Abstract

Betti numbers are topological objects which were proved to be invariants by Poincaré, and used by him to extend the polyhedral formula to higher dimensional spaces. Informally, the Betti number of a surface is the maximum number of cuts that can be made without dividing the surface into two separate pieces. Formally, the nth Betti number is the rank of the nth homology group of a topological space.

On the other hand, the theory of Hilbert functions of finitely generated graded modules over Noetherian rings are among important topics in commutative algebra. In fact the properties of an algebra that can be read off from its Hilbert functions are its *linear* properties. Those that cannot, such as its reduction numbers, are taken to be *nonlinear* properties. Therefore, it is not unexpected that relationships between Hilbert functions and such nonlinear invariants are expressed by inequalities. More concretely, the question is: If the syzygies of an ideal I code its linear invariants, where we should look for the nonlinear invariants of I? Some immediate partial answers can be obtained by looking at the syzygies of the powers I^n of I and considering the algebraic relations among the elements of I. Another sets of invariants of an algebra are the suitably-interpreted multiplicities.

One of the goals of this thesis is to explain certain properties of the powers I^n of an equigenerated graded ideal I by testing the initial ideal (w.r.t. some term order) of the defining equations of the Rees ring of I. More specifically, we provide an upper bound for the Castelnuovo-Mumford regularity of powers of I and give a simple criterion in terms of the Rees algebra of I to check regularity of its powers. This gives some information on the nonlinear invariants of I.

Another goal of thesis is to study the Betti numbers of the canonical module of a Cohen-Macaulay ring. In other words, for the growth of the Betti sequence of the canonical module, we provide some generalizations of the known case where the radical cube of the maximal ideal is zero. This leads to some new results on even more general contexts. We note that this study is worthwhile since canonical modules play a central role in the local duality theorem relating local cohomology with certain Ext functors.

The study of the graded minimal free resolution of some ideals over polynomial rings and the finiteness of the Bass numbers, the dual notion of the Betti numbers, of local cohomology modules are the other goals of this thesis.

The organization of this thesis is as follows. In Chapter 1 we recall some basic definitions and known facts on Betti numbers and graded Betti numbers, Hilbert functions, local cohomology modules, Castelnuovo-Mumford regularity, Koszul complexes, Taylor resolutions, and Stanley-Reisner rings. The Castelnuovo-Mumford regularity provides links between local cohomology theory and the syzygies of finitely generated graded modules over polynomial rings over a field. In Chapter 2 as we mentioned above, we provide a careful overview of the Castelnuovo-Mumford regularity and its asymptotic behavior. We also recall the Rees ring of a homogenous ideal with special attention to its bigraded structure. This leads us to derive a criterion and algorithms to test and check the linear resolution of equigenerated ideals. Chapter 3 is devoted to the problem of growth of the Betti sequence of the canonical module where we provide some generalization of the case of radical cube zero. In Chapter 5 we study the Bass numbers of local cohomology modules in detail and give some answers to the problem of finiteness of this number which is equivalent with the finiteness of the set of associated primes of Artinian and minimax local cohomology modules. The graded minimal free resolution of some ideals are studied in Chapter 4. We also provide a mechanism to construct pure Cohen-Macaulay simplicial complexes.

The results in Chapter 2, 3, 4 and 5 have been published in [9, 11, 12] or are submitted to [10, 13].