Betti numbers of modules over Noetherian rings with applications to local cohomology

A dissertation presented

by

Keivan Borna

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Abstract

Betti numbers are topological objects which were proved to be invariants by Poincaré, and used by him to extend the polyhedral formula to higher dimensional spaces. Informally, the Betti number of a surface is the maximum number of cuts that can be made without dividing the surface into two separate pieces. Formally, the nth Betti number is the rank of the nth homology group of a topological space.

On the other hand, the theory of Hilbert functions of finitely generated graded modules over Noetherian rings are among important topics in commutative algebra. In fact the properties of an algebra that can be read off from its Hilbert functions are its *linear* properties. Those that cannot, such as its reduction numbers, are taken to be *nonlinear* properties. Therefore, it is not unexpected that relationships between Hilbert functions and such nonlinear invariants are expressed by inequalities. More concretely, the question is: If the syzygies of an ideal I code its linear invariants, where we should look for the nonlinear invariants of I? Some immediate partial answers can be obtained by looking at the syzygies of the powers I^n of I and considering the algebraic relations among the elements of I. Another sets of invariants of an algebra are the suitably-interpreted multiplicities.

One of the goals of this thesis is to explain certain properties of the powers I^n of an equigenerated graded ideal I by testing the initial ideal (w.r.t. some term order) of the defining equations of the Rees ring of I. More specifically, we provide an upper bound for the Castelnuovo-Mumford regularity of powers of I and give a simple criterion in terms of the Rees algebra of I to check regularity of its powers. This gives some information on the nonlinear invariants of I.

Another goal of thesis is to study the Betti numbers of the canonical module of a Cohen-Macaulay ring. In other words, for the growth of the Betti sequence of the canonical module, we provide some generalizations of the known case where the radical cube of the maximal ideal is zero. This leads to some new results on even more general contexts. We note that this study is worthwhile since canonical modules play a central role in the local duality theorem relating local cohomology with certain Ext functors.

The study of the graded minimal free resolution of some ideals over polynomial rings and the finiteness of the Bass numbers, the dual notion of the Betti numbers, of local cohomology modules are the other goals of this thesis.

The organization of this thesis is as follows. In Chapter 1 we recall some basic definitions and known facts on Betti numbers and graded Betti numbers, Hilbert functions, local cohomology modules, Castelnuovo-Mumford regularity, Koszul complexes, Taylor resolutions, and Stanley-Reisner rings. The Castelnuovo-Mumford regularity provides links between local cohomology theory and the syzygies of finitely generated graded modules over polynomial rings over a field. In Chapter 2 as we mentioned above, we provide a careful overview of the Castelnuovo-Mumford regularity and its asymptotic behavior. We also recall the Rees ring of a homogenous ideal with special attention to its bigraded structure. This leads us to derive a criterion and algorithms to test and check the linear resolution of equigenerated ideals. Chapter 3 is devoted to the problem of growth of the Betti sequence of the canonical module where we provide some generalization of the case of radical cube zero. In Chapter 5 we study the Bass numbers of local cohomology modules in detail and give some answers to the problem of finiteness of this number which is equivalent with the finiteness of the set of associated primes of Artinian and minimax local cohomology modules. The graded minimal free resolution of some ideals are studied in Chapter 4. We also provide a mechanism to construct pure Cohen-Macaulay simplicial complexes.

The results in Chapter 2, 3, 4 and 5 have been published in [9, 11, 12] or are submitted to [10, 13].

Contents

List of Tables

List of Figures

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my mother,
and my wife.

Chapter 1

Preliminaries

In this chapter we collect some basic facts which will be used throughout of this thesis. Our sources for most of this material are: for basic commutative algebra, [26, 51]; for the theory of Cohen-Macaulay rings and modules, [16]; for homological algebra, [69]; and for Rees rings, [67]. In this dissertation we will be concerned primarily with commutative rings R (with identity) which are of either of the following types

- Noetherian local rings; or
- homogeneous K-algebras, K a field.

A homogeneous K-algebra (so called a standard graded algebra) is a positively graded ring $R = \bigoplus_{i=0}^{\infty} R_i$, where R_i denotes the ith graded component of R, such that $R_0 = K$, each R_i is a finite dimensional vector space over K, and R is generated as a K-algebra in degree 1: $R = R_0[R_1]$. Equivalently, R is of the form $R = S/I$, where $S = K[x_1, \cdots, x_n]$ is a polynomial ring over K with $\deg x_i = 1$ and I is a homogeneous ideal. If R is local, then we let $\mathfrak{m}$ denote the maximal ideal and we let K denote the residue class field $R/\mathfrak{m}$. If R is homogeneous, then $\mathfrak{m}$ will denote the irrelevant maximal ideal $\mathfrak{m} = R_1 \oplus R_2 \oplus \cdots$.

When R is a homogeneous K-algebra, automatically I is a homogeneous ideal of R. Furthermore M stands for a finitely generated R-module unless otherwise specified.

1.1 Betti numbers and graded Betti numbers

The Betti numbers of a finitely generated module M over a commutative Noetherian local unit ring R are the minimal numbers b_i for which there exists a long exact sequence

$$0 \to R^{b_n} \xrightarrow{f_n} R^{b_{n-1}} \to \ldots \to R^{b_1} \xrightarrow{f_1} R^{b_0} \xrightarrow{f_0} M \to 0,$$

which is called a minimal free resolution of M. The Betti numbers are uniquely determined by requiring that b_i be the minimal number of generators of $\mathrm{Ker} f_{i-1}$ for all $i \geq 0$.

Graded modules arise naturally in homology. For example for every integer i, there exists an ith singular homology group of a space $H_i(X)$, and usually the "total homology" of the space is considered to be the direct sum $\oplus_i H_i(X)$. This makes the "total" homology of X a $\mathbb{Z}$- module graded over the natural numbers $\mathbb{N}$. There are several reasons to support studying Graded Betti numbers in detail. A minimal free resolution of a finitely generated graded module M over a commutative Noetherian $\mathbb{N}$-graded ring R in which all maps are homogeneous module homomorphisms, i.e., the homomorphisms that map every homogeneous element to a homogeneous element of the same degree, is called the *Graded free resolution.* It is usually written in the form

$$\cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{s\,j}} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{1\,j}} \longrightarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{0\,j}} \longrightarrow M \longrightarrow 0, \tag{1.1}$$

where $R(-j)$ indicates the ring R with the shifted graduation such that, for all $a \in \mathbb{Z}$,

$$(R(-j))_a = R_{a-j}.$$

For all nonnegative integers i and all integers j, $\beta_{i\,j}$ is the number of copies of $R(-j)$ appearing in the ith module of the resolution, and is called a *graded Betti number,* i.e., F_i requires $\beta_{i\,j}$ minimal generators of degree j. The ordinary ith Betti number is $\beta_i = \sum_{j\in\mathbb{Z}} \beta_{i\,j}$. Furthermore, if R is a polynomial ring over a field K we have $\beta^R_{i\,j}(M) = \dim_K \operatorname{Tor}^R_i(M,K)_j$.

Example 1.1.1. Let R be the polynomial ring $K[x_1, x_2, x_3]$ over a field K, with the usual graduation. Then the graded free resolution of $M = R/(x_1^2, x_2^3)$ is

$$0 \longrightarrow R(-5) \xrightarrow{d_2} R(-2) \oplus R(-3) \xrightarrow{d_1} R \xrightarrow{d_0} M \longrightarrow 0,$$

where

$$d_0 : 1 \mapsto \overline{1},$$

$$d_1 : (1,0) \mapsto (-x_1^2), (0,1) \mapsto (-x_2^3),$$

$$d_2 : 1 \mapsto (-x_2^3, x_1^2).$$

In $R(-2)$, the constant polynomials have degree 2. It follows that $-x_2^3$ has degree 5. Similarly, x_1^2 has degree 5 in $R(-3)$.

Let $F^\bullet$ be a graded free resolution of M. The ranks and twists of the modules F_i is kept track of in "Betti diagram". Since the numerical invariants of minimal free resolutions contain more information than Hilbert functions, a convenient way of displaying this information in a compact form is provided in Betti diagram.

The computer algebra system CoCoA [20] has such facility and in fact the command *BettiDiagram* returns the ("Macaulay style") Betti diagram for module M. The graded Betti number $\beta_{i\,i+j}$ is displayed in column i and row j, and the rows and columns are numbered starting from zero. By convention, we often write a dash "-" in place of a 0. The j-th column specifies the degrees of the generators of F_j and the last row labeled "Tot" represents the ordinary Betti numbers.

	0	1	$\cdots$	s
0	$\beta_{0\,0}$	$\beta_{1\,1}$	$\cdots$	$\beta_{s\,0+s}$
1	$\beta_{0\,1}$	$\beta_{1\,2}$	$\cdots$	$\beta_{s\,1+s}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
j	$\beta_{0\,j}$	$\beta_{1\,j+1}$	$\cdots$	$\beta_{s\,j+s}$
Tot	β_0	β_1	β_2	$\cdots$

For instance, in the resolution in Example 1.1.1, the corresponding Betti diagram is,

	0	1	2
0	1	-	-
1	-	1	-
2	-	1	-
3	-	-	1
Tot	1	2	1

As a second example consider the following situation.

