L:F)$. Hence, by 12.6 and the fact that $(\alpha_j \eta_i(\gamma))_{j \in J, 1 \leq i \leq S}$ is a basis of of L over k, $n_{\lambda(\theta)(1)j} = n_{1j}$ for all $j \in J$ and for all $\theta \in$ Gal $(L:F)$. Hence $n_{ij} = n_{1j}$ for all $j \in J$ and for all $i = 1, \cdots, s$. Hence

$$x = \sum_{j \in J} n_{1j} \otimes \alpha_j (\sum_{i=1}^{s} \eta_i(\gamma)) = \sum_{j \in J} (n_{1j} \otimes \alpha_j).$$

consequently, $x \in Im(Id_N \otimes \rho)$.

12.8 COROLLARY. Let A and B be Gorenstein rings, and let $[F:k]$ be the dimension of F considered as a vector space over k. Then each $\sigma \in G$ induces an A-automorphism σ^* on $\bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{B})} E(B/\mathfrak{q})$ such that $G^* = \{\sigma^* : \sigma \in G\}$ is a group and if

$$(\bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q}))^{\mathfrak{G}} := \{\mathfrak{x} \in (\bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})} \mathfrak{E}(\mathfrak{B}/\mathfrak{q})) : \sigma^*(\mathfrak{x}) = \mathfrak{x} \text{ for all } \sigma \in \mathfrak{G}\},$$

then $\oplus[F:k]E(A/\mathfrak{p})$ and $(\bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q}))^{\mathfrak{G}}$ are A-isomorphic, where $\oplus[F:k]E(A/\mathfrak{p})$ denotes a direct sum of $[F:k]$ copies of $E(A/\mathfrak{p})$.

Proof. It follows from 5.21 and the fact that $\mathrm{ht}_A(f^{-1}(\mathfrak{q})) = \mathrm{ht}_B\mathfrak{q}$ for all $\mathfrak{q} \in$ Spec (B) that there is an isomorphism of complexes of B-modules and B-homomorphisms from $\mathcal{C}(\mathcal{H}(A), A) \otimes_A B$ to $C(\mathcal{H}(B), B)$. Hence, by 11.11,

$$E(A/\mathfrak{p}) \otimes_A B \cong \bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q})(\text{as } \mathfrak{B} - \text{modules}).$$

Hence

$$E(A/\mathfrak{p}) \otimes_A K \cong \bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q})(\text{as } \mathfrak{A} - \text{modules}).$$

Hence $Id_{E(A/\mathfrak{p})} \otimes \sigma$ induces an A-automorphism σ^* on $\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q})$ for all $\sigma \in G$. It is clear that $G^* = \{\sigma^* : \sigma \in G\}$ is a group and

$$(E(A/\mathfrak{p}) \otimes_A K)^G \cong (\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q}))^{\mathfrak{G}}(\text{as } \mathfrak{A} - \text{modules}),$$

where

$$(E(A/\mathfrak{p}) \otimes_A K)^G := \{x \in (E(A/\mathfrak{p}) \otimes_{\mathfrak{k}} \mathfrak{K}) : (\mathfrak{Id}_{\mathfrak{E}(\mathfrak{A}/\mathfrak{p})} \otimes \sigma)(\mathfrak{x}) = \mathfrak{x} \text{ for all} \sigma \in \mathfrak{G}\}.$$

Hence, by 12.7,

$$E(A/\mathfrak{p}) \otimes_k F \cong (\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/(\mathfrak{q}))^{\mathfrak{G}}(\text{as } \mathfrak{A} - \text{modules}).$$

since tensor product commutes with direct sum,

$$\bigoplus [F : k] E(A/\mathfrak{p}) \cong (\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/(\mathfrak{q}))^{\mathfrak{G}}(\text{as } \mathfrak{A} - \text{modules}).$$

12.9 REMARK. Note that, if A is a Gorenstein ring and K is a finitely generated extension field over k, then, by [41, proposition 2], B is also a Gorenstein ring.

12.8 is a generalization of the final result of [43] (that is, 3.7). In 12.13, we shall present another proof for 12.8 by use of a method which involve the results of [43]. We shall need to use the following lemmas (12.11 and 12.12). Note that, the construction of modules of generalized fractions was introduced briefly in section 6.

12.10 PROPOSITION. (See [5, II, (1.1)]) Let R' be a commutative ring and let $\varphi : R \longrightarrow R'$ be a ring homomorphism. Let U be a triangular subset of R^n and let Y be an R'-module. Let $E \subseteq D_n(R)$ be such that

(i) E is closed under multiplication;

(ii) diag $(u_1, \cdots, u_n) \in E$ for all $(u_1, \cdots, u_n) \in U$;

(iii) whenever $u, v \in U$, then there exist $w \in U$ and $H, K \in E$ with $Hu^T = w^T = Kv^T$.

Then $\varphi(U) := \{(\varphi(u_1), \cdots, \varphi(u_n)) : (u_1, \cdots, u_n) \in U\}$ is a triangular subset of $(R')^n$ and, for $a, b \in Y$ and $u = (u_1, \cdots, u_n), v = (v_1, \cdots, v_n) \in U$

$$\frac{a}{(\varphi(u_1), \cdots \varphi(u_n))} = \frac{b}{(\varphi(v_1), \cdots \varphi(v_n))},$$

in $\varphi(U)^{-n}Y$, if and only if there exist $w = (w_1, \cdots w_n) \in U$ and $H, H' \in E$ such that $Hu^T = w^T = H'v^T$ and $\varphi(|H|)a - \varphi(|H'|)b \in (\sum_{i=1}^{n-1} R'\varphi(w_i))Y$.

It is easy to deduce the following lemma from 12.10

12.11 LEMMA. Let R' be a commutative ring and let $\varphi : R \longrightarrow R'$ be a ring homomorphism. Let U be a triangular subset of R^n and let Y be an R'-module. Then $\varphi(U) = \{(\varphi(u_1), \cdots, \varphi(u_n)) : (u_1, \cdots, u_n) \in U\}$ is a triangular subset of (R'^n) and there is an R'-isomorphism

$$g : U^{-n}Y \longrightarrow \varphi(U)^{-n}Y$$

for which

$$g(\frac{y}{(u_1, \cdots, u_n)}) = \frac{y}{(\varphi(u_1), \cdots, \varphi(u_n))}$$

for all $y \in Y$ and $(u_1, \cdots u_n) \in U$.

It is easy to deduce the following lemma.

12.12 LEMMA. Let R' be a commutative ring and let $\varphi : R \longrightarrow R'$ be a ring homomorphism. Let U be a triangular subset of R^n. Then the natural R'-homomorphism

$$h : U^{-n}R \otimes_R R' \longrightarrow U^{-n}R'$$

given by

$$h(\frac{r}{(u_1, \cdots, u_n)} \otimes r') = \frac{\varphi(r)r'}{(u_1, \cdots, u_n)}$$

for all $r' \in R', r \in R$ and $(u_1, \cdots, u_n) \in U$ is an isomorphism.

12.13 REMARK. For any Noetherian ring C, we adopt to convention that $ht_C(C) = \infty$, and for $n \in N$, we shall use C_n to denote the set

$$\{(x_1, \cdots, x_n) \in C^n : ht_C(\sum_{r=1}^{i} Cx_r) \geq i \text{ for all } i = 1, \cdots, n\}.$$

Note that C_n is a triangular subset C^n, for each $n \in \mathbb{N}$ (See [37, 5.2]). We will let $ht_A\mathfrak{p} = \mathfrak{n}$.

