Steps toward almost Cohen-Macaulayness

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Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring with its system of parameters $\underline{x} := x_1, \ldots, x_d$. Hochster's *Monomial Conjecture* states that $x_1^t \ldots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})$ for all $t \geq 0$.

The Monomial Conjecture is known for all equi-characteristic local rings and for all local rings of dimension ≤ 3. Heitmann's proof of this conjecture in dimension 3 has opened a new approach to the study of homological conjectures in mixed characteristic. This approach is an example of *Almost Ring Theory*.

Let R^+ denote the integral closure of integral domain R in an algebraic closure of the fraction field of R. Using extraordinarily difficult methods, it was recently shown by Heitmann that R^+ is almost Cohen-Macaulay for a three-dimensional complete local domain R of mixed characteristic.

Let $(R, \mathfrak{m})$ be a Noetherian complete local domain. There is a valuation map

$$v : R^+ \longrightarrow \mathbb{Q} \bigcup \{\infty\}$$

which is nonnegative on R^+ and positive on $\mathfrak{m}_{R^+}$. Now consider the following:

- **Definition.** Let $\mathfrak{m} \subseteq rad(A)$ be an ideal of an integral domain A with a valuation map

$$v : A \longrightarrow \mathbb{R} \bigcup \{\infty\}$$

which is nonnegative on A.

i) (Roberts-Sing-Sirinivas) M is called almost zero with respect to v, if for all $m \in M$ and for all $\epsilon > 0$ there is an element $a \in A$ with $v(a) < \epsilon$ such that $am = 0$.

ii) (Gaber-Ramero) Assume that $\mathfrak{m}^2 = \mathfrak{m}$. Then M is called almost zero with respect to $\mathfrak{m}_{R+}$, if $\mathfrak{m}M = 0$.

iii) (Faltings) Assume that $x^{1/n} \in A$ for some $x \in A$ and all n. M is called almost zero with respect to x, if $x^{1/n}$ kills M for arbitrarily large n.

• **Definition.** Let $\underline{x} := x_1, \ldots, x_r$ be a sequence of elements of a ring A.

i) $\underline{x}$ is called system of parameters if $\underline{x}$ is a system of parameters for a Netherian and local subring of A.

ii) $\underline{x}$ is called almost regular sequence if

$$((x_1, \ldots, x_{i-1})A :_A x_i)/(x_1, \ldots, x_{i-1})A$$

is almost zero for each $i = 1, \ldots, r$.

iii) A is called almost Cohen-Macaulay if every system of parameters for A is an almost regular sequence.

The following observation motivate us to study non-Noetherian ring as first step toward almost Cohen-Macaulayness:

- **Fact:** Adopt the above notation and assumptions. If there exists an almost zero module, then the base ring is non-Noetherian.

Now consider the following conjecture of Glaz:

- **Conjecture.** There exists a definition of the notion of non-Noetherian Cohen-Macaulay rings such that it satisfies the following three conditions.

(i) The definition coincides with the original definition in the Noetherian case.

(ii) Coherent regular rings are Cohen-Macaulay.

(iii) For a coherent regular ring R and a group G of automorphisms of R, assume that there exists a module retraction map $\rho : R \longrightarrow R^G$ and that R is a finitely generated R^G-module. Then R^G is Cohen-Macaulay.

Perhaps it is worth pointing out that there are many characterizations of Noetherian Cohen-Macaulay rings and modules. In the non-Noetherian case, these are not necessarily equivalent. All of these characterizations have been chosen as candidates for definition of non-Noetherian Cohen-Macaulay rings.

• **Definition.** Let R be a ring and M an R-module.

(i) R is called Cohen-Macaulay in the sense of Hamilton-Marley, if each strong parameter sequence on R becomes a regular sequence on R. We denote this property by **HM**.

(ii) M is called Cohen-Macaulay in the sense of Glaz, if for each prime ideal $\mathfrak{p}$ of R,

$$ht_M(\mathfrak{p}) = Kgrad_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p})$$

and denote this by **Glaz**.(Here Kgrad is Koszul grad).

(iii) Let $\mathfrak{a}$ be a finitely generated ideal of R. Set $\mu(\mathfrak{a})$, the minimal number of elements of R that need to generate $\mathfrak{a}$. Assume that for each ideal $\mathfrak{a}$ with the property $ht\mathfrak{a} \geq \mu(\mathfrak{a})$, we have $\min(\mathfrak{a}) = wAss_R(R/\mathfrak{a})$. A ring with such property is called weak Bourbaki unmixed. We denote this property by **WB**.