Example 1.1.2. Let $R = K[x_1, x_2, x_3]$, and $\mathfrak{m} = (x_1, x_2, x_3)$. The resolution of $R/\mathfrak{m}^5$ is

$$0 \longrightarrow R(-7)^{15} \longrightarrow R(-6)^{35} \longrightarrow R(-5)^{21} \longrightarrow R \longrightarrow R/\mathfrak{m}^5 \longrightarrow 0,$$

represented by this Betti diagram:

	0	1	2	3
0	1	-	-	-
1	-	-	-	-
2	-	-	-	-
3	-	-	-	-
4	-	21	35	15
Tot	1	21	35	15

1.2 Hilbert functions and multigraded Hilbert-Poincaré series

If R is homogeneous and $M = \bigoplus_{i\in\mathbb{Z}} M_i$ is a finitely generated graded R-modules of dimension d, then the Hilbert function of M is the map $\mathrm{H}(M,-) : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that

$$\mathrm{H}(M,n) = \dim_k(M_n) = \ell(M_n).$$

The Hilbert function agrees with a polynomial for large enough values of n :

$$\mathrm{H}(M,n) = \alpha.n^{d-1} + \text{terms of lower order.}$$

The generating function

$$\mathrm{H}_M(t) = \sum_{n\in\mathbb{Z}} \mathrm{H}(M,n)t^n$$

of the Hilbert function is called the Hilbert series. The Hilbert series is a rational function of t and can be written in the form

$$\mathrm{H}_M(t) = \frac{Q_M(t)}{(1-t)^d},$$

where $d = \dim M$ and $Q_M(t) \in \mathbb{Z}[t,t^{-1}]$ satisfies $Q_M(1) = \deg M$. The *multiplicity* (or degree) of R/I, denoted $e(M)$, is $Q_M(1)$. If M is positively graded, i.e., $M_i = 0$ for all

$i < 0$, then $Q_M(t)$ is an ordinary polynomial with integer coefficients in the variable t. If moreover $d = 0$, then $\mathrm{H}_M(t) = Q_M(t)$, i.e., the Hilbert series is a polynomial.

If M has a finite graded free resolution

$$\cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{s\,j}} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{1\,j}} \longrightarrow \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{0\,j}} \longrightarrow M \longrightarrow 0,$$

then

$$\mathrm{H}_M(t) = \mathrm{H}_R(t) \sum_{i,j} (-1)^i \beta_{i\,j}^R(M) t^j. \tag{1.2}$$

Moreover, if $x_1, \cdots, x_r$ is a regular sequence over M of homogeneous elements of degree 1, then the Hilbert function of the $n-r$-dimensional quotient module $\bar{M} = M/(x_1, \cdots, x_r)M$ is

$$\mathrm{H}_{\bar{M}}(t) = \frac{Q_M(t)}{(1-t)^{n-r}}, \tag{1.3}$$

and in particular, $Q_{\bar{M}}(t) = Q_M(t)$.

These properties suggest effective methods for computing the Hilbert series of a finitely generated graded module over the polynomial ring $R = K[X_1, \cdots, X_n]$, where K is a field.

The Hilbert series of R, which has dimension n, can be obtained by considering the maximal regular sequence $X_1, \cdots, X_n$ of R, and the Hilbert function of the 0-dimensional quotient ring $\bar{R} = R/(X_1, \cdots, X_r)$, which is the same as K. Now $\mathrm{H}(K, 0) = 1$, and $\mathrm{H}(K, i) = 0$ for all $i \neq 0$. Hence $\mathrm{H}_K(t) = 1$. It follows that $Q_R(t)$ is the constant polynomial 1, so that

$$\mathrm{H}_R(t) = \frac{1}{(1-t)^n}.$$

This approach can be applied to all Cohen-Macaulay quotient rings $S = R/I$, where I is an ideal generated by homogeneous polynomials. The first step is to find a maximal regular sequence $f_1, \cdots, f_m$ of S composed of homogeneous polynomials of degree 1; here, by virtue of the Cohen-Macaulay property, $m = \dim S$. This will produce a 0-dimensional ring $\bar{S} = S/(f_1, \cdots, f_m)$ whose Hilbert series is the polynomial $Q_{\bar{S}}$. By (1.2) and (1.3) we get that

$$\mathrm{H}_S(t) = \frac{Q_{\bar{S}}(t)}{(1-t)^m}.$$

Example 1.2.1. Let f be a homogenous polynomial of $R = K[x_1, \cdots, x_n]$ of degree $d > 0$ and $I = (f)$. The graded free resolution of R/I is

$$0 \longrightarrow R(-d) \xrightarrow{1 \mapsto f} R \longrightarrow R/I \longrightarrow 0,$$

which yields $\beta_{0\,0}, \beta_{1\,d} = 1$, whereas the remaining $\beta_{i\,j}$ are zero. Hence (1.2) implies that,

$$\begin{aligned} \mathrm{H}_{R/I}(t) &= \mathrm{H}_R(t)(1-t^d) \\ &= \frac{(1-t^d)}{(1-t)^n} \\ &= \frac{(1+t+t^2+\cdots+t^{d-1})}{(1-t)^{n-1}}, \end{aligned}$$

so the Hilbert series of R/I is

$$\frac{1+t+t^2+\cdots+t^{d-1}}{(1-t)^{n-1}}.$$

For more complicated ideals I, the computation requires the use of Gröbner bases; see [45, 58] for a fruitful resource of techniques.

In [29, Chapter 6] some techniques for computing the multiplicity is provided. Thanks to Francisco, we include them in this section.

Remark 1.2.2. Let I be a homogeneous ideal in $R = K[x_1, \cdots, x_n]$. If R/I is Artinian, then the multiplicity is just $dim_K(R/I)$, or the sum of the values of the Hilbert function of R/I. If R/I is not Artinian, and $P = a_d t^d + \cdots + a_0$, where $a_d \neq 0$, is its Hilbert polynomial, then $e(R/I) = a_d.d!$.

1. Let $I = (x^2, xy^3, xy^2z, y^4, z^5) \subset R = k[x, y, z]$. To find the Hilbert function of R/I, we count the number of monomials of R in each degree not in I. The only monomial of degree zero in R is 1, and it is not in I, so $\mathrm{H}(R/I, 0) = 1$. Non of x, y and z is in I and so $\mathrm{H}(R/I, 1) = 3$. In degree 2, there are five monomial of R not in I, y^2, z^2, xy, xz, yz; thus $\mathrm{H}(R/I, 2) = 5$. It is also easy to see that $\mathrm{H}(R/I, 3) = 7$, $\mathrm{H}(R/I, 4) = 6$, $\mathrm{H}(R/I, 5) = 5$, $\mathrm{H}(R/I, 6) = 3$ and $\mathrm{H}(R/I, 7) = 1$. All the monomials of S of degree greater than or equal to 8 are in I, and therefore $\mathrm{H}(R/I, d) = 0$ for $d \geq 8$. We abbreviate this data by writing $\mathrm{H}(R/I) = (1, 3, 5, 7, 6, 5, 3, 1)$. Hence $e(R/I) = 31$, the K-vector space dimension of S/I.

2. If we remove the generators y^4 and z^5, we obtain an ideal $J = (x^2, xy^3, xy^2z)$ that is not Artinian. The Hilbert series of R/J is $\frac{1-t^2-2t^4+3t^5-t^6}{(1-t)^3} = \frac{1+t-2t^4+t^5}{(1-t)^2}$, and thus $e(R/J) = 1 + 1 - 2 + 1 = 1$. Alternatively, the Hilbert polynomial of R/J is $t + 3$, and hence $e(R/J) = 1.1! = 1$.

Finally we recall the definition of *multigraded Hilbert-Poincaré* series. Let K be a field and $S = K[x_1, \cdots, x_n]$ the polynomial ring with its natural $\mathbb{N}$-grading. When I is a homogenous ideal generated by monomials in the variables $x_1, \cdots, x_n$, the ring $R = S/I$ is $\mathbb{N}$-graded for the induced grading. Let M be an $\mathbb{N}$-graded finite R-module. For each $i \geq 0$, the K-vector space $\mathrm{Tor}_i^R(M, K)$ is multigraded with homogeneous basis elements of degrees $a_{i,j}$ for $j = 1, \cdots, \beta_i$. The multigraded Poincaré series of M in the variables t and $s_1, \cdots, s_n$ is defined by $P_M^R(s,t) = \sum_{i \geq 0} r_i(s) t^i$, where

$$r_i(s) = \sum_{d \in \mathbb{Z}^n} \dim_K \operatorname{Tor}_i^R(M,K)_d \, s_1^{d_1} \cdots s_n^{d_n}$$

is a Laurent polynomial in $\mathbb{Z}[s_1^{\pm 1}, \cdots, s_n^{\pm 1}]$. Note that by setting $s_1 = s_2 = \cdots = s_n = u$ one obtains the usual Poincaré series of the graded R-module associated to the n-graded R-module M; see [40] for some results on shifts in resolutions of multigraded modules.

1.3 Koszul complex and Taylor resolution

Let R be a graded ring and $\underline{x} = x_1, \cdots, x_n$ a sequence of homogeneous elements. The Koszul complex of R with respect to $\underline{x}$, usually denoted $K_\bullet(x_1, x_2, \cdots, x_n)$, is a free complex. There are exactly $\binom{n}{j}$ copies of the ring R in the jth slot in the complex $(0 \leq j \leq n)$. The matrices involved in the maps can be written down precisely. Letting $e_{i_1 \cdots i_n}$ denote a free-basis generator in K_p, $d : K_p \to K_{p-1}$ is defined by:

$$d(e_{i_1 \ldots i_n}) := \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} e_{i_1 \ldots \hat{i_j} \ldots i_n}.$$

For the case of two elements x and y, the Koszul complex can then be written down quite succinctly as $0 \to R \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} -y & x \end{pmatrix}} R \to 0$. The cycles in slot 1 are then exactly the linear relations on the elements x and y while the boundaries are the trivial relations. The first Koszul homology $H_1(K_\bullet(x,y))$ therefore measures exactly the relations mod the trivial relations. With more elements the higher-dimensional Koszul homologies measure the higher level versions of this.

In the case that the elements $x_1, x_2, \cdots, x_n$ form a regular sequence, the higher homology modules of the Koszul complex are all zero, so $K_\bullet(x_1, x_2, \cdots, x_n)$ forms a free resolution of the R-module $R/(x_1, x_2, \cdots, x_n)R$. For example if K is a field and $X_1, X_2, \cdots, X_d$ are indeterminates and R is the polynomial ring $k[X_1, X_2, \cdots, X_d]$, the Koszul complex on the X_i's, $K_\bullet(X_i)$, forms a concrete free R-resolution of k.

Theorem 1.3.1. *If $(R, \mathfrak{m})$ is local and M is a finitely-generated R-module with $x_1, x_2, \cdots, x_n$ in $\mathfrak{m}$, then the following are equivalent:*

1. *The (x_i) form an M-sequence,*

2. $H_1(K_\bullet(x_i)) = 0,$

3. $H_j(K_\bullet(x_i)) = 0$ *for all* $j \geq 1$.