Let $f(A_n) = \{(f(x_1), \cdots, f(x_n)) : (x_1, \cdots, x_n) \in A_n\}$. It follows from [43,3.2], [43,3.4] that there is a B-isomorphism

$$\psi : (f(A_n) \times S)^{-n-1}B \longrightarrow \bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})} (B_n \times (B \backslash \mathfrak{q}))^{-\mathfrak{n}-1}\mathfrak{B}$$

which is such that, for $Z = \frac{b}{(f(x_1), \cdots, f(x_n), t)} \in (f(A_n) \times S)^{-n-1}B$ (with $b \in B, t \in s$ and $(x_1, \cdots, x_n) \in A_n$) and $q \in F(\mathfrak{p})$, the companent of $\psi(Z)$ in $(B_n \times (B \backslash \mathfrak{q}))^{-\mathfrak{n}-1}\mathfrak{B}$ is

$Z = \frac{b}{(f(x_1), \cdots, f(x_n), t)}$. Hence by [43, page 19], σ induces an A-module authomorphism σ^* of $M = \bigoplus_{\mathfrak{q} \in \mathfrak{F}(\mathfrak{p})} (B_n \times (B \backslash \mathfrak{q}))^{-\mathfrak{n}-1}\mathfrak{B}$, for each $\sigma \in G$ such that if

$$M^G = \{m \in M : \sigma^*(m) = m \text{ for all } \sigma \in G\}$$

and

$((f(A_n) \times S)^{-n-1}B)^G = \{\frac{b}{(f(x_1), \cdots, f(x_n), t)} \in (f(A_n) \times S)^{-n-1}B :$
$\frac{(Id_A \otimes \sigma)(b) - b}{(f(x_1), \cdots, f(x_n), t)} = 0$ for all $\sigma \in G\}$,

then

$$((f(A_n) \times S)^{-n-1}B)^G \cong M^G \text{ (as } A-\text{modules)}. \quad (1)$$

Also, by 12.11 and 12.12, there is a B-isomorphism

$$\theta : (A_n \times (A \backslash \mathfrak{p}))^{-\mathfrak{n}-1}\mathfrak{A} \otimes_{\mathfrak{A}} \mathfrak{B} \longrightarrow (\mathfrak{f}(\mathfrak{A}_\mathfrak{n}) \times \mathfrak{S})^{-\mathfrak{n}-1}\mathfrak{B}$$

which is such that

$$\theta(\frac{a}{(x_1, \cdots, x_n, s)} \otimes b) = \frac{f(a)b}{(f(x_1), \cdots, f(x_n), f(s))}$$

for all $a \in A, b \in B, s \in A \backslash \mathfrak{p}$ and $(x_1, \cdots x_n) \in A_n$. Let $L = (A_n \times (A \backslash \mathfrak{p}))^{-\mathfrak{n}-1}\mathfrak{A}$. Then

$$(L \otimes_A B)^G \cong ((f(A_n) \times S)^{-n-1}B)^G \text{ (as } A-\text{modules)}, \quad (2)$$

where

$$(L \otimes_A B)^G = \{x \in (L \otimes_A B) : (Id_L \otimes (Id_A \otimes \sigma))(x) = x \text{ for all } \sigma \in G\}.$$

It is easy to check that

$$(L \otimes_k K)^G \cong (L \otimes_A B)^G \text{ (as } A-\text{modules)}, \quad (3)$$

where

$$(L \otimes_k K)^G = \{x \in (L \otimes_k K) : (Id_L \otimes \sigma)(x) = x \text{ for all } \sigma \in G\}.$$

By 12.7, $(L \otimes_k K)^G$ and $L \otimes_k F$ are A-isomorphic. Hence if $[F : k]$ is the dimension F considered as a vector space over k, then

$$(L \otimes_k K)^G \cong \oplus[F : k]L \text{ (as } A-\text{modules).} \quad (4)$$

It follows from (1), (2), (3) and (4) that

$$\oplus[F : k]L \cong M^G \text{ (as } A-\text{modules).} \quad (5)$$

By [43, 3.6], if A and B are Gorenstein rings, then

$$M \cong \bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B\backslash\mathfrak{q}) \text{ (as } \mathfrak{B}-\text{modules) } \mathfrak{and}\ \mathfrak{L} \cong \mathfrak{E}(\mathfrak{A}/\mathfrak{p}) \text{ (as } \mathfrak{A}-\text{modules).} \quad (6)$$

It therefore follows from (5) and (6) that σ induces an A-module automorphism σ^{**} of $\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q})$, for each $\sigma \in G$ such that if

$$(\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q}))^{\mathfrak{G}} = \{\mathfrak{x} \in (\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} \mathfrak{E}(\mathfrak{B}/\mathfrak{q})) : \sigma^{**}(\mathfrak{x}) = \mathfrak{x} \text{ for all } \sigma \in \mathfrak{G}\},$$

then

$$(\bigoplus_{\mathfrak{q}\in\mathfrak{F}(\mathfrak{p})} E(B/\mathfrak{q}))^{\mathfrak{G}} \cong \oplus[\mathfrak{F} : \mathfrak{k}]\mathfrak{E}(\mathfrak{A}/\mathfrak{p}) \text{ (as } \mathfrak{A}-\text{modules).}$$

§ 13 Properties of A^G.

In this section, we consider various questions of the following type: when does a good property of A pass to A^G?

Throughout this section, G is a finite group of automorphisms of A such that $|G|$, the order of G, is invertible in A.

13.1 REMARKS. Let $\mathfrak{b}$ an ideal of A and M be a finitely generated A-module. Recall that a sequence $a_1, \cdots, a_n$ of A are said to be a *poor M-sequence* (of length n) if, for all $i = 1, \cdots, n$, the element a_i is not a zero-divisor on $M / \sum_{j=1}^{i-1} a_j M$; it is an M-sequence if, in addition, $M \neq (Aa_1 + \cdots Aa_n)M$.

(i) It is well known that, if $M \neq \mathfrak{b}\mathfrak{M}$, then $\text{grade}_M \mathfrak{b}$, that is, the common lenght of all maximal M-sequences contained in $\mathfrak{b}$, is equal to the last integer is such that $H^i_{\mathfrak{b}}(M) \neq 0$.

(ii) Let $M \neq \mathfrak{b}\mathfrak{M}$. Then the notation $\text{ht}_M \mathfrak{b}$ denotes the M-height of $\mathfrak{b}$, that is,

$$inf\{ht_M \mathfrak{p} : \mathfrak{p} \in \text{ Supp}_{\mathfrak{A}}(\mathfrak{M}) \text{ and } \mathfrak{p} \supseteq \mathfrak{b}\}.$$

It is well known that $\text{grade}_M\ \mathfrak{b} \leq \mathfrak{ht}_{\mathfrak{M}}\mathfrak{b}$.

(iii) It is also well known that, M is Cohen-Macaulay if and only if $ht_M \mathfrak{p} = \text{grade}_M\ \mathfrak{p}$, for every $\mathfrak{p} \in \text{Supp}_{\mathfrak{A}}(\mathfrak{M})$.

13.2 LEMMA. Let $n \in \mathbb{N}$ and $y_1, \cdots, y_n$ be elements of A^G. Then $y_1, \cdots, y_n$ is an A-sequence if and only if $y_1, \cdots, y_n$ is an A^G-sequence.