(iv) Let $\mathcal{A}$ be a non empty subclass of the class of all ideals of a ring R. We say that M is Cohen-Macaulay in the sense of $\mathcal{A}$, if $ht_M(\mathfrak{a}) = Kgrad_R(\mathfrak{a}, M)$ for all ideals $\mathfrak{a}$ in $\mathcal{A}$. We denote this property by $\mathcal{A}$. The classes we are interested in are $Supp_R(M)$, $Supp_R(M) \cap \max(R)$, the class of all ideals and the class of all finitely generated ideals. We denote them respectively by **Spec**, **Max**, **ideals** and **f.g.**.

Relations. The following diagram illustrates their relations:

$$\textbf{Max} \Leftarrow \textbf{Spec} \Leftrightarrow \textbf{ideals} \Rightarrow \textbf{Glaz} \Rightarrow \textbf{f.g.} \Rightarrow \textbf{HM} \Leftarrow \textbf{WB}$$

Also, when the base ring is coherent, we show that **Spec** $\Rightarrow$ **WB**.

• **Remark.** By $Egrad_A(\mathfrak{a}, M)$ we mean Ext grade. While $Egrad_A(\mathfrak{a}, M)$ has many common properties with $Kgrad_A(\mathfrak{a}, M)$, one difference is that a ring A may satisfies the inequality

$$Egrad_A(\mathfrak{a}, M) > ht_M(\mathfrak{a}).$$

It is worth to note that

$$Kgrad_A(\mathfrak{a}, M) \leq ht_M(\mathfrak{a})$$

has an important role in our work.

Non examples. Here, we provide some counterexamples to show that none of the following implications are valid:

$$\begin{array}{ccccc} & & & \textbf{WB} & \\ & & & \Uparrow & \\ \textbf{f.g. ideals} & \Leftarrow \textbf{Max} & \Leftrightarrow & \textbf{HM} & \Rightarrow \textbf{f.g. ideals} \end{array}$$

The next result show that the following can not solve Glaz's conjecture: Let $(R, \mathfrak{m})$ be a valuation domain. Then, the following are equivalent:

(i) R satisfies the property **ideals**.

(ii) R satisfies the property **Spec**.

(iii) R satisfies the property **Glaz**.

(iv) R satisfies the property **f.g. ideals**.

(v) $\dim R \leq 1$.

(vi) R satisfies the property **WB**.

(vii) R satisfies the property **Max**.

We present another definition. We call it weakly Cohen-Macaulay.

- **Theorem.** The following assertions hold.

(i) A Noetherian ring is Cohen-Macaulay with original definition in Noetherian case if and only if it is weakly Cohen-Macaulay.

(ii) Coherent regular rings are weakly Cohen-Macaulay.

(iii) Let R be a weakly Cohen-Macaulay ring and G a finite group of automorphisms of

R such that the order of G is a unit in R. Assume that R is finitely generated as an R^G-module. Then R^G is weakly Cohen-Macaulay.

(iv) Let R be a Noetherian Cohen-Macaulay ring. Then the polynomial ring $R[X_1, X_2, \cdots]$ is weakly Cohen-Macaulay.

(v) If $R_{\mathfrak{p}}$ is weakly Cohen-Macaulay for all prime ideals $\mathfrak{p}$ of R, then R is weakly Cohen-Macaulay.

We also study the behavior of rings of invariants of different types of non-Noetherian Cohen-Macaulay rings.

The following observation motivate us to study Cohen-Macaulayness w.r.t Serre classes as second step toward almost Cohen-Macaulayness:

- **Fact:** Classes of almost zero modules are hereditary torsion theories:

- **Definition**Let A be a ring and let $\mathfrak{T}$ be a subclass of A-modules.

i) $\mathfrak{T}$ is called a Serre class, if for any exact sequence of A-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

the module $M \in \mathfrak{T}$ if and only if $M' \in \mathfrak{T}$ and $M'' \in \mathfrak{T}$.

ii) A Serre class which is closed under taking the directed limit of any directed system of its objects is called a torsion theory.

One may introduce the notion of weak M-sequences with respect to a Serre class $\mathcal{S}$. We define the $\mathcal{S} - cgrad_R(\mathfrak{a}, M)$ as the supremum of the lengths of weak M-sequences with respect to $\mathcal{S}$ in $\mathfrak{a}$. We characterize it via

$$\mathcal{S}{-}Egrad_R(\mathfrak{a}, M) := \inf\{i \in \mathbb{N}_0 | Ext^i_R(R/\mathfrak{a}, N) \notin \mathcal{S}\},$$

and also we characterize it via Koszul complex and local cohomology modules.

Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. Then M is called an $\mathcal{S}$-*Cohen-Macaulay* R-module if any system of parameters of M form a weak M-sequence with respect to $\mathcal{S}$. The ring R is called $\mathcal{S}$-Cohen-Macaulay if R is $\mathcal{S}$-Cohen-Macaulay over itself.

Notation. For an R-module L, we denote $\{\mathfrak{p} \in Supp_R L | R/\mathfrak{p} \notin \mathcal{S}\}$ by $\mathcal{S} - Supp_R L$ and $\{\mathfrak{p} \in Ass_R L | R/\mathfrak{p} \notin \mathcal{S}\}$ by $\mathcal{S} - Ass_R L$.

The following theorem, which provides some equivalent conditions to Definition 3.4.

- **Theorem** Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. Then the following are equivalent:

(i) M is an $\mathcal{S}$-Cohen-Macaulay R-module.

(ii) For any $\mathfrak{p} \in \mathcal{S} - Supp_R(M)$, $M_\mathfrak{p}$ is Cohen-Macaulay and

$$ht_M(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim M.$$

(iii) For any $\mathfrak{p} \in \mathcal{S}{-}Supp_R(M)$, $Egrad_{\mathcal{S}}(\mathfrak{p}, M) = \mathcal{S} - ht_M(\mathfrak{p}) = ht_M(\mathfrak{p})$ and

$$ht_M(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim M.$$

(iv) $depth_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim M - \dim(R/\mathfrak{p})$ for any $\mathfrak{p} \in \mathcal{S} - Supp_R(M)$.

(v) If $x_1, \cdots, x_d$ is a system of parameters for M, then

$$\dim(R/\mathfrak{p}) = \dim M - i$$

for all $\mathfrak{p} \in \mathcal{S} - Ass_R(M/(x_1, \cdots, x_i)M)$ and all $0 \leq i \leq d := \dim M$.

(vi) $ht(\mathfrak{p}/\mathfrak{q}) + ht_M(\mathfrak{q}) = ht_M(\mathfrak{p})$ for all $\mathfrak{q} \subseteq \mathfrak{p}$ of $\mathcal{S} - Supp_R(M) \cup \{\mathfrak{m}\}$, $M_\mathfrak{p}$ is Cohen Macaulay for all $\mathfrak{p} \in \mathcal{S} - Supp_R(M)$ and $\dim(R/\mathfrak{p}) = \dim M$ for all $\mathfrak{p} \in \min(\mathcal{S} - Supp_R(M))$.

• **Remark** We say that an ideal $\mathfrak{a}$ of R is unmixed on M with respect to $\mathcal{S}$, if $\mathcal{S} - Ass_R(\frac{M}{\mathfrak{a}M})$ is equal with $\{\mathfrak{p} \in \min(Supp_R(\frac{M}{\mathfrak{a}M})) : R/\mathfrak{p} \notin \mathcal{S}\}$. A connection between this and the property that $M_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p} \in \mathcal{S} - Supp_R(M)$ are given.

Let $f : R \longrightarrow A$ be a flat homomorphism of rings and $\mathcal{S}$ a Serre class of A-modules. Set $\mathcal{S}^c = \{M \in R\text{-}\mathbf{Mod} | M \otimes_R A \in \mathcal{S}\}$.

• **Theorem** Let $f : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings. Let $\mathcal{S}$ be a Serre class of A-modules and M a finitely generated R-module. Then

$M \otimes_R A$ is $\mathcal{S}$-Cohen-Macaulay if and only if the following three conditions are satisfied:

(i) M is $\mathcal{S}^c$-Cohen-Macaulay.

(ii) $\frac{A_{\mathfrak{q}}}{f^{-1}(\mathfrak{q})A_{\mathfrak{q}}}$ is Cohen-Macaulay for all $\mathfrak{q} \in \mathcal{S} - Supp_A(M \otimes_R A)$.

(iii) $ht(\frac{\mathfrak{q}}{f^{-1}(\mathfrak{q})A}) + \dim(\frac{A}{\mathfrak{q}}) = \dim(\frac{A}{f^{-1}(\mathfrak{q})A})$ for all $\mathfrak{q} \in \mathcal{S} - Supp_A(M \otimes_R A)$.