Such presentation is engineered so that the homological algebra of the ring, in particular a minimal projective resolution of the trivial module, K, is easier to understand.

However there is a nice nonminimal resolution which was discovered by Diana Taylor in [18]. It generalizes the Koszul complex in a natural way. Let $S = A[x_1, \cdots, x_r]$, where A is any ring and the x_i are indeterminates. Let I be a monomial ideal of S, and $R = S/I$. Suppose that I is minimally generated by the monomials $m_1, \ldots, m_t$. Let X denote the exterior algebra of a rank t free module. The standard basis elements of X are written $T_{i_1 \ldots i_p}$ where $i_1 < \cdots < i_p$. Then X becomes a free resolution of S/I using the differentials:

$$d(T_{i_1\ldots i_p}) = \sum_{j=1}^{p}(-1)^{j-1}\frac{m_{i_1\ldots i_p}}{m_{i_1\ldots\hat{i_j}\ldots i_p}},$$

where $m_{i_1\ldots i_p}$ denotes the least common multiple of the monomials $m_{i_1},\ldots,m_{i_p}$ and $\hat{}$ denotes omission. This resolution is usually far from being minimal. An obstruction to minimality occurs every time $m_{i_1\ldots i_p} = m_{i_1\ldots\hat{i_j}\ldots i_p}$. Let J be the indexed set of the minimal monomial generating set of I. If K is any subset of J we use the notation m_k to denote the least common multiple of the monomials indexed by K. Of course when $K' \subseteq K$, $m_{K'}$ divides m_K. It is also known that the Taylor resolution is minimal if and only if $m_K \neq m_{K'}$ for all subsets $K' \subseteq K$; see [26, pp. 439] and [18] for more details.

Example 1.3.2. Let $R = K[x,y]$ and $I = (x^2, xy, y^2)$. The Koszul complex of R/I is as follows:

$$0 \longrightarrow R(-6) \xrightarrow{\begin{pmatrix} x^2 \\ -xy \\ y^2 \end{pmatrix}} R(-4)^3 \xrightarrow{\begin{pmatrix} 0 & y^2 & xy \\ y^2 & 0 & -x^2 \\ -xy & -x^2 & 0 \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}} R \longrightarrow 0,$$

and the **Taylor resolution** of R/I:

$$0 \longrightarrow R(-4) \xrightarrow{f_2} R(-3)^2 \oplus R(-4) \xrightarrow{f_1} R(-2)^3 \xrightarrow{f_0} R \to R/I \to 0,$$

where

$$f_2 = \begin{pmatrix} -y \\ 1 \\ -x \end{pmatrix}, f_1 = \begin{pmatrix} y & y^2 & 0 \\ -x & 0 & y \\ 0 & -x^2 & -x \end{pmatrix}, f_0 = \begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}.$$

1.4 Local cohomology and Local Duality

Let I be an ideal of a commutative Noetherian ring R and M an R-module. We define

$$\mathrm{H}^0_I(M) = \{m \in M : (\exists i) I^i m = 0\}$$

It is easy to see that $\mathrm{H}^0_I(M) = \varinjlim_{n \geq 0} \mathrm{Hom}_R(R/I^n, M)$. Since each $\mathrm{Hom}(R/I^i, -)$ is left exact and $\varinjlim$ is exact, we see that H^0_I is an additive left exact functor from R-**mod** to itself. The $q$the right derived functor of $\mathrm{H}^0_I(-)$ applied on M is shown by $\mathrm{H}^q_I(M)$ and since the direct limit is exact, we have

$$\mathrm{H}^q_I(M) = \varinjlim_{n \geq 0} \mathrm{Ext}^q_R(R/I^n, M)$$

So that $\mathrm{H}^i_I(M)$ is the ith cohomology module of the complex obtained by applying $\mathrm{H}^0_I(-)$ to an injective resolution of M.

Our standard reference for local cohomology is the book of Brodmann and Sharp [15] which gives a detailed and comprehensive account of this material. It covers important applications and uses detailed examples designed to illustrate the geometrical significance of aspects of local cohomology. In the following we recall some preliminary facts.

Theorem 1.4.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring or a homogeneous K-algebra, and let M be a finitely generated R-module. Local cohomology detects depth and dimension.*

$$\mathrm{depth}(M) = \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\},$$

$$\dim(M) = \max\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\} \quad \textit{Grothendieck's non-anishing.}$$

The following result which is known as the *local duality* theorem will be a handy toll:

Theorem 1.4.2. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay complete local ring of dimension d with canonical module ω_R. Then for all finitely generated R-modules M and all integers i there exists natural isomorphisms*

$$\begin{aligned} H^i_{\mathfrak{m}}(M) &\cong \mathrm{Hom}_R(\mathrm{Ext}^{d-i}_R(M, \omega_R), E(k)), \quad \textit{and} \\ \mathrm{Ext}^i_R(M, \omega_R) &\cong \mathrm{Hom}_R(H^{d-i}_{\mathfrak{m}}(M), E(k)). \end{aligned} \tag{1.4}$$

In particular, $\mathrm{Ext}^{d-i}_R(M, \omega_R) = 0$ *for all* $i < \dim M$.

Theorem 1.4.3. *Let R, S be two Noetherian rings, I an ideal of R and $\varphi : R \longrightarrow S$ be a ring map. Then $H^i_I(M) \cong H^i_{IS}(M)$ for every S-module M. Consequently, $H^i_I(M) = 0$ for $i > \dim(R/ann_R(M))$.*

1.5 Castelnuovo-Mumford regularity

Let K be a field and $S = K[x_1, \cdots, x_r]$ and let

$$\mathbb{F} : \cdots \to F_i \to F_{i-1} \to \cdots$$

be a graded complex of free S-modules, with $F_i = \bigoplus_j S(-a_{i,j})$. The Castelnuovo-Mumford regularity, or simply regularity, of F is the supremum of the numbers $a_{i,j} - i$. The regularity of a finitely generated graded S-module M is the regularity of a minimal graded free resolution of M. We will write $\mathrm{reg}(M)$ for this number. The regularity of an ideal is an important measure of how complicated the ideal is. The above definition of regularity shows how the regularity of a module governs the degrees appearing in a minimal resolution. As Eisenbud mentions in [27] Mumford defined the regularity of a coherent sheaf on projective space in order to generalize a classic argument of Castelnuovo. Mumford's definition [53] which is given in terms of sheaf cohomology. The definition for modules, which extends that for sheaves, and the equivalence with the condition on the resolution used the above definition, come from Eisenbud and Goto [28]. Alternate formulations in terms of Tor, Ext and local cohomology are given in the following. Let M be a graded S-module, $\mathfrak{m} = (x_1, \cdots, x_r)$ the maximal ideal of S, and $n = \dim(M)$. Let

$$a_i(M) = \max\{t; \mathrm{H}^i_{\mathfrak{m}}(M)_t \neq 0\}, 0 \leq i \leq n,$$

where $\mathrm{H}^i_{\mathfrak{m}}(M)$ is the ith local cohomology module with the support in $\mathfrak{m}$ (with the convention $\max \emptyset = -\infty$). Then the regularity is the number

$$\mathrm{reg}(M) = \max\{a_i(M) + i; 0 \leq i \leq n\}.$$

The following now are easy to be seen:

1. $\mathrm{reg}(I) = \mathrm{reg}(S/I) + 1$ for a graded ideal I of S,

2. For an Artinian graded S-module M, we have $\mathrm{H}^0_{\mathfrak{m}}(M) = M$, and hence

$$\mathrm{reg}(M) = \max\{j; M_j \neq 0\}.$$

We may also compute $\mathrm{reg}(I)$ in terms of Tor by the formula

$$\mathrm{reg}(I) = \max_k\{t_k(I) - k\},$$

where $t_p(I) := \max\{\text{degree of the minimal } p\text{th syzygies of } I\}$. Simply this definition may be rewritten as

$$\operatorname{reg}(I) = \max_{i,j}\{j - i : \operatorname{Tor}_i(I, K)_j \neq 0\},$$

$$= \max_{i,j}\{j - i ; \beta_{i\,j}(I) \neq 0\}.$$

Anyway, from local duality one sees that the two ways of expressing the regularity are also connected termwise by the inequality $t_k(I) - k \geq a_i(S/I) + n - k$.

More precisely, in [15, Example 13.4.6] it is mentioned that over a polynomial ring $R = K[x_1, \cdots, x_n]$ over a field K, graded in the usual way, if $\mathfrak{m}$ denotes the unique *maximal graded ideal $(x_1, \cdots, x_n)$ of R, for each f.g. graded R-module M and for each i, there homogenous isomorphisms

$$\mathrm{H}^i_{\mathfrak{m}}(M) \cong^* \operatorname{Hom}_K({}^*\operatorname{Ext}^{n-i}_R(M, R(-n)), K)$$

and (since $R_0 = K$ is a complete local ring)

$${}^*\operatorname{Hom}_K(\mathrm{H}^i_{\mathfrak{m}}(M), K) \cong^* \operatorname{Ext}^{n-i}_R(M, R(-n)).$$

We say M has a p-linear resolution if and only if M is generated in degree p and $\operatorname{Tor}^S_i(M, K)_j = 0, \ \forall j - i \neq p$. That is, $\operatorname{Tor}^S_i(M, K)_{i+j} = 0$ for all $j \neq p$, i.e., $\beta^S_{i\,j}(M) = 0$ for all $j \neq p$. Therefore M has a free resolution of the form

$$0 \to \oplus S(-p-k)^{\beta^S_{k\,k+p}(M)} \to \cdots \to \oplus\_k S(-p-1)^{\beta^S_{1\,1+p}(M)} \to \oplus S(-p)^{\beta^S_{0\,p}(M)} \to 0.$$

So we can bring the following definition of regularity in terms of linear resolutions.

$$\operatorname{reg}(M) = \min\{c : M_c \text{ has linear resolution}\},$$

where naturally $M_c = \bigoplus_{i \geq c} M_i$.