Proof. Let $1 \leq i \leq n$. Let $Ay_1 + \cdots + Ay_{i-1} = \bigcap_{j=1}^{m} Q_j$ be minimal primary decomposition of $Ay_1, + \cdots + Ay_{i-1}$ in A, and let $r(Q_j) = \mathfrak{q}_\mathfrak{j} (\mathfrak{1} \leq \mathfrak{j} \leq \mathfrak{m})$. By 10.1

(ii), $A^G y_1 + \cdots + A^G y_{i-1} = \bigcap_{j=1}^{m} (Q_j \cap A^G)$. It is easy to see that $Q_j \cap A^G$ is $(\mathfrak{q}_i \cap \mathfrak{A})^{\mathfrak{G}}$-primary, for each $1 \leq j \leq m$. Hence $A^G y_1 + \cdots + A^G y_{i-1} = \bigcap_{j=1}^{m} (Q_j \cap A^G)$ is a primary decomposition of $A^G y_1 + \cdots + A^G y_{i-1}$ in A^G. Therefore,

$$Z_A(\frac{A}{Ay_1 + \cdots + Ay_{i-1}}) = \bigcup_{j=1}^{m} \mathfrak{q}_j \text{ and } \mathfrak{Z}_{\mathfrak{A}^{\mathfrak{G}}}(\frac{\mathfrak{A}^{\mathfrak{G}}}{\mathfrak{A}^{\mathfrak{G}}\mathfrak{y}_1 + \cdots + \mathfrak{A}^{\mathfrak{G}}\mathfrak{y}_{i-1}}) = \bigcup_{\mathfrak{j}=1}^{\mathfrak{m}} (\mathfrak{q}_\mathfrak{j} \cap \mathfrak{A}^{\mathfrak{G}}).$$

Hence $y_i \in Z_A(\frac{A}{Ay_1 + \cdots + Ay_{i-1}})$ if and only if $y_i \in Z_{A^G}(\frac{A^G}{A^G y_1 + \cdots + A^G y_{i-1}})$. Also, by 10.1 (ii), $Ay_1 + \cdots Ay_n = A$ if and only if $A^G y_1 + \cdots + A^G y_n = A^G$.

13.3 LEMMA. Let $\mathfrak{a}$ be an ideal of A^G such that $\mathfrak{a} \neq \mathfrak{A}^{\mathfrak{G}}$. Then

(i) grade_{A^G} $(\mathfrak{a}) = \text{grade}_A(\mathfrak{a}\mathfrak{A})$;

(ii) ht_{A^G} $(\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}) = \mathfrak{ht}_{\mathfrak{A}}(\mathfrak{q})$, for every $\mathfrak{q} \in \text{Spec}$ (A);

(iii) ht_{A^G} $\mathfrak{a} = \text{ht}_A$ $(\mathfrak{a}\mathfrak{A})$.

Proof. (i) It follows from 13.2 that grode_{A^G} $(\mathfrak{a}) \leq$ $\text{grade}_{\mathfrak{A}}(\mathfrak{a}\mathfrak{A})$. Therefore it is enough for us to show that grade_A $(\mathfrak{a}\mathfrak{A}) \leq$ $\text{grade}_{\mathfrak{A}^{\mathfrak{G}}}(\mathfrak{a})$.

By 13.1 (i), $H^i_{\mathfrak{a}\mathfrak{A}}(A) = 0$ for all $i < \text{grode}_A(\mathfrak{a}\mathfrak{A})$. Hence, by 9.3 and 9.4 (ii), $H^i_{\mathfrak{a}\mathfrak{A} \cap \mathfrak{A}^{\mathfrak{G}}}(A^G) = 0$ for all $i < \text{grade}_A(\mathfrak{a}\mathfrak{A})$. Hence, by 10.1 (ii), $H^i_{\mathfrak{a}}(A^G) = 0$ for all $i < \text{grade}_A(\mathfrak{a}\mathfrak{A})$. Hence, by 13.1(i), $\text{grade}_A(\mathfrak{a}\mathfrak{A}) \leq \text{grade}_{A^G}(\mathfrak{a})$.

(ii) This follows from 10.2 (vii).

(iii) Let $\mathfrak{p} \in \text{Spec}$ (A^G) such that $\mathfrak{a} \subseteq \mathfrak{p}$. Then, by 10.2 (i) and the Lying - over theorem [1, 5.10], there exists $\mathfrak{q} \in \text{Spec}$ (A) such that $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$. Thus $\mathfrak{a}\mathfrak{A} \subseteq \mathfrak{q}$. It therefore follows from (ii) that $\text{ht}_A(\mathfrak{a}\mathfrak{A}) \leq \mathfrak{ht}_{\mathfrak{A}^{\mathfrak{G}}}(\mathfrak{p})$. Hence

$$ht_A(\mathfrak{a}\mathfrak{A}) \leq \mathfrak{inf}\{\mathfrak{ht}_{\mathfrak{A}^{\mathfrak{G}}}(\mathfrak{p}) : \mathfrak{p} \in \text{ Spec}(\mathfrak{A}^{\mathfrak{G}}) \text{ and } \mathfrak{a} \subseteq \mathfrak{p}\} = \mathfrak{ht}_{\mathfrak{A}^{\mathfrak{G}}}(\mathfrak{a}). \quad (1)$$

Let $\mathfrak{q} \in \text{Spec}$ (A) such that $\mathfrak{a}\mathfrak{A} \subseteq \mathfrak{q}$. Then, by 10.1 (ii), $\mathfrak{a} \subseteq \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}$. Hence, by (ii), $\mathrm{ht}_{A^G}(\mathfrak{a}) \leq \mathfrak{ht}_{\mathfrak{A}}(\mathfrak{q})$. Therefore

$$ht_{A^G}(\mathfrak{a}) \leq \mathfrak{inf}\{\mathfrak{ht}_{\mathfrak{A}}(\mathfrak{q}) : \mathfrak{q} \in \text{ Spec}(\mathfrak{A}) \text{ and } \mathfrak{a}\mathfrak{A} \subseteq \mathfrak{q}\} = \mathfrak{ht}_{\mathfrak{A}}(\mathfrak{a}\mathfrak{A}). \quad (2)$$

The claim follows from (1) and (2).

13.4 PROPOSITION. A is Cohen-Macaulay if and only if A^G is Cohen-Macaulay

Proof. It is easy to deduce from 13.1 (iii) that A^G is Cohen-Macaulay if and only if $ht_{A^G}(\mathfrak{a}) = \mathrm{grade}_{A^G}(\mathfrak{a})$, for every $\mathfrak{a}$ ideal of A^G. Hence, by 13.3, if A is Cohen-Macaulay, then A^G is Cohen-Macaulay.

Let A^G be Cohen-Macaulay. Then, by 13.1 (iii), 13.2 and 13.3 (ii),

$$ht_A(\mathfrak{q}) = \mathfrak{ht}_{\mathfrak{A}^{\mathfrak{G}}}(\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}) = \mathrm{grade}_{\mathfrak{A}^{\mathfrak{G}}}(\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}) \leq \mathrm{grade}_{\mathfrak{A}}(\mathfrak{q})$$

for every $\mathfrak{q} \in$ Spec (A). On the other hand, by 13.1 (ii), $\mathrm{grade}_A(\mathfrak{q}) \leq \mathfrak{ht}_{\mathfrak{A}}(\mathfrak{q})$, for every $\mathfrak{q} \in$ Spec (A). Hence $\mathrm{grade}_A(\mathfrak{q}) = \mathfrak{ht}_{\mathfrak{A}}(\mathfrak{q})$, for every $\mathfrak{q} \in$ Spec (A). Hence, by 13.1 (iii), A is Cohen-Macaulay.