Let $(R, \mathfrak{m})$ be a local ring and M a d-dimensional finitely generated R-module. Set

$$\mathfrak{a}(M) := \prod_{i=1}^{d-1} Ann_R(H^i_{\mathfrak{m}}(M)).$$

• **Theorem.** Let $(R, \mathfrak{m})$ be a local ring and M a d-dimensional finitely generated R-module. Consider the following conditions:

i) $R/\mathfrak{a}(M) \in \mathcal{S}$.

ii) M is an $\mathcal{S}$-Cohen-Macaulay R-module.

Always $i)$ implies $ii)$ and if R is a quotient of a Gorenstein local ring, then $ii)$ implies $i)$.

Denote by $\mathcal{NCM}(M)$ the non Cohen-Macaulay locus of M. It consists of the following:

$$\{\mathfrak{p} \in SpecR | M_{\mathfrak{p}} \textit{ is not Cohen-Macaulay}\}.$$

• **Theorem.** Let $(R, \mathfrak{m})$ be a local ring which is a quotient of a Cohen-Macaulay ring and M a finitely generated R-module. Then the following are equivalent:

(i) M is an $\mathcal{S}$-Cohen-Macaulay R-module.

(ii) $R/\mathfrak{a}_M \in \mathcal{S}$ and $\dim(R/\mathfrak{p}) = \dim M$ for all prime ideals $\mathfrak{p} \in \min(\mathcal{S} - Supp_R(M))$.

Next by combining the preceding steps we give the notion of almost Cohen-Macaulayness.

- **Definition.** Let $\mathfrak{T}$ be a torsion theory of A-modules and M an A-module. A sequence $\underline{x} := x_1, \ldots, x_r$ of elements of A is called M-regular sequence w.r.t. $\mathfrak{T}$ if

$$((x_1, \ldots, x_{i-1})M :_M x_i)/(x_1, \ldots, x_{i-1})M \in \mathfrak{T}$$

for all $i = 1, \ldots, r$.

We define $\mathfrak{T}-cgrad_A(\mathfrak{a}, M)$ as the supremum length of M-sequences with respect to $\mathfrak{T}$ in $\mathfrak{a}$. The notions of Cech grade, Ext grade, Koszul grade and local cohomology grade w.r.t. $\mathfrak{T}$ are defined in a a natural way. We denote them by $\mathfrak{T}-Cgrad_A(\mathfrak{a}, M)$, $\mathfrak{T}-Egrad_A(\mathfrak{a}, M)$, $\mathfrak{T}-Kgrad_A(\mathfrak{a}, M)$ and $\mathfrak{T}-Hgrad_A(\mathfrak{a}, M)$.

We denoted the category of all A-modules by A-Mod. We say that a torsion theory $\mathfrak{T}$ is representable if there exists an R-module M such that

$$\mathfrak{T} = \{N \in \text{A-Mod} | Supp(N) \subseteq Supp(M)\}.$$

One of our results says that any torsion theory over a Noetherian ring is representable. The Noetherian assumption is really needed.

• **Theorem** Let $\mathfrak{T}$ be a torsion theory and M an A-module. Then

(i) $\mathfrak{T} - Cgrad_A(\mathfrak{a}, M) = \mathfrak{T} - Kgrad_R(\mathfrak{a}, M)$.

(ii) If either $\mathfrak{T}$ is representable or A is coherent, then

$$\begin{aligned} \mathfrak{T} - Egrad_R(\mathfrak{a}, M) &= \mathfrak{T} - Kgrad_R(\mathfrak{a}, M) \\ &= \mathfrak{T} - Hgrad_R(\mathfrak{a}, M). \end{aligned}$$

A sequence $\underline{x} := x_1, \ldots, x_r$ of elements of A is called system of parameters for A if $\underline{x}$ is system of parameters for a Noetherian local subring $(R, \mathfrak{m})$ of A. If every system of parameters for A is a regular sequence w.r.t. $\mathfrak{T}$, then A is said to be almost Cohen-Macaulay w.r.t. $\mathfrak{T}$.

We close our talk by the following result which motivate us not only to write a thesis in commutative algebra but also for our future study of commutative algebra.

- **Theorem**(Roberts-Singh-Srinivas)

Let R be a local domain with an integral extension which is almost Cohen-Macaulay w.r.t. a valuation. Then the Monomial Conjecture holds for R.

Thank you for your consideration