Example 1.5.1. Note that $\operatorname{reg}(M) = 8$ is simply equivalent with $\beta_{i\,j}(M) = 0$ where $j - i > 8$ and so actually we have the following $(\text{proj.dim}(M) + 1)$ conditions on M and its syzygies:

M is generated in degrees ≤ 8,

$\Omega_1(M)$ is generated in degrees ≤ 9,

$\Omega_2(M)$ is generated in degrees ≤ 10,

$\cdots$,

$\Omega_{\text{proj.dim}(M)}(M)$ is generated in degrees $\leq 8 + \text{proj.dim}(M)$.

Example 1.5.2. In Example 1.1.2, since no cancelation is possible in its Betti diagram, and therefore, since $\mathrm{H}(R/\mathfrak{m}^5) = (1; 3; 6; 10; 15)$, there is only one possible resolution for a module with Hilbert function $(1; 3; 6; 10; 15)$ which was given in Example 1.1.2. Note that $\mathfrak{m}^d$ has a d-linear resolution for all $d \geq 1$; that is, for $i \geq 1$, the i-th free module in the resolution, if nonzero, is a direct sum of copies of $R(-d - i)$ and so its Castelnuovo-Mumford regularity is simply $\operatorname{reg}(\mathfrak{m}^d) = d$.

Proposition 1.5.3. *Given a short exact sequence of modules* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, *we have the following inequalities.*

- $\operatorname{reg}(A) \leq \max\{\operatorname{reg}(B), \operatorname{reg}(C) + 1\}$
- $\operatorname{reg}(B) \leq \max\{\operatorname{reg}(A), \operatorname{reg}(C)\}$
- $\operatorname{reg}(C) \leq \max\{\operatorname{reg}(A) - 1, \operatorname{reg}(B)\}$

Proposition 1.5.4. *If M is a finitely generated R-module and $x \in R_1$ is a linear form such that $\ell(0 :_M x) < \infty$, then*

$$\operatorname{reg}(M) = \max\{\operatorname{reg}(0 :_M x), \operatorname{reg}(M/xM)\}.$$

In particular, if $x \in R_1$ *is regular on* M*, then* $\operatorname{reg}(M) = \operatorname{reg}(M/xM)$.

1.6 Stanley-Reisner rings

A simplicial complex Δ over a set of vertices $V = \{v_1, \cdots, v_n\}$ is a collection of subsets of V, for which $\{v_i\} \in \Delta$ for all i and if $F \in \Delta$ then all subsets of F are also in Δ. An element of Δ is called a face of Δ, and the dimension of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The faces of dimensions 0 and 1 are called vertices and edges, respectively, and $\dim \emptyset = -1$. The maximal faces of Δ under inclusion are called facets of Δ. The dimension of the simplicial complex Δ is the maximal dimension of its facets. Let Δ be a simplicial complex on the vertex set $V = \{v_1, \cdots, v_n\}$, and K be a field. The *Stanley-Reisner* ring of the complex Δ is the graded K-algebra $K[\Delta] = K[X_1, \cdots, X_n]/I_\Delta$, where I_Δ is the ideal generated by all monomials $X_{i_1} X_{i_2} \cdots X_{i_k}$ such that $\{v_{i_1}, v_{i_2}, \cdots, v_{i_k}\} \notin \Delta$. The dimension of a Stanley-Reisner ring can be easily determined.

Given a simplicial complex Δ, in order to reach I_Δ we may use the primary decomposition of the Stanley- Reisner ideal of Δ

$$I_\Delta = \bigcap_F P_F, \tag{1.5}$$

where the intersection is taken over all facets F of Δ, and P_F denotes the face ideal generated by all x_i such that $x_i \notin F$. In particular, $\dim K[\Delta] = \dim R/I_\Delta = \dim \Delta + 1$; see, for instance [16, Theorem 5.1.4].

The simplicial complex Δ is said to be *pure* if all its facets are of the same dimension, namely $\dim \Delta$. A Cohen-Macaulay simplicial complex is pure. Our terminology

Figure 1.1: The simplicial complex associated to R in Example 1.6.1

and comments comes from [16, 63].

Note that any quotient of a polynomial ring R over an ideal generated by square free monomials of degree greater than 1 (say I), is the Stanley-Reisner ring of a simplicial complex (say Δ). In the literature, the ring R/I is also known as the Stanley-Reisner ring of the simplicial complex Δ; see [35, 63].

Example 1.6.1. Let $S = K[X_1, X_2, Y_1, Y_2]$ be a K-algebra and consider

$$R = \frac{K[X_1, X_2, Y_1, Y_2]}{(X_1 X_2 Y_1 Y_2)}.$$

We denote by x_i and y_j the residue classes of X_i and Y_j for $i, j = 1, 2$ and denote the vertices corresponding to the X_i by v_i and those corresponding to the Y_j by w_j. We may view $R = K[x_1, x_2, y_1, y_2]$ as the Stanley-Reisner ring (associated to the) simplicial complex Δ whose facets are given by the sets

$$\{v_1, v_2, w_1\}, \{v_1, w_1, w_2\}, \{w_1, w_2, v_2\}, \text{ and } \{w_2, v_2, v_1\}.$$

See Figure 1 and [56, Example 5.12] for more details.

Chapter 2

Linear resolution of powers of an ideal

In this chapter we give a generalization of a recent result of Herzog, Hibi, and Zheng providing an upper bound for regularity of powers of an ideal. As the main result of the chapter, we give a simple criterion in terms of Rees algebra of a given ideal to show that high enough powers of this ideal have linear resolution. We apply the criterion to two important ideals J, J_1 for which we show that J^k, and J_1^k have linear resolution if and only if $k \neq 2$. The procedures we include in this work is encoded in computer algebra package CoCoA [20].

2.1 Upper bounds on Castelnuovo-Mumford regularity

Castelnuovo-Mumford regularity is a kind of universal bound for important invariants of graded algebras, such as the maximum degree of the syzygies and the maximum non-vanishing degree of the local cohomology modules. One has often tried to find upper bounds for the Castelnuovo-Mumford regularity in terms of simpler invariants which reflect the complexity of a graded algebra like dimension and multiplicity. Clearly $t_0(I^k) \leq k\, t_0(I)$ and one may expect to have the same inequality for regularity, that is, $\operatorname{reg}(I^k) \leq k \operatorname{reg}(I)$. Unfortunately this is not true in general. However, in [22] Cutkosky, Herzog, and Trung and in [44] Kodiyalam studied the asymptotic behavior of the Castelnuovo-Mumford regularity and independently showed that the regularity of I^k is a linear function for large k, i.e., there exist integers $a(I)$ and $b(I)$ such that

$$\operatorname{reg}(I^k) = a(I)k + b(I), \quad \forall k \geq c(I). \tag{2.1}$$

Now assume that I is an equigenerated ideal, that is, generated by forms of the same degree d. Then one has $a(I) = d$ and hence, $\operatorname{reg}(I^{k+1}) - \operatorname{reg}(I^k) = d$ for all $k \geq c(I)$. Hence we have

$$\operatorname{reg}(I^k) = (k - c(I))d + \operatorname{reg}(I^{c(I)}), \quad \forall k \geq c(I). \tag{2.2}$$

One says that the regularity of the powers of I jumps at place k if $\operatorname{reg}(I^k) - \operatorname{reg}(I^{k-1}) > d$. In [21] Conca gives several examples of ideals generated in degree d ($d = 2, 3$), with linear resolution (i.e., $\operatorname{reg}(I) = d$), and such that the regularity of the powers of I jumps at place 2, i.e., such that $\operatorname{reg}(I^2) > 2d$. As it is indicated in [21], the first example of such an ideal was given by Terai. Throughout this chapter we use J for this ideal. Geometrically speaking, this is an example of Reisner which corresponds to the (simplicial complex of a) triangulation of the real projective plane $\mathbb{P}^2$; see Fig. 2.1 and [16] for more details. Let $R := K[x_1, \cdots, x_6]$ one has

$$\begin{aligned} J = (&x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, \\ &x_3x_4x_6). \end{aligned} \tag{2.3}$$

Figure 2.1: The ideal of triangulation of the real projective plane $\mathbb{P}^2$.

It is known that J is a square-free monomial ideal whose Betti numbers, regularity and projective dimension depend on the characteristic of the base field. Indeed whenever $\operatorname{char}(K) \neq 2$, R/J is Cohen-Macaulay (and otherwise not), moreover one has $\operatorname{reg}(J) = 3$ and $\operatorname{reg}(J^2) = 7$ (which is of course $> 2 \times 3$). If $\operatorname{char}(K) = 2$, then J itself has no linear resolution. So the following natural question arises:

Question 2.1.1. *How it goes on for the regularity of powers of J?*

By the help of (2.1) we are able to write $\operatorname{reg}(J^k) = 3k + b(J)$, $\forall k \geq c(J)$. But what are $b(J)$ and $c(J)$? In this thesis we give an answer to this question and prove that J^k has linear resolution (in $\operatorname{char}(K) = 0$) $\forall k \neq 2$, that is, $b(J) = 0$ and $c(J) = 3$. That is

$$\operatorname{reg}(J^k) = 3k, \quad \forall k \neq 2.$$

To answer Question 2.1.1 we develop a general strategy and to this end we need to follow the literature a little bit. In [57] Römer proved that

$$\operatorname{reg}(I^n) \leq nd + \operatorname{reg}_x(\mathcal{R}(I)), \tag{2.4}$$

where $\mathcal{R}(I)$ is the Rees ring of I, which is naturally bigraded, and reg_x refers to the x-regularity of $\mathcal{R}(I)$, that is,

$$\operatorname{reg}_x(\mathcal{R}(I)) = \max\{b - i : \operatorname{Tor}_i(\mathcal{R}(I), K)_{(b,d)} = 0\},$$

as defined by Aramova, Crona and De Negri [4]. It follows from (2.4) that if $\operatorname{reg}_x(\mathcal{R}(I)) = 0$, then each power of I admits a linear resolution. Based on Römer's formula, in [34, Theorem 1.1 and Corollary 1.2] Herzog, Hibi and Zheng showed the following:

Theorem 2.1.2. *Let $I \subseteq K[x_1, \cdots, x_n] := S$ be an equigenerated graded ideal. Let m be the number of generators of I and let $T := S[t_1, \cdots, t_m]$, and let $\mathcal{R}(I) = T/P$ be the Rees algebra associated to I. If for some term order $<$ on T, P has a Gröbner basis G whose elements are at most linear in the variables $x_1, \cdots, x_n$, that is $\deg_x(f) \leq 1$ for all $f \in G$, then each power of I has a linear resolution.*

Theorem 2.1.2 is subject to condition that $\mathrm{in}(P) = (u_1, \cdots, u_m)$ and $\deg_x(u_i) \leq 1$. So the natural way to generalize it is to change the upper bound for x-degree of u_i with some number t. As one may expect, we end up with $\mathrm{reg}(I^n) \leq nd + (t - 1)\,\mathrm{proj.dim}(T/\mathrm{in}(P))$. The proof is mainly as that of Theorem 2.1.2 but for the sake of convenience of reader we bring it here.