13.5 DEFINITIONS. Let A be a local ring having the maximal ideal $\mathfrak{m}$ and a positive dimension, and let M be a non-zero finitely generated A-module of dimension $n \geq 1$. Recall that n elements $x_1, \cdots x_n$ of $\mathfrak{m}$ form a system of parameters (s.o.p) for M if $M/\sum_{i=1}^{n} x_i M$ has finite lenght.

(i) ([22, (3.2) and (3.3)]) We say that M is a *generalized Cohen-Macaulay* A-module if there exists $r \in \mathbb{N}$ such that, for each system of parameters $x_1, \cdots, x_n$ for M, and for all $i = 1, \cdots, n$,

$$\mathfrak{m}^{\mathfrak{r}}[((\mathfrak{A}\mathfrak{x}_1 + \cdots + \mathfrak{A}\mathfrak{x}_{\mathfrak{i}-1})\mathfrak{M} : \mathfrak{x}_{\mathfrak{i}})/(\mathfrak{A}\mathfrak{x}_1 + \cdots + \mathfrak{A}\mathfrak{x}_{\mathfrak{i}-1})\mathfrak{M}] = 0.$$

(ii) ([22, (4.1)]) We say that M is *Buchsbaum* A-module if, for each system of parameters $x_1, \cdots x_n$ for M and for all $i = 1, \cdots, n$,

$$\mathfrak{m}[((\mathfrak{A}\mathfrak{x}_1 + \cdots + \mathfrak{A}\mathfrak{x}_{i-1})\mathfrak{M} : \mathfrak{x}_i)/(\mathfrak{A}\mathfrak{x}_1 + \cdots + \mathfrak{A}\mathfrak{x}_{i-1})\mathfrak{M}] = \mathfrak{o}.$$

13.6 LEMMA. Let A be a semi local ring with maximal ideals $\mathfrak{m}_1, \cdots \mathfrak{m}_t$, that dim$A = n(\geq 1)$, and let $a_1, \cdots a_n$ be elements of A^G.

(i) A^G is semi-local ring with maximal ideals $\mathfrak{m}_1 \cap \mathfrak{A}^{\mathfrak{G}}, \cdots \mathfrak{m}_t \cap \mathfrak{A}^{\mathfrak{G}}$ (not necessarily $|G|$ is invertable in A).

(ii) $a_1, \cdots a_n$ is a system of parameters (s.o.p) for A if and only if $a_1, \cdots a_n$ is a system of parameters for A^G.

Proof. (i) It follows from 10.2 (i) and [1, 5.8] that $\mathfrak{m}_1 \cap \mathfrak{A}^{\mathfrak{G}}, \cdots, \mathfrak{m}_t \cap \mathfrak{A}^{\mathfrak{G}}$ are maximal ideals of A^G.

Let $\mathfrak{p}$ be an maximal ideal of A^G. Then, by 10.2 (i) and the Lying over Theorem [1, 5.10], there exists $\mathfrak{q} \in$ Spec (A) such that $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$. Hence, by [1, 5.8], there exists $1 \leq j \leq t$ such that $\mathfrak{m}_j = \mathfrak{q}$.

(ii) Note that, by 10.2 (vi), dim $A=$ dim A^G. Therefore it is enough to show that $r_A(Aa_1 + \cdots + Aa_n) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t$ if and only if $r_{A^G}(A^G a_1 + \cdots + A^G a_n) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t \cap \mathfrak{A}^{\mathfrak{G}}$.

Let $r_A(Aa_1 + \cdots + a_n) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t$. We show that if $\mathfrak{p} \in$ Spec (A^G) such that $A^G a_1 + \cdots + A^G a_n \subseteq \mathfrak{p}$, then there exists $\mathfrak{m}_i(\mathfrak{1} \leq \mathfrak{i} \leq \mathfrak{t})$ such that $\mathfrak{m}_i \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$. Let $\mathfrak{p} \in$ Spec (A^G) such that $A^G a_1 + \cdots + A^G a_n \subseteq \mathfrak{p}$. Then, by 10.2 (i) and the Lying over theorem [1, 5.10], there exists $\mathfrak{q} \in$ Spec (A) such that $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}$. Hence $Aa_1 + \cdots + Aa_n \subseteq \mathfrak{q}$. By the hypothesis, there exists $\mathfrak{m}_j(\mathfrak{1} \leq \mathfrak{j} \leq \mathfrak{t})$ such that $\mathfrak{m}_j = \mathfrak{q}$. Hence $\mathfrak{m}_j \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{p}$.

Now, let $r_{A^G}(A^G a_1 + \cdots + A^G a_n) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t \cap \mathfrak{A}^{\mathfrak{G}}$. We show that if $\mathfrak{q} \in$ Spec (A) such that $Aa_1 + \cdots + Aa_n \subseteq \mathfrak{q}$, then there exists $\mathfrak{m}_j(\mathfrak{1} \leq \mathfrak{j} \leq \mathfrak{t})$

such that $\mathfrak{q} = \mathfrak{m}_j$. Let $\mathfrak{q} \in$ Spec (A) such that $Aa_1 + \cdots + Aa_n \subseteq \mathfrak{q}$. Then, by the hypothesis, there exists $\mathfrak{m}_j(1 \leq j \leq t)$ such that $\mathfrak{q} \subseteq \mathfrak{m}_j$. Hence, by 10.1 (ii), $A^G a_1 + \cdots A^G a_n \subseteq \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} \subseteq \mathfrak{m}_j \cap \mathfrak{A}^{\mathfrak{G}}$. Hence there exists $\mathfrak{m}_{\mathfrak{k}}(1 \leq \mathfrak{k} \leq t)$ such that $\mathfrak{m}_{\mathfrak{k}} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}}$. Hence, by (i), $\mathfrak{m}_{\mathfrak{k}} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{m}_j \cap \mathfrak{A}^{\mathfrak{G}}$. Hence $\mathfrak{q} \cap \mathfrak{A}^{\mathfrak{G}} = \mathfrak{m}_j \cap \mathfrak{A}^{\mathfrak{G}}$. Hence, by 10.2 (i), the Incomparability theorem [1, 5.9] and the fact that $\mathfrak{q} \subseteq \mathfrak{m}_j$, $\mathfrak{q} = \mathfrak{m}_j$.

13.7 PROPOSITION. Let A be a local ring having the maximal indeal $\mathfrak{m}$ and a positive dimension. Then

(i) If M is a generalized Cohen-Macaulay A-module of dimension n and there exists H a finite group of A^G-module automorphisms of M such that $|H|$, the order of H, is invertible in A and M^H is a non-zero finitely A^G-module of dimension n, then M^H is a generalized Cohen-Macaulay A^G-module.

(ii) If A is a generalized Cohen-Macaulay ring, then A^G is a generalized Cohen-Macaulay with maximal ideal $\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}}$.

(iii) If A is a Bachsbaum ring, then A^G is a Buchsbaum ring with maximal ideal $\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}}$.

Proof. It follows from 13.6 (i) that A^G is a local ring with maximal ideal $\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}}$.