Proposition 2.1.3. *Let $I \subseteq S$ be an equigenerated graded ideal and let $\mathcal{R}(I) = T/P$. If $\mathrm{in}(P) = (u_1, \cdots, u_m)$ and $\deg_x(u_i) \leq t$, then $\mathrm{reg}(I^n) \leq nd+(t-1)$ proj.dim$(T/\mathrm{in}(P))$.*

Proof. Let $C_\bullet$ be the Taylor resolution of $\mathrm{in}(P)$. The module C_i has the basis e_σ with $\sigma = j_1 < j_2 < \cdots < j_i \subseteq [m]$. Each basis element e_σ has the multidegree (a_σ, b_σ) where $x^{a_\sigma}.y^{b_\sigma} = \mathrm{lcm}\{u_{j_1}, \cdots, u_{j_m}\}$. It follows that $\deg_x(e_\sigma) \leq ti$ for all $e_\sigma \in C_i$. Since the shifts of $C_\bullet$ bound the shifts of a minimal multigraded resolution of $\mathrm{in}(P)$, we conclude that

$$\begin{aligned}\mathrm{reg}_x(T/P) \leq \mathrm{reg}_x(T/\mathrm{in}(P)) &= \max_{i,j}\{a_{ij} - i\} \\ &\leq ti - i = (t-1)i \\ &\leq (t-1)\,\mathrm{proj.dim}(T/\mathrm{in}(P)).\end{aligned}$$

Now (2.4) completes the proof. □

	$\underline{x} > \underline{t}$	$\underline{t} > \underline{x}$
DegRevLex	(1,2):2,(2,2):2	(1,2):2,(2,2):1
Lex	(1,2):2,(2,2):1	(1,2):2,(2,2):1

Table 2.1: Count of elements of in(P) with $deg_x > 1$ for the ideal of (2.3).

One can see that now Theorem 2.1.2 is the special case of Proposition 2.1.3 with $t = 1$.

Remark 2.1.4. The case where Q is not eqigenerated is also interesting. Let $Q = (q_1, \cdots, q_k)$ be an ideal of S and let $d(I)$ denote the minimum degree of the homogeneous generators of I and $D(I)$ denote the maximum degree of the homogeneous generators of I then for sure we have the following bounds:

$$d(I)k \leq reg(Q^k) \leq D(I)k.$$

Throughout this chapter we simply write $S = K[\underline{x}]$ and $T = S[\underline{t}]$. One can easily see that for J, (2.3), one has at least 3 elements in in(P) with $deg_x > 1$, no matter if we take initial ideal w.r.t. term ordering $\underline{x} > \underline{t}$ or $\underline{t} > \underline{x}$ in either Lex or DegRevLex order as it is reported in Table 2.1. Note that for example if one starts in DegRevLex order and $\underline{x} > \underline{t}$ then there is 4 elements in in(P) which have x-degree > 1 ($= 2$ actually) and among them 2 term has t-degree 1 and 2 term is in t-degree 2.

The main motivation for this work is to generalize Herzog, Hibi and Zheng's techniques in order to apply them to a wider class. Furthermore, we will indicate the least exponent k_0 for which I^k has linear resolution for all $k \geq k_0$. Indeed our generalization works for all ideals which admit the following condition:

Theorem 2.1.5. *Let $Q \subseteq S = K[x_1, \cdots, x_r]$ be a graded ideal which is generated by m polynomials all of the same degree d, and let $I = \text{in}(g(P))$ for some linear bi-*

transformation $g \in \mathrm{GL}_r(K) \times \mathrm{GL}_m(K)$*. Write* $I = G + B$ *where* G *is generated by elements of* $\deg_x \leq 1$ *and* B *is generated by elements of* $\deg_x > 1$*. If* $I_{(k,j)} = G_{(k,j)}$ *for all* $k \geq k_0$ *and for all* $j \in \mathbb{Z}$*, then* Q^k *has linear resolution for all* $k \geq k_0$*. In other words,* $\mathrm{reg}(Q^k) = kd$ *for all* $k \geq k_0$*.*

2.2 Rees algebra of an ideal and its bigrded structure

Let K be a field, $I = (f_1, \ldots, f_m)$ be a graded ideal of $S = K[x_1, \ldots, x_r]$ generated in a single degree, say d. The Rees algebra of I is known to be

$$R(I) = \bigoplus_{j \geq 0} I^j t^j = S[f_1 t, \ldots, f_m t] \subseteq S[t],$$

t an indeterminate. Let $T = S[t_1, \ldots, t_m]$. Then there is a natural surjective homomorphism of bigraded K-algebras $\varphi : T \longrightarrow R(I)$ with $\varphi(x_i) = x_i$ for $i = 1, \ldots, r$ and $\varphi(t_j) = f_j t$ for $j = 1, \ldots, m$. These equations carry most of the information one might want to have about the algebra $\mathcal{R}(I)$. Write down $\mathcal{R}(I) = T/P$. In this thesis we consider T, and so $\mathcal{R}(I)$, as a standard bigraded polynomial ring with $\deg(x_i) = (0, 1)$ and $\deg(t_j) = (1, 0)$. Indeed if we start with the natural bigraded structure $\deg(x_i) = (0, 1)$ and $\deg(f_j t) = (d, 1)$ then $\mathcal{R}(I)_{(k,vd)} = (I^k)_{vd}$, but the standard bidegree normalizes the bigrading in the following sense:

$$\mathcal{R}(I)_{(k,j)} = (I^k)_{kd+j} \tag{2.5}$$

There is a different platform from which to look at $\mathcal{R}(I)$ through the study of the reductions of the ideal I. However describing properties of the Rees algebra of the ideal

I, in any of these ways, emphasizes the structure of the polynomial relations amongst the elements of a generating set of I. Here we start with a matrix of presentation of the ideal $I = (f_1, \ldots, f_m)$, and using elimination theory directly or via Gröbner basis computations, one seeks to describe P.

As it was indicated in [67] studying the Rees algebra of an ideal I is focused on the degrees of a generating set for the presentation ideal P and seeks to obtain those equations from the syzygies of I. The ideal P, which we refer to as the *equations* of $R[It]$, is graded (in fact, a bigraded ideal of T):

$$P = P_1 + P_2 + \cdots ,$$

where P_1 is the R-module of linear forms $\sum a_i t_i$ such that $\sum a_i f_i = 0$. The module P_r is the module of syzygies of the r-products of the f_i.

Example 2.2.1. From [67, Example 1.2], if the ideal I is generated by a regular sequence $f_1, \cdots, f_m$, the equations of $\mathcal{R}(I)$are nice:

$$\mathcal{R}(I) \cong S[t_1, \ldots, t_m]/I_2 \begin{pmatrix} t_1 & \cdots & t_m \\ f_1 & \cdots & f_m \end{pmatrix}.$$

In other words, P is generated by the Koszul relations of the f_i.

Now let R be a ring and M a finitely generated R-module. We discuss *presentations* of the symmetric algebra of M. The symmetric algebra $S_R(M)$, or simply $S(M)$, is the algebra

$$S_R(M) = T_R(M)/(x \otimes y - y \otimes x, x, y \in M)$$

where $T_R(M)$ is the tensor algebra of M over R. It is convenient to give it in terms of generators and relations. This arises directly from a free presentation of M, as follows. Given the first order syzygies of M,

$$0 \longrightarrow L \longrightarrow R^n \longrightarrow M \longrightarrow 0,$$

$$S(M) = S(R^n)/(L) = R[t_1, \cdots, t_n]/(L),$$

where $L = (a_1t_1 + \cdots + a_mt_m : a_1f_1 + \cdots + a_mf_m = 0)$. Now take $a = (a_1, \cdots, a_m) \in T$ such that $a_1f_1 + \cdots + a_mf_m = 0$. In other words, $a_1t_1 + \cdots + a_mt_m \in \mathrm{Ker}\varphi = P$. So looking only at t-degrees we get

$$P_1 \overset{\text{as } R\text{-module}}{\cong} L = \text{First syzygy of } (f_1, \cdots, f_m). \tag{2.6}$$

In addition for an ideal I of R we have the following short exact sequence:

$$S(R^n) = R[x_1, \cdots, x_n] \xrightarrow{\varphi^1} S(I) \longrightarrow \mathcal{R}(I),$$

where φ^1 is the natural surjective map induced from a free presentation $R^m \xrightarrow{\rho} I$. So $\varphi^1(\sum a_it_i) = \sum a_if_i$. Thus by (2.6) we actually have $\mathrm{Ker}\varphi^1 = P_1$.
The difficulty in finding the equations is partly governed by the notion of *relation type*.

Definition 2.2.2. *The ideal I is said to be of linear type if $P = P_1$. More generally, I is said to be of relation type r if P can be generated by forms of degree $\leq r$.*

Hence; I is linear type $\iff \mathrm{Ker}\varphi^1 = \mathrm{Ker}\varphi \iff S(I) \cong \mathcal{R}(I)$.

We remark that any ideal which is generated by a regular sequence is of linear type. The same is also true for ideals generated by a d-sequence; see [38, 66] for the source of information.