(i) Let N be a non-zero finitely generated A-module of dimension $n \geq 1$. Then, by [22, (3.2) and (3.3)], N is a generalized Cohen-Macauley A-module if and only if there exists $s \in \mathbb{N}$ such that $\mathfrak{m}^{\mathfrak{s}} \mathfrak{H}^{\mathfrak{i}}_{\mathfrak{m}}(\mathfrak{N}) = \mathfrak{o}$ for all $i = 0, \cdots, n-1$. Hence there exists $t \in \mathbb{N}$ such that $\mathfrak{m}^{\mathfrak{t}} \mathfrak{H}^{\mathfrak{i}}_{\mathfrak{m}}(\mathfrak{M}) = \mathfrak{o}$ for all $i = 0, \cdots, n-1$. Hence $(\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}})^{\mathfrak{t}} (\mathfrak{H}^{\mathfrak{i}}_{\mathfrak{m}}(\mathfrak{M}))^{\mathfrak{H}} = \mathfrak{o}$ for all $i = 0, \cdots, n-1$ (we are using notation introduced in 9.1). Hence, by 9.2 and 9.4 (iv), $(\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}})^{\mathfrak{t}} \mathfrak{H}^{\mathfrak{i}}_{\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}}}(\mathfrak{M}^{\mathfrak{H}}) = \mathfrak{o}$ for all $i = 0, \cdots, n-1$. Hence M^H is a generalized Cohen-Macaulay A^G-module.

(ii) This follows from (i) and the fact dim $A =$ dim A^G.

(iii) Let $a_1, \cdots, a_n$ be a system of parameters for A^G. Then, by 13.6 (ii), $a_1, \cdots, a_n$ is a system of parameters for A. Hence, for each $i = 1, \cdots, n$,

$$\mathfrak{m}[((\mathfrak{A}\mathfrak{a}_1 + \cdots + \mathfrak{A}\mathfrak{a}_{i-1}) :_{\mathfrak{A}} \mathfrak{a}_i)/(\mathfrak{A}\mathfrak{a}_1 + \cdots + \mathfrak{A}\mathfrak{a}_{i-1})] = \mathfrak{o}.$$

Hence, for each $i = 1, \cdots, n$,

$$\mathfrak{m}((\mathfrak{A}\mathfrak{a}_1 + \cdots + \mathfrak{A}\mathfrak{a}_{i-1}) :_{\mathfrak{A}} \mathfrak{a}_i) \subseteq \mathfrak{A}\mathfrak{a}_1 + \cdots + \mathfrak{A}\mathfrak{a}_{i-1}.$$

Hence, for each $i = 1, \cdots, n$,

$$(\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}})[((\mathfrak{A}\mathfrak{a}_1 + \cdots + \mathfrak{A}\mathfrak{a}_{i-1}) :_{\mathfrak{A}} \mathfrak{a}_i) \cap \mathfrak{A}^{\mathfrak{G}}] \subseteq (\mathfrak{A}\mathfrak{a}_1 + \cdots + \mathfrak{A}\mathfrak{a}_{i-1}) \cap \mathfrak{A}^{\mathfrak{G}}.$$

Hence, by 10.1 (ii), for each $i = 1, \cdots, n$,

$$(\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}})((\mathfrak{A}^{\mathfrak{G}}\mathfrak{a}_1 + \cdots + \mathfrak{A}^{\mathfrak{G}}\mathfrak{a}_{i-1}) :_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{a}_i) \subseteq \mathfrak{A}^{\mathfrak{G}}\mathfrak{a}_1 + \cdots + \mathfrak{A}^{\mathfrak{G}}\mathfrak{a}_{i-1}.$$

Therefore, for each $i = 1, \cdots, n$,

$$(\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}})[((\mathfrak{A}^{\mathfrak{G}}\mathfrak{a}_1 + \cdots + \mathfrak{A}^{\mathfrak{G}}\mathfrak{a}_{i-1}) :_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{a}_i)/\mathfrak{A}^{\mathfrak{G}}\mathfrak{a}_1 + \cdots + \mathfrak{A}^{\mathfrak{G}}\mathfrak{a}_{i-1}] = \mathfrak{o}.$$

The claim follows immediately from preceding paragraph and 13.5(ii).

13.8 REMARKS AND DEFINITIONS. Let A be a local ring with the maximal ideal $\mathfrak{m}$, and dim $A = n$, and let $a_1, \cdots a_n$ be a system of parameters (s.o.p) for A. An (not necessarily finitely generated) A-module M is said to be a *big Cohen-Macaulay A-module with respect* to $a_1, \cdots a_n$ if $a_1, \cdots a_n$ is an M-sequence. The reader is refered to the work and writings of Hochster, such as [9], for details of the relationship between this concept and the various homological conjectures in commutative algebra. Furthemore, we say that M is a *balanced big Cohen Macaulay* A-module if it is a big Cohen-Macaulay module with respect to every s.o.p for A(See

[32]). If an A-module M is a big Cohen-Macaulay A-module with respect to some s.o.p for A and M is finitely generated, then it is well known that M is a balanced big Cohen-Macaulay A-module.

Let $\mathcal{D}(A) = (D_i)_{i \in \mathbb{N}_{\nvdash}}$ be the dimension filtration of Spec (A), which is defined by

$$D_i = \{\mathfrak{q} \in \text{ Spec } (\mathfrak{A}) : \mathfrak{dim}(\mathfrak{A}/\mathfrak{q}) \leq \mathfrak{dim}\mathfrak{A} - \mathfrak{i}\},$$

for each $i \in \mathbb{N}_{\nvdash}$, and let M be an A-module. Then M is a balanced big Cohen-Macaulay A-module if and if $M \neq \mathfrak{m}\mathfrak{M}$ and the Cousin complex $C(\mathcal{D}(A), M)$ for M with respect to the $\mathcal{D}(A)$ is exact (See [33, 3.7]).

13.9 PROPOSITION. Let A be a local ring with the maximal ideal $\mathfrak{m}$, and dim $A = n$, so that, by 13.6 (i) and 10.2 (vi), A^G is a local ring with the maximal ideal $\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}}$ and dim $A^G = n$. Let M be an A-module. Then

(i) If $b_1, \cdots, b_n$ is a system of parameters for A^G, then $b_1, \cdots, b_n$ is a system of parameters for A and M, when regarded as an A-module, is a big Cohen-Macaulay with respect to $b_1, \cdots b_n$ if and only if M, when regorded as an A^G-module, is a big Cohen-Macaulay with respect to $b_1, \cdots b_n$.

(ii) If M is a balanced big Cohen-Macaulay A-module, then M is a balanced big Cohen-Macaulay A^G-module.

(iii) If N is an A^G-module such that $N \otimes_{A^G} A$ is a balanced big Cohen-Macaulay A-module, then N is a balanced big Cohen-Macaulay A^G-module.

Proof. (i) Note that, $b_i \in Z_A(M/b_1M + \cdots + b_{i-1}M)$ if and only if $b_i \in Z_{A^G}(M/b_1M + \cdots + b_{i-1}M)$ for all $i = 1, \cdots, n$. Hence $b_1, \cdots, b_n$ is a M-sequence (as A-module) if and only if $b_1, \cdots b_n$ is a M-sequence (as A^G-module).

The claim follows from 13.6 (ii) and above note.

(ii) This follows from (i).

(iii) By 13.8, the Cousin complex $C(\mathcal{D}(A), N \otimes_{A^G} A)$ for $N \otimes_{A^G} A$ with respect to the dimension filtration of Spec (A) is exact and $\mathfrak{m}(\mathfrak{N} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{A}) \neq (\mathfrak{N} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{A})$. Hence, by 10.13 (i), the Cousin complex $C(\mathcal{D}(A^G), N)$ for N with respect to the dimension filtration of Spec (A^G) is exact. Hence, by 13.8, it is enough for us to show that $(\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}})\mathfrak{N} \neq \mathfrak{N}$.