Definition 2.2.3. *Let $\underline{x} = \{x_1. \cdots, x_n\}$ be a sequence of elements in a ring R generating the ideal I. Then $\underline{x}$ is called a d-sequence if $(x_1, \cdots, x_i) : x_{i+1}x_k = (x_1, \cdots, x_i) : x_k$ for $i = 0, \cdots, n-1$ and $k \geq i+1$.*

Remark 2.2.4. A useful observation in checking whether an ideal is of linear type is the following. Let $I = (a_1, \cdots, a_m)$ be an ideal of an integral domain R and let $(J_1) \subseteq R[t_1, \cdots, t_m]$ be the ideal defining the symmetric algebra $S_R(I)$. Suppose that for $0 \neq x \in R$, I_x is of linear type (say, generated by a regular sequence or more generally by a d-sequence). Then I is of linear type if and only if $J_1 : x = (J_1)$.

If I is not of linear type, there is a beginning of a theory for ideals of quadratic type and for certain families of ideals whose equations are obtained from elimination and are concentrated in degrees 1 and another degree. This provides the ingredients of studying our next section. Indeed we study the general situation where the initial ideal of P w.r.t. some term order has some elements of x-degree > 1.

2.3 The proof of our criterion for linear resolutions

Let K be a field, $I = (f_1, \ldots, f_m)$ be a graded ideal of $S = K[x_1, \ldots, x_r]$ generated in a single degree, say d. Let $T = S[t_1, \cdots, t_m]$. For each $k \in \mathbb{Z}$ we define a functor F_k from the category of bigraded T-modules to the category of graded S-modules with bigraded maps of degree zero. Let M be a bigraded T-module, define

$$F_k(M) = \bigoplus_{j \in \mathbb{Z}} M_{(k,j)},$$

obviously F_k is an exact functor and associates to each free $K[\underline{x}, \underline{t}]$-module a free $K[\underline{x}]$-module. Sometimes we simply write $M_{(k,\star)}$ instead of $F_k(M)$. Using (2.5) we get

$$[T/P]_{(k,\star)} = \mathcal{R}(I)_{(k,\star)} = \bigoplus_{j \in \mathbb{Z}} \mathcal{R}(I)_{(k,j)} = \bigoplus_{j \in \mathbb{Z}} (I^k)_{kd+j} = I^k(kd), \tag{2.7}$$

which provides the link between I and its Rees ring $\mathcal{R}(I)$. In the sequel we need to know what is $F_k(T(-a,-b))$. For the convenience of reader we provide a proof.

Remark 2.3.1. For each integer k we have

$$T(-a,-b)_{(k,\star)} = \begin{cases} 0 & \text{if } k < a, \\ S(-b)^N & \text{otherwise.} \end{cases} \tag{2.8}$$

Where $N := \#\{\underline{t}^\alpha : |\alpha| = k-a\} = \binom{m-1+k-a}{m-1}$.

Proof.

$$\begin{aligned} T(-a,-b)_{(k,\star)} &= \bigoplus_{j\in\mathbb{Z}} T(-a,-b)_{(k,j)} = \bigoplus_{j\in\mathbb{Z}} T_{(k-a,j-b)} \\ &= \bigoplus_{j\in\mathbb{Z}} < \underline{\mathbf{t}}^\alpha \underline{\mathbf{x}}^\beta : |\alpha| = k-a, |\beta| = j-b >, \end{aligned} \tag{2.9}$$

where the last equality is as vector spaces. From (2.9) the proof is immediate when $k < a$. Considering as an $S = K[\underline{x}]$-module the last module in (2.9) is free. Since $|\beta| = j-b$ could be any integer where j changes over $\mathbb{Z}$, a shift by $-b$ is required for the representation of the graded free module $T(-a,-b)_{(k,\star)}$ and finally the proposed N will take care of the required copies. □

Note that in the spacial case $a = b = 0$, we have

$$T_{(k,\star)} = S^{\binom{m-1+k}{m-1}}. \tag{2.10}$$

Our approach to generalize Theorem 2.1.2 is to change P with an isomorphic image $g(P)$ so that $\text{in}(g(P))_{(k,\star)}$ only consists of terms with x-degree≤ 1, for some k. To this end, we need a simple fact.

Let $<$ be any term order on $S = K[\underline{x}]$ and let $V \subseteq S$ be a K-vector space. Then with respect to the monomial order on S obtained by restricting $<$, by definition V

is homogenous if for any element f of V, $f = \sum_{i=0}^{n} f_i$, where f_i is an element of S of degree i, we have $f_i \in V$, $\forall i = 0, \cdots, n$. That is to say $V = \bigoplus_{i=0}^{\infty} V_i$, $V_i = V \bigcap S_i$. It yields that $\mathrm{in}(V) = \bigoplus_{i=0}^{\infty} \mathrm{in}(V_i)$ and so, $\mathrm{in}(V)_i = \mathrm{in}(V_i)$. Generalizing this idea to bigraded (or multigraded) situation is also well understood. Let F be a free S-module with a fixed basis and M a bigraded subvector space of it. Then

$$\mathrm{in}(M)_{(i,j)} = \mathrm{in}(M_{(i,j)}),$$

and so

$$\mathrm{in}(M)_{(k,\star)} := \bigoplus_{j\in\mathbb{Z}} \mathrm{in}(M)_{(k,j)} = \bigoplus_{j\in\mathbb{Z}} \mathrm{in}(M_{(k,j)}) = \mathrm{in}(M_{(k,\star)}). \tag{2.11}$$

See [26, Chapter 15.2] for more details. Furthermore since $\beta^S_{ij}(F/M) \leq \beta^S_{ij}(F/\mathrm{in}(M))$, it is easy to conculde with

$$\mathrm{reg}(F/M) \leq \mathrm{reg}(F/\mathrm{in}(M)). \tag{2.12}$$

Lemma 2.3.2. *Let P be the associated ideal of Rees ring $\mathcal{R}(I)$ and let $T = R/P$. Then*

$$\mathrm{reg}([T/P]_{(k,\star)}) \leq \mathrm{reg}([T/\mathrm{in}(P)]_{(k,\star)}).$$

Proof. Since P is a naturally bigraded ideal of T, and since easily $T_{(k,\star)}$ is a free S-module (2.10) and (2.11) imply that $\mathrm{in}(P)_{(k,\star)} = \mathrm{in}(P_{(k,\star)})$. Applying (2.12) for $F := T_{(k,\star)}$ and $M := P$ we obtain $\mathrm{reg}(T_{(k,\star)}/P_{(k,\star)}) \leq \mathrm{reg}(T_{(k,\star)}/\mathrm{in}(P_{(k,\star)}))$. Finally putting all together we get the required inequality.

$$\begin{aligned}\mathrm{reg}([T/P]_{(k,\star)}) = \mathrm{reg}(T_{(k,\star)}/P_{(k,\star)}) &\leq \mathrm{reg}(T_{(k,\star)}/\mathrm{in}(P_{(k,\star)})) \\ &= \mathrm{reg}(T_{(k,\star)}/\mathrm{in}(P)_{(k,\star)}) \\ &= \mathrm{reg}([T/\mathrm{in}(P)]_{(k,\star)}).\end{aligned}$$

□

In the following the proof of Theorem 2.1.5 is given.

Proof. First of all notice that, since $g : K[\underline{\mathrm{x}}, \underline{\mathrm{t}}] \longrightarrow K[\underline{\mathrm{x}}, \underline{\mathrm{t}}]$ is an invertible bi-homogenous transformation, we have the following bi-homogenous isomorphism:

$$\frac{K[\underline{\mathrm{x}}, \underline{\mathrm{t}}]}{P} \simeq \frac{K[\underline{\mathrm{x}}, \underline{\mathrm{t}}]}{g(P)},$$

and so we can simply take $g = id$ in the rest of proof. Write down the so-called Taylor resolution of T/G:

$$\cdots \longrightarrow \begin{matrix} F_{2,0} \\ \bigoplus \\ F_{2,1} \\ \bigoplus \\ F_{2,2} \end{matrix} \longrightarrow \begin{matrix} F_{1,0} \\ \bigoplus \\ F_{1,1} \end{matrix} \longrightarrow T \longrightarrow T/G \longrightarrow 0, \tag{2.13}$$

where $F_{i,j} = \bigoplus_{a \in \mathbb{Z}} T(-a, -j)^{\beta_{i,(a,j)}(T/G)}$. Note that $\beta_{i,(a,j)}(T/G)$, (a, j) is an integer number which depends on i, a, and j. Since $(k, \star)$ is an exact functor, the following complex of $K[\underline{\mathrm{x}}]$-modules is exact:

$$\cdots \longrightarrow \begin{matrix} (F_{2,0})_{(k,\star)} \\ \bigoplus \\ (F_{2,1})_{(k,\star)} \\ \bigoplus \\ (F_{2,2})_{(k,\star)} \end{matrix} \longrightarrow \begin{matrix} (F_{1,0})_{(k,\star)} \\ \bigoplus \\ (F_{1,1})_{(k,\star)} \end{matrix} \longrightarrow T_{(k,\star)} \longrightarrow [T/G]_{(k,\star)} \longrightarrow 0. \tag{2.14}$$

Using formula (2.8) we obtain $T(-a, -b)_{(k,\star)} = S(-b)^{N_{a,k}}$, so for $F_{i,j}$ we get

$$(F_{i,j})_{(k,\star)} = \bigoplus_{a \in \mathbb{Z}} S(-j)^{N_{a,k}\, \beta_{i,(a,j)}(T/G)}. \tag{2.15}$$

It follows that (2.14) is a (possibly non-minimal) graded free $K[\underline{x}]$- resolution of $[T/G]_{(k,\star)}$. Since $\deg_x(G) \leq 1$, from (2.14) and (2.15) we conclude that

$$\operatorname{reg}([T/G]_{(k,\star)}) = 0 \quad \text{for all } k. \tag{2.16}$$

Now we have

$$\begin{aligned} dk \leq \operatorname{reg}(Q^k) \leq \operatorname{reg}([T/P]_{(k,\star)}) + dk &\leq \operatorname{reg}([T/\operatorname{in}(P)]_{(k,\star)}) + dk \\ &= \operatorname{reg}([T/G]_{(k,\star)}) + dk \quad \text{for all } k \geq k_0 \\ &= 0 + dk = dk, \end{aligned} \tag{2.17}$$

where the second (in)equality in (2.17) follows from (2.7), the third inequality is due to Lemma 2.3.2, and the forth comes from the easy argument $[T/\operatorname{in}(P)]_{(k,\star)} = T_{(k,\star)}/\operatorname{in}(P)_{(k,\star)} = T_{(k,\star)}/G_{(k,\star)} = [T/G]_{(k,\star)}$.