Let $(m \cap A^G)N = N$. Then $(\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}})(\mathfrak{N} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{A}) = \mathfrak{N} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{A}$. Hence $\mathfrak{m}(\mathfrak{N} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{A}) = \mathfrak{N} \otimes_{\mathfrak{A}^{\mathfrak{G}}} \mathfrak{A}$. This contradiction shows that $(\mathfrak{m} \cap \mathfrak{A}^{\mathfrak{G}})\mathfrak{N} \neq \mathfrak{N}$.

13.10 REMARK AND DEFINITION. (i) (See [17, page 183]) Suppose $k \in \mathbb{N}_{\not{k}}$. The ring A is said to *satisfy condition* (S_k) if and only if, for all $\mathfrak{p} \in$ Spec (A),

$$\text{grade}_{A_{\mathfrak{p}}}(\mathfrak{p}\mathfrak{A}_{\mathfrak{p}}) \geq \mathfrak{min}(\mathfrak{htp}, \mathfrak{k})$$

(ii) (See [26, 2.2]) Suppose $k \in \mathbb{N}_{\not{k}}$, and

$$C(\mathcal{H}(A), A) : 0 \longrightarrow A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \longrightarrow \cdots \longrightarrow A^n \xrightarrow{d^n} A^{n+1} \longrightarrow \cdots$$

is the Cousin complex for A with respect to the height filtration $\mathcal{H}(A)$ (See 5.4). Then the following statements are equivalent:

(i) A satisfies condition (S_k)

(ii) $C(\mathcal{H}(A), A)$ is exact at $A^{-1} = A, A^0, \cdots, A^{k-2}$.

It is easy to deduce the following proposition from 10.13(ii) and 13.10 (ii).

13.11 PROPOSITION. Suppose $k \in \mathbb{N}_{\not{k}}$, and A statisfies condition (S_k), then A^G satisfies condition (S_k).

We shall need to use the following proposition.

13.12 PROPOSITION. (See [28, (3.5)]) Let $f : A \longrightarrow B$ be a homomorphism of commutative Noetherian rings. Assume that f makes B into a flat A-module. Let I be an injective B-module, then I, when regarded as an A-module by means of f, is A-injective.

13.13 PROPOSITION. Let A be a Gorenstien ring and $\mathfrak{a}$ be an ideal of A^G. Then

$(i)C(\mathcal{H}(A^G), A^G)$ the Cousin complex for A^G with respect to the height filtration $\mathcal{H}(A^G)$ (See 5.4) is a $L_{\mathfrak{a}}$-acyclic resolution of A^G.

(ii) If A is a flat A^G-module, then A^G is a Gorenstien ring.

Proof. Suppose that $C(\mathcal{H}(A), A)$ the Cousin complex for A with respect to the height filtration $\mathcal{H}(A)$ has the form

$$0 \longrightarrow A \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \longrightarrow \cdots \longrightarrow A^n \xrightarrow{d^n} A^{n+1} \longrightarrow \cdots .$$

By 5.4, $C(\mathcal{H}(A), A)$ is a minimal injective resolution for A. It follows from 10.13 (ii) that there is an isomorphism of complexes of A^G-modules and A^G-homomorphisms from $(C(\mathcal{H}(A), A))^G$ to $C(\mathcal{H}(A^G), A^G)$ and furthermore, $C(\mathcal{H}(A^G), A^G)$ is exact (we are using notation introduced in 10.11).

(i) It is enough for us to show that $(A^i)^G$ is a $L_{\mathfrak{a}}$-acyclic, for each $i \in \mathbb{N}_{\not{\,0}}$ (we are using notation introduced in 10.10).

Let $i \in \mathbb{N}_{\not{\,0}}$. Since A^i is an injective A-module, it follows from 1.2 and 1.4 (1) that $H^n_{\mathfrak{a}\mathfrak{A}}(A^i)$ for all $n \in \mathbb{N}$. Therefore $(H^n_{\mathfrak{a}\mathfrak{A}}(A^i))^G = 0$ for all $n \in \mathbb{N}$ (we are using notation introduced in 9.1). Hence, by 9.2 and 9.4 (ii), $H^n_{\mathfrak{a}\mathfrak{A}\cap\mathfrak{A}^{\mathfrak{G}}}((A^i)^G) = 0$ for all $n \in \mathbb{N}$. Hence, by 10.1 (ii), $H^n_{\mathfrak{a}}((A^i)^G) = 0$ for all $n \in \mathbb{N}$. Hence $(A^i)^G$ is a $L_{\mathfrak{a}}$-acyclic.

(ii) By 5.4, it is enough for us to show that $C(\mathcal{H}(A)^G, A^G)$ is a injective resolution for A^G. Since $(C(\mathcal{H}(A), A))^G$ and $C(\mathcal{H}(A^G), A^G)$ are isomorphic as complexes of A^G-modules and A^G-homomorphisms and $C(\mathcal{H}(A^G), A^G)$ is exact, it is enough for us to show that $(A^i)^G$ is an injective A^G-module, for each $i \in \mathbb{N}_0$.

Let $i \in \mathbb{N}_0$. By 8.3, $(A^i)^G$ is a direct summand of A^i as A^G-module. On the other hand, by 13.12, A^i is an injective A^G-module. Hence $(A^i)^G$ is an injective A^G-module.

13.14 LEMMA. Let $C = R[x_1, \cdots, x_n]$ be the polynomial ring in n indeterminates over R. Let S be a subring of C such that $R \subseteq S$. Then C is flat over S.

Proof. Let N be a S-module. Then there is a natural S-isomorphism

$$\varphi : N \otimes_S S[x_1, \cdots, x_n] \longrightarrow N[x_1, \cdots, x_n]$$

which is such that

$$\varphi(\alpha \otimes \sum_{i=1}^{t} s_i x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}) = \sum_{i=1}^{t} (s_i \alpha) x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$$

for all $\alpha \in N$ and $\sum_{i=1}^{t} s_i x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} \in S[x_1, \cdots x_n]$. Also, there is a natural S-isomorphism

$$\psi : R[x_1, \cdots, x_n] \otimes_R S \longrightarrow S[x_1, \cdots, x_n]$$

which is such that

$$\varphi(\sum_{i=1}^{t} r_i x_1^{\beta_{i1}} \cdots x_n^{\beta_{in}} \otimes s) = \sum_{i=1}^{l} (r_i s) x_1^{\beta_{i1}} \cdots x_n^{\beta_{in}}$$

for all $s \in S$ and $\sum_{i=1}^{l} r_i x_1^{\beta_{i1}} \cdots x_n^{\beta_{in}} \in R[x_1, \cdots, x_n]$. Hence there is a natural S-isomorphism

$$\theta_N : (N \otimes_S R[x_1, \cdots, x_n]) \otimes_R S \longrightarrow N[x_1, \cdots x_n]$$

which is such that

$$\theta_N((\alpha \otimes \sum_{i=1}^{l} r_i x_1^{\beta_{i1}}, \cdots, x_n^{\beta_{in}}) \otimes s) = \sum_{i=1}^{l}((r_i s)\alpha) x_1^{\beta_{i1}} \cdots x_n^{\beta_{in}}$$

for all $\alpha \in N, s \in S$ and $(\sum_{i=1}^{l} r_i x_1^{\beta_{i1}} \cdots x_n^{\beta_{in}} \in R[x_1, \cdots, x_n]$.