Finally (2.17) implies that $\operatorname{reg}(Q^k) = kd$ for all $k \geq k_0$ as desired. □

In Appendix A, we will develop some further details of Theorem 2.1.5.

2.4 Examples and applications

In this section we provide some applications of Theorem 2.1.5. Using the strategy introduced in the last section and as an application for our main result we give an answer to the Question 2.1.1.

Example 2.4.1. Let $S = \mathbb{Q}[x_1, \cdots, x_6]$ and let J be the ideal of (2.3). J has 10 generators, so let $T = \mathbb{Q}[x_1, \cdots, x_6, t_1, \cdots, t_{10}]$ and with term order $\underline{x} > \underline{t}$ (and DegRevLex). We also use J for the ideal of T generated by the same generators as of J in S. Let P be the defining ideal of the Rees ring of J, so $R(J) = T/P$. One can check that P has 15 elements of bidegree (1,1), 10 elements of bidegree (3,0), and

15 elements of bidegree (4,0). Take G and B as in Theorem 2.1.5. We have checked that $\mid G \mid = 60$, $B = \mathrm{Ideal}(t_6x_4x_5, t_4x_3x_5, t_4t_6x_5^2)$, and so $\max\{\deg_t(h) \mid h \in B\} = 2$. But $(\underline{t})^2(t_6x_4x_5) \nsubseteq G, (\underline{t})^2(t_4x_3x_5) \nsubseteq G, (\underline{t})(t_4t_6x_5) \nsubseteq G$. So in DegRevLex (also Lex) order and $\underline{x} > \underline{t}$, we were unable to admit the conditions of Theorem 2.1.5. We have observed that the same story happens for ordering $\underline{t} > \underline{x}$ either DegRevLex or Lex. We observed that it is better served if we continue in DegRevLex order and $\underline{t} > \underline{x}$. Using algorithm 7 we look for a desired upper triangular bi-change of coordinates (say g). The following g is fine, but note that there exists many of such g indeed:

$$g := g_1 \times g_2 \in \mathrm{GL}_6(\mathbb{Q}) \times \mathrm{GL}_{10}(\mathbb{Q}),$$

where $g_1 : \mathbb{Q}[\underline{x}] \longrightarrow \mathbb{Q}[\underline{x}]$ is given by

$$x_4 \longmapsto x_1 + x_4,$$

$$x_6 \longmapsto x_3 + x_6,$$

and sends x_i for $i \neq 4, 6$ to itself and let g_2 to be the identity map over $\mathbb{Q}[\underline{t}]$. One can compute that $\mid G \mid = 98$, $B = (t_7x_3^2, t_4t_6x_5^2)$. It is easy to verify that

$$I_{(k,\star)} = G_{(k,\star)}, \text{ for } k > 2 \Longleftrightarrow \begin{cases} (t_7x_3^2)(t_1, \cdots, t_{10})^2 \subseteq G, \\ (t_4t_6x_5^2)(t_1, \cdots, t_{10}) \subseteq G, \end{cases} \tag{2.18}$$

and since in the right side of (2.18) both containments are valid we conclude with $\mathrm{reg}(J^k) = 3k$ for all $k > 2$.

Another motivation for this work is an example that Conca considered in [21].

Example 2.4.2. Let J_1 be the ideal of 3-minors of a 4×4 symmetric matrix of linear

	$\underline{x} > \underline{t}$	$\underline{t} > \underline{x}$
DegRevLex	(1,2):6,(2,2):5,(1,3):1,(4,2):1	(1,2):6,(2,2):3,(1,3):1
Lex	(1,2):6,(2,2):3	(1,2):6,(2,2):5

Table 2.2: Count of elements of $\mathrm{in}(P_1)$ with $deg_x > 1$ for the ideal of (2.19).

forms in 6 variables, that is, 3-minors of

$$\begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & x_4 & x_5 \\ x_2 & x_4 & 0 & x_6 \\ x_3 & x_5 & x_6 & 0 \end{bmatrix}.$$

As an ideal of $S = \mathbb{Q}[x_1, \cdots, x_6]$ one has:

$$\begin{aligned} J_1 := (&2x_1x_2x_4, 2x_1x_3x_5, 2x_2x_3x_6, 2x_4x_5x_6, x_1x_3x_4 + x_1x_2x_5 - x_1^2x_6, x_3x_4x_6 + \\ &x_2x_5x_6 - x_1x_6^2, -x_2x_3x_4 + x_2^2x_5 - x_1x_2x_6, -x_3^2x_4 + x_2x_3x_5 + x_1x_3x_6, -x_3x_4^2 + \\ &x_2x_4x_5 + x_1x_4x_6, -x_3x_4x_5 + x_2x_5^2 - x_1x_5x_6). \end{aligned} \tag{2.19}$$

As Conca mentioned in his paper [21, Remark 3.6] and as we will show here, the ideals J, J_1 are very closely related. But similar to the ideal of (2.3), one can easily check that $\mathrm{in}(P_1)$, where P_1 is the associated ideal to Rees ring of J_1, has at least 9 elements with $deg_x > 1$, no matter if we take initial ideal w.r.t. term ordering $\underline{x} > \underline{t}$ or $\underline{t} > \underline{x}$ in Lex or DegRevLex order; see Table 2.2 for more details. So in this example Theorem 2.1.2 is not applicable as well.

In the following we show that $\mathrm{reg}(J_1^k) = 3k$, for all $k > 2$.

Example 2.4.3. Let $S = \mathbb{Q}[x_1, \cdots, x_6]$ and let J_1 be the ideal of (2.19). Again J_1 has 10 generators so let $T = \mathbb{Q}[t_1, \cdots, t_{10}, x_1, \cdots, x_6]$ and in DegRevLex order. Let

P_1 be the defining ideal of the Rees ring of J_1, so $R(J_1) = T/P_1$. One can observe that P has 15 elements of bidegree (1,1), 10 elements of bidegree (3,0), and 12 elements of bidegree (4,0). Take g to be the following simple upper triangular bi-transformation:

$$g := g_1 \times g_2 \in \mathrm{GL}_6(\mathbb{Q}) \times \mathrm{GL}_{10}(\mathbb{Q}),$$

where $g_1 : \mathbb{Q}[\underline{x}] \longrightarrow \mathbb{Q}[\underline{x}]$ shall be given by

$$x_4 \longmapsto x_2 + x_4,$$

$$x_6 \longmapsto x_1 + x_6,$$

and sending the rest to themselves and take $g_2 : \mathbb{Q}[\underline{t}] \longrightarrow \mathbb{Q}[\underline{t}]$ to be

$$t_8 \longmapsto t_7 + t_8,$$

and for $i \neq 8$, $t_i \longmapsto t_i$. Computations by CoCoA shows that $\mid G \mid= 144$, $B = (t_{10}x_2x_3, t_2t_4x_5^2)$. Since $I := \mathrm{in}(g(P)) = G + B$, we have

$$I_{(k,\star)} = G_{(k,\star)}, \text{ for } k > 2 \Longleftrightarrow \begin{cases} (t_{10}x_2x_3)(t_1, \cdots, t_{10})^2 \subseteq G, \\ (t_2t_4x_5^2)(t_1, \cdots, t_{10}) \subseteq G, \end{cases} \tag{2.20}$$

and since it is easy to check that the right side of (2.20) is holding, we obtain that $\mathrm{reg}(J_1^k) = 3k$ for all $k > 2$.

We conclude with the following corollary which indicates that ideals J, (2.3), and J_1, (2.19), are very tightly related.

Corollary 2.4.4. *All the powers of J, and J_1, but the second power have linear resolution.*

Since the least exponent k_0 for J^k, and also for J_1^k in order to have linear resolution for all $k > k_0$ is 2, the following question seems to be interesting to discover:

Question 2.4.5. *Does there exist an ideal Q with generators of the same degree d over some polynomial ring $S = K[x_1, \cdots, x_r]$, for which $\mathrm{reg}(Q^k) = kd$, $\forall k \neq 3$ or $\forall k \neq 2, 3$?*

We also show that J and J_1 and their powers have the same Hilbert series (HS for short) respectively:

$$\mathrm{HS}(S/J^k) = \mathrm{HS}(S/J_1^k), \quad \forall k.$$

For this we have computed the multigraded Hilbert series of the corresponding ideals to the Rees algebra of J and J_1 and observed that they are the same. i.e., T/P and T/P_1 have the same multigraded Hilbert series, where P, and P_1 are the defining ideals of Rees rings of J and J_1 correspondingly. As a result we conclude that all of the powers of J and J_1 have the same graded Betti numbers as well:

Corollary 2.4.6. $\mathrm{HS}(S/J^k) = \mathrm{HS}(S/J_1^k)\,\forall k$, *and so* $\beta_{i,j}(J^k) = \beta_{i,j}(J_1^k)\,\forall i, j, \forall k$.

Our next example is devoted to another important ideal which was first discovered by Sturmfels [64]. Indeed Sturmfels provided this example as a counterexample to Chandler's question [17] whether the Castelnuovo-Mumford regularity of a homogeneous ideal I in a polynomial ring $S = k[x_0, \cdots, x_m]$ satisfies the inequality $\mathrm{reg}(I^n) \leq n\,\mathrm{reg}(I)$. This inequality holds true if $\dim S/I \leq 1$. Sturmfels then constructed a 2-dimensional Cohen-Macaulay ideal I generated by 8 square-free monomials in 6 variables such that $\mathrm{reg}(I) = 3$ but $\mathrm{reg}(I^2) = 7$ for any base field k. According to Sturmfels [64], there are no such examples with less than 8 generators.

Example 2.4.7. Let $S = \mathbb{Q}[x_1, \cdots, x_6]$ and

$$J_2 := (x_4x_5x_6, x_3x_5x_6, x_3x_4x_6, x_3x_4x_5, x_2x_5x_6, x_2x_3x_4, x_1x_3x_6, x_1x_4x_5). \qquad (2.21)$$

	$\underline{x} > \underline{t}$	$\underline{t} > \underline{x}$
DegRevLex	$\mid \text{in}(P_2) \mid = 34; (1,2) : 6, (2,2) : 3$	$\mid \text{in}(P_2) \mid = 33; (1,2) : 6, (2,2) : 3$
Lex	$\mid \text{in}(P_2) \mid = 32; (1,2) : 6, (2,2) : 3$	$\mid \text{in}(P_2) \mid = 32; (1,2) : 6, (2,2) : 3$

Table 2.3: Count of elements of $\text{in}(P_2)$ with $deg_x > 1$ for the ideal of (2.21).