Let $f : N_1 \longrightarrow N_2$ be a monomorphism of S-modules. Then there is a natural S-monomorphism

$$f^* : N_1[x_1, \cdots, x_n] \longrightarrow N_2[x_1, \cdots x_n]$$

which is such that

$$f^*(\sum_{i=1}^{k} \alpha_i x_1^{\gamma_{i1}} \cdots x_n^{\gamma_{in}}) = \sum_{i=1}^{k} f(\alpha_i) x_1^{\gamma_{i1}}, \cdots, x_n^{\gamma_{in}}$$

for all $\sum_{i=1}^{k} \alpha_i x_1^{\gamma_{i1}} \cdots x_n^{\gamma_{in}} \in N_1[x_1, \cdots, x_n]$. Since the diagram

$$\begin{array}{ccc} (N_1 \otimes_s R[x_1, \cdots x_n]) \otimes_R S & \xrightarrow{\theta_{N_1}} & N[x_1, \cdots x_n] \\ \downarrow {\scriptstyle (f \otimes Id_C) \otimes Id_S} & & \downarrow {\scriptstyle f^*} \\ (N_2 \otimes_S R[x_1, \cdots x_n] \otimes_R S & \xrightarrow{\theta_{N_2}} & N[x_2, \cdots x_n] \end{array}$$

Commutes, $(f \otimes Id_C) \otimes Id_S$ is a S-monomorphism.

Let L be a S-module. Then there are the natural S-homomorphisms

$$\mu_L : L \longrightarrow L \otimes_R S \text{ and } \lambda_L : L \otimes_R S \longrightarrow L$$

which is such that

$$\mu_L(x) = x \otimes 1_S \text{ and } \lambda_L(x \otimes s) = sx$$

for all $x \in L$ and $s \in S$. It is clear that $\lambda_L o \mu_L = Id_L$. Hence μ_L is a S-monomorphism.

Let $f : N_1 \longrightarrow N_2$ be a S-monomorphism. Since $(f \otimes Id_C) \otimes Id_S$ and $\mu_{N_1 \otimes_S C}$ are S-monomorphicms and the diagram

$$\begin{array}{ccc} N_1 \otimes_S R[x_1, \cdots x_n] & \xrightarrow{\mu_{N_1 \otimes_S C}} & (N_1 \otimes_S R[x_1, \cdots x_n]) \otimes_R S \\ \Big\downarrow f \otimes Id_C & & \Big\downarrow (f \otimes Id_C) \otimes Id_S \\ N_2 \otimes_S R[x_1, \cdots, x_n] & \xrightarrow{\mu_{N_2 \otimes_S C}} & (N_2 \otimes_S R[x_1, \cdots x_n]) \otimes_R S \end{array}$$

Commutes, it follows that $f \otimes Id_C$ is a S-monomorphism.

It is easy to deduce the following corallary from 13.13 and 13.14.

13.15 COROLLARY. Let B be a commutative ring with identity, and let $A = B[x_1, \cdots, x_n]$ be the polynomial ring in n indeterminates over B. Then

(i) If A is a Gorenstein ring and $B \subseteq A^G$, then A^G is a Gorenstein ring.

(ii) If G is a group of B-algebra automorphisms of A and A is a Gorenstein ring, then A^G is a Gorenstein ring.

13.16 REMARK. Let k be a field and A be a polynomial ring in n indeterminates over k. In section two of [11], the following theorems are stated (from K. Watanabe and R. Stanley) to show that under what condition A^G is a Gorenstein ring, whenever G is a finite linear algebric group over k with $|G|$ invertible in k such that it acts linearly on A.

(i) (K. Watanabe, See [11, (2.2)]) Let G be a finite linear algebraic group over k (See [10, page 164]) whose order is invertible in the field k acting linearly (See [10, page 164]) on the polynomial ring $A = k[x_1, \cdots x_n]$. Assume that $G \subseteq GL(V)$, where $V = \sum_{i=1}^{n} kx_i$. Then

(a) A^G is Gorenstein, whenever $G \subseteq SL(V)$.

(b) A^G is Gorenstein if and only if $G \subseteq SL(V)$, whenever G contains no psecutorefiections.

(ii) (R. Stanley, see [11, 2.4]) Let G be a finite linear algebraic group over k acting linearly on a polynomial ring $A = k[x_1, \cdots x_n]$ with $|G|$ invertible in k. Then A^G is Gorenstein if and only if

$$\sum_g 1/det(I - tg) = t^{-r} \sum_g det(g)/det(I - tg),$$

where r is the number of pseudorefiections in G. Both sides are to be regarded as rational functions of the indeterminate t.

Note that, we have already established in 13.15 (ii), under weaker conditions, a result similler to the results stated in 13.16.

13.17 REMARK AND DEFINITIONS. (i) (see [29]) the category of all complexes of A-modules and translations of such complexes is denoted by $Y(A)$. Also $Y^b(A)$ will denote the full subcategory of $Y(A)$ whose objects are the complexes which are bounded. $Y_c^b(A)$ will denote the full subategroup of $Y^b(A)$ whose objects are those complexes in $Y^b(A)$ all of whose cohomology modules are finitely generated.

If n is an integer, $H^n : Y(A) \longrightarrow \mathcal{C}(A)$ will denote the n th cohomology functor. So, if $X \in Y(A)$, then $H^n(X^\bullet) = Ker\ d^n_{X^\bullet}/Im\ d^{n-1}_{X^\bullet}$. Now suppose that $X^\bullet, Y^\bullet \in Y(A)$. A translation of complexes $u^\bullet : X^\bullet \longrightarrow Y^\bullet$ is said to be a *quasi isomorphism* (abbreviated as quism) if, for all $i \in \mathbb{Z}$, the induced homomorphism of cohomology modules.

$$H^i(u^\bullet) : H^i(X^\bullet) \longrightarrow H^i(Y^\bullet)$$

is an isomorphism

A *dualizing complex* for A is an injective complex $I^\bullet \in Y_c^b(A)$ such that the translation of complexes

$$\theta(X^\bullet, I^\bullet) : X^\bullet \longrightarrow Hom_A([Hom_A(X^\bullet, I^\bullet)], I^\bullet)$$

of 2.3 (ii) of [29] is a quism whenever $X^\bullet \in Y_c^b(A)$.

(ii) (See [29, 3.9]) Suppose $f : A \longrightarrow B$ is a homomorphism of commutative Noetherian rings such that B, when regarded as an A-module by means of f, is finitely generated. Suppose A has a dualizing complex $J^\bullet$ given by:

$$J^\bullet : \cdots O \longrightarrow J^t \longrightarrow J^{t+1} \longrightarrow \cdots \longrightarrow J^{t+n} \longrightarrow 0 \longrightarrow 0 \cdots .$$

Then the complex $I^\bullet$ of B-modules and homomorphisms given by:

$$I^\bullet : \cdots 0 \longrightarrow Hom_A(B, J^t) \longrightarrow Hom_A(B, J^{t+1}) \longrightarrow \cdots \longrightarrow Hom_A(B, J^{t+n}) \longrightarrow 0 \longrightarrow \cdots$$

(in which it is to be understood that $Hom_A(B, J^t)$ is the t-th term) is a dualizing complex for B.

13.18 PROPOSITION. Let A, when regarded as an A^G-module, be finitely generated. Suppose A^G has a dualizing complex $J^\bullet$ given by:

$$J^\bullet : \cdots 0 \longrightarrow J^t \longrightarrow J^{t+1} \longrightarrow \cdots \longrightarrow J^{t+n} \longrightarrow 0 \longrightarrow 0 \cdots .$$

Then there is a dualizing complex $I^\bullet$ for A such that each $\sigma \in G$ induces an isomorphism of complexes of A^G-modules and A^G-homomorphisms $\sigma_+ : I^\bullet \longrightarrow I^\bullet$. Furthermore, $G^+ = \{\sigma_+ : \sigma \in G\}$ is a group of isomorphisms of complexes from $I^\bullet$ to $I^\bullet$ and $(I^\bullet)^{G^+}$ (we are using notation introduced in 10.10) and $J^\bullet$ are isomorphic of complexes (as complexes of A^G-modules and A^G-homomorphisms).

Proof. By 13.17 (ii), the complex $I^\bullet$ of A-modules and A-homomorphisms given by:

$$I^\bullet : \cdots 0 \longrightarrow Hom_{A^G}(A, J^t) \longrightarrow Hom_{A^G}(A, J^{t+1}) \longrightarrow \cdots \longrightarrow Hom_{A^G}(A, J^{t+n}) \longrightarrow 0 \longrightarrow \cdots .$$

is a dualizing complex for A. Each $\sigma \in G$ induces an isomorphism of compexes of A^G-modules and A^G-homomorphisms

$$\sigma_+ = (\sigma_+^i)_{i\in\mathbb{Z}} : I^\bullet \longrightarrow I^\bullet$$

which is such that $\sigma_+^i = Hom_{A^G}(\sigma, Id_{J^i})$ for all $i \in \mathbb{Z}$. Set $G^+ = \{\sigma_+ : \sigma \in G\}$. Then it is clear that G^+ is a group of isomorphism of complexes from $I^\bullet$ to $I^\bullet$. There is an A^G-homomorphism $\eta' : A \longrightarrow A^G$ which is such that

$$\eta'(\alpha) = \frac{1}{|G|}\sum_{\sigma\in G}\sigma(\alpha)$$

for all $\alpha \in A$. Hence, by 8.9, the homomorphism of complexes of A^G-modules and A^G-homomorphisms

$$(Hom_{A^G}(\eta', Id_{J^i}))_{i\in\mathbb{Z}} : Hom_{A^G}(A^G, J^\bullet) \longrightarrow (Hom_{A^G}(A, J^\bullet))^{G+}$$

is an isomorphism. It therefore follows from the fact that the complexes $Hom_{A^G}(A^G, J^\bullet)$ and $J^\bullet$ are isomorphic that $(I^\bullet)^{G^+}$ and $J^\bullet$ are isomorphic (as complexes of A^G-modules and A^G-homomorphisms).

13.19 REMARK AND DEFINITION. (i) (See [13, (1.1)]) Let $x_1, \cdots, x_n \in R$; the sequence of elements $x_1, \cdots, x_n$ is said to be a *d-sequence*

(1) $x_i \notin Rx_1 + \cdots + Rx_{i-1} + Rx_{i+1} + \cdots + Rx_n$ for $i = 1, \cdots n$,

(2) For all $k \geq i+1$ and all $i \geq 0$, $(x_0 = 0)$,

$$((x_0R + \cdots x_iR) : x_{i+1}x_k) = ((x_0R + \cdots + x_iR) : x_k).$$

(ii) Let $x_1, \cdots x_n \in R$. It is easy to check that $x_1, \cdots x_n$ is a d-sequence on R if and only if for all $k \geq i+1$ and all $i \geq 0, (x_0 = 0)$,

$$x_{i+1} \notin Z_R(\frac{Rx_k + Rx_0 + \cdots + Rx_i}{Rx_0 + \cdots + Rx_i}),$$

that is, x_{i+1} is not a zero-divisor on R-module $Rx_k + Rx_1 + \cdots + Rx_i/Rx_1 + \cdots Rx_i$.

13.20 PROPOSITION. Let $n \in \mathbb{N}$ and $y_1, \cdots, y_n \in A^G$. Then $y_1, \cdots, y_n$ is a d-sequence on A if and only if $y_1, \cdots y_n$ is a d-sequence on A^G.

Proof. Let $0 \leq i \leq n-1$, and let $k \geq i+1$. Let $y_0 = 0$. Suppose $Ay_0 + \cdots + Ay_i = \bigcap_{j=1}^{m} Q_j$ is a minimal primary decomposition of the submodule $Ay_0 + \cdots + Ay_i$ of A-module $Ay_k + Ay_0 + \cdots + Ay_i$, and let $r_A(Q_j : Ay_k + Ay_0 + \cdots + Ay_i) = \mathfrak{q}_\mathfrak{j} (\mathfrak{1} \leq \mathfrak{j} \leq \mathfrak{m})$. Let $1 \leq j \leq m$. Since Q_j is a A-submodule of $Ay_k + Ay_0 + \cdots + Ay_i, Q_j$ is an ideal of A and Q_j contained in $Ay_k + Ay_0 + \cdots + Ay_i$. Hence, by 10.1 (ii), $Q_j \cap A^G$ is an ideal of A^G and $Q_j \cap A^G \subseteq A^G y_k + A^G y_0 + \cdots + A^G y_i$. Hence $Q_j \cap A^G$ is an A^G-submodule of $A^G y_k + A^G y_0 + \cdots + A^G y_i$. It follows from 10.1 (ii) that $A^G y_0 + \cdots + A^G y_i = \bigcap_{j=1}^{m} (Q_j \cap A^G)$. It is easy to see that $r_{A^G}((Q_j \cap A^G) : A^G y_k + A^G y_0 + \cdots + A^G y_i) = (\mathfrak{q}_\mathfrak{j} \cap \mathfrak{A}^\mathfrak{G})$ and $(Q_j \cap A^G)$ is a primary submodule of $A^G y_k + A^G y_0 + \cdots + A^G y_i$, for each $j = 1, \cdots, m$. Hence $A^G y_0 + \cdots + A^G y_i = \bigcap_{j=1}^{m} (Q_j \cap A^G)$ is a primary decomposition of the submodule $A^G y_0 + \cdots A^G y_i$ of A^G-module $A^G y_k + A^G y_0 + \cdots + A^G y_i$. Therefore,

$$Z_A\left(\frac{Ay_k + Ay_0 + \cdots + Ay_i}{Ay_0 + \cdots + Ay_i}\right) = \bigcup_{j=1}^{m} \mathfrak{q}_\mathfrak{j} \text{ and } Z_{A^G}\left(\frac{A^G y_k + A^G y_0 + \cdots + A^G y_i}{A^G y_0 + \cdots + A^G y_i}\right) = \bigcup_{j=1}^{m} (\mathfrak{q}_\mathfrak{j} \cap \mathfrak{A}^\mathfrak{G}).$$

Hence

$$y_{i+1} \in Z_A\left(\frac{Ay_k + Ay_0 + \cdots + Ay_i}{Ay_0 + \cdots + Ay_i}\right) \text{ if and only if } y_{i+1} \in Z_{A^G}\left(\frac{A^G y_k + A^G y_0 + \cdots + A^G y_i}{A^G y_0 + \cdots + A^G y_i}\right).$$

Also, by 10.1 (ii), $y_{i+1} \in Ay_1 + \cdots + Ay_i + Ay_{i+2} + \cdots + Ay_n$ if and only if $y_{i+1} \in A^G y_1 + \cdots + A^G y_i + A^G y_{i+2} + \cdots A^G y_n$.

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