J_2 has 8 generators so let $T = \mathbb{Q}[x_1, \cdots, x_6, t_1, \cdots, t_8]$ and in DegRevLex order. Let P_2 be the defining ideal of the Rees ring of J_2, i.e., $R(J_2) = T/P_2$. One can observe that P_2 is consisting of 11 elements of bidegree $(1,1)$, 1 element of bidegree $(2,2)$, 2 elements of bidegree $(3,0)$, and 2 elements of bidegree $(4,0)$. In Table 2.3 we give a report of elements of $\text{in}(P_2)$ w.r.t. term ordering $\underline{x} > \underline{t}$ or $\underline{t} > \underline{x}$ in either Lex or DegRevLex order. Due to existence of guys with x-degree > 1 we are unable to apply Theorem 2.1.2 (in four ordinary term orderings discussed in Table 2.3 at least) to deduce the linear resolution of powers of J_2. Hence we try to make a use of Algorithm 7 in order to find a suitable upper triangular bi-change of $\underline{x}$ and $\underline{t}$ that fulfils the requirements of our criterion. It was interesting (for us at least) to report that after more or less $122,000$ times of tests we were unable to find such a bi-change. Indeed we believe that J_2 is one whose powers have non-linear resolution. In fact, $\text{reg}(J_2) = 3$ and we have checked that

$$\text{reg}(J_2^2) = 7, \text{reg}(J_2^3) = 10, \text{reg}(J_2^4) = 13, \text{reg}(J_2^5) = 16 \text{ and } \text{reg}(J_2^6) = 19$$

it attracts our interests to the following question:

Question 2.4.8. *Is it true that* $\text{reg}(J_2^k) = 3k + 1,\ \forall k \geq 2$?

Chapter 3

Growth of the Betti sequence of the canonical module

Throughout this chapter $(R, \mathfrak{m}, k)$ is a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k and of dimension d. If R has a canonical module, we use ω_R to denote it. $\mathrm{E}_R(k)$, as usual, represents the injective hull of the residue field k.

It is well-known that the growth of the Betti sequence of the residue field k of a local ring R characterizes its regularity. The foundational Auslander-Buchsbaum-Serre theorem states that: R is regular if and only if the Betti sequence of k is finite.

Gullikson ([31], [32]) proved that R is a complete intersection if and only if the Betti sequence of k grows polynomially. Furthermore, concerning the following Question of Huneke actually little is known:

Question 3.0.1. If R is a Cohen-Macaulay ring but not Gorenstein, must the Betti numbers of the canonical module of R grow exponentially?

In [41, Proposition 1.1] the authors stated and proved some cases where the answer to Question 3.0.1 is positive:

1. R is a Golod ring, cf. [48], [55];

2. R has codimension ≤ 3, cf. [6], [65];

3. R is one link from a complete intersection, cf. [6], [65];

4. R is radical cube zero, cf. [47].

Our analysis to study Question 3.0.1 is motivated by the case 4 above which refer to it as (†) during this chapter. Indeed in Section 3.1 we provide some criteria guaranteeing that the answer to this question is positive (resp. negative), that is, the growth of the Betti sequence of ω_R is exponential (resp. polynomial). To understand the connection note that for a finitely generated R-module M, $\mathfrak{m}^2$ kills M if and only if there are non-negative integers a, b such that the sequence

$$0 \longrightarrow (R/\mathfrak{m})^a \longrightarrow M \longrightarrow (R/\mathfrak{m})^b \longrightarrow 0 \quad (\ddagger)$$

is exact. This situation which we discuss here is, in a sense, more general than (†) because, when $\mathfrak{m}^3 = 0$, we have $\mathfrak{m}^2 M = 0$ whenever M is a syzygy, and so M fits into an exact sequence of the form (‡). On the other hand, our situation is, in a sense, less general than (†) because we need to assume some Ext-vanishing or some Tor-vanishing. Using this terminology and as an application of our results, in Section 3.3 we argue for the finiteness of the flat dimension of $\mathrm{E}(k)$ and deduce a criterion to check its finiteness. Section 3.4 is devoted to further analysis of the vanishing of certain Ext and Tor modules. In fact in Theorems 4.2.1 and 4.3.1 we study the cases where the ideals in the end sides of (‡) are different from $\mathfrak{m}$, especially when some of them are generated by regular sequences.

3.1 The canonical module and Growth of its Betti sequence

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with unique maximal ideal $\mathfrak{m}$ and residue field k and of dimension d. A *canonical* module of R is a finitely generated R-module ω_R for which

$$\operatorname{Ext}^i_R(k, \omega_R) \cong \begin{cases} 0 & \text{if } i \neq d, \\ k & \text{if } i = d. \end{cases}$$

A canonical module of R, if it exists, is unique up to isomorphism. Obviously a canonical module of R is *maximal Cohen-Macaulay* (MCM for short). If R is a Gorenstein local ring, $\omega_R = R$, and if $d = 0$, $\omega_R \cong \mathrm{E}(k)$. It is known that if ω_R is a canonical module of R, then for any $\mathfrak{p} \in \operatorname{Spec} R$, $\omega_{R_\mathfrak{p}}$ is a canonical module of $R_\mathfrak{p}$. Also, it is easy to check that $\widehat{\omega_R}$, the $\mathfrak{m}-$adic completion of ω_R, is a canonical module of $\widehat{R}$ and $\omega_R/\mathfrak{a}\,\omega_R$ is a canonical module of $R/\mathfrak{a}$ if $\mathfrak{a}$ is an ideal which is generated by an R-sequence. If the sequence is maximal, then $\dim R/\mathfrak{a} = 0$ and so $\omega_R/\mathfrak{a}\,\omega_R \cong \mathrm{E}_{R/\mathfrak{a}}(k)$. We refer the reader to Chapter 3 of [16] for standard facts about canonical modules.

Recall that by *polynomial growth* of a sequence $\{b_i\}$ we mean that there is an integer d and a positive constant c such that $b_i \leq cd^i$. Also a sequence $\{b_i\}$ has *exponential growth* if there exist real numbers $1 < \alpha < \beta$ such that $\alpha^i < b_i < \beta^i$ for all $i \gg 0$.

Our investigation for studying the growth of the Betti sequence of ω_R is started with the following result:

Theorem 3.1.1. *Let R be a local Cohen-Macaulay ring with canonical module ω_R and let M be a non zero R-module for which $0 \to k^n \to M \to k^m \to 0$ is an exact sequence. Let t be an integer.*

1. *Let $\operatorname{Tor}_i^R(\omega_R, M) = 0$ for $i = t, t+1$. Then one has $n\beta_t^R(\omega_R) = m\beta_{t+1}^R(\omega_R)$. In particular, if $\operatorname{Tor}_i^R(\omega_R, M) = 0$ for all $i = 1, \cdots, t$, then*

$$\beta_t^R(\omega_R) = (n/m)^{t-1} \cdot \beta_1^R(\omega_R).$$

2. *Let $\operatorname{Ext}_R^i(\omega_R, M) = 0$ for $i = t, t+1$. Then one has $m\beta_t^R(\omega_R) = n\beta_{t+1}^R(\omega_R)$. In particular, if $\operatorname{Ext}_R^i(\omega_R, M) = 0$ for all $i = 1, \cdots, t$, then*

$$\beta_t^R(\omega_R) = (m/n)^{t-1} \cdot \beta_1^R(\omega_R).$$

3. *Assume that $\operatorname{Tor}_i^R(\omega_R, M) = 0$ for all $i > t$.*

   - *If $n > m$, then the Betti sequence $\{\beta_i^R(\omega_R)\}$ grows exponentially, provided that $\beta_i^R(\omega_R) \neq 0$.*
   - *If $n = m$, then the Betti numbers $\beta_i^R(\omega_R)$ are eventually constant.*
   - *If $n < m$, then the Betti numbers $\beta_i^R(\omega_R)$ are eventually zero and so R is Gorentein.*

4. *Assume that $\operatorname{Ext}_R^i(\omega_R, M) = 0$ for all $i > t$.*

   - *If $n < m$, then the Betti sequence $\{\beta_i^R(\omega_R)\}$ grows exponentially, provided that $\beta_i^R(\omega_R) \neq 0$.*
   - *If $n = m$, then the Betti numbers $\beta_i^R(\omega_R)$ are eventually constant.*
   - *If $n > m$, then the Betti numbers $\beta_i^R(\omega_R)$ are eventually zero and so R is Gorentein.*

Proof. 1. For each $i > 0$, put $B_i = \mathrm{Tor}_i^R(\omega_R, k)$. Applying the functor $\omega_R \otimes_R -$ to the given exact sequence, we obtain a long exact sequence part of which is $\mathrm{Tor}_{j+1}^R(\omega_R, M) \to B_{j+1}^m \to B_j^n \to \mathrm{Tor}_j^R(\omega_R, M)$. Note that $B_i = k^{r_i}$ where $r_i = \beta_i^R(\omega_R)$. Thus for each j indeed we have the following exact sequence,

$$\mathrm{Tor}_{j+1}^R(\omega_R, M) \to k^{mr_{j+1}} \to k^{nr_j} \to \mathrm{Tor}_j^R(\omega_R, M).$$

Since $\mathrm{Tor}_t^R(\omega_R, M) = \mathrm{Tor}_{t+1}^R(\omega_R, M) = 0$, it yields that $nr_t = mr_{t+1}$.

In particular, if $\mathrm{Tor}_j^R(\omega_R, M) = 0$ for all $j \le t$, it turns out recursively that $r_t = (n/m)^{t-1} r_1$, that is, $\beta_t^R(\omega_R) = (n/m)^{t-1} \cdot \beta_1^R(\omega_R)$.

2. This is proved the same way as (1), using $\mathrm{Ext}_R(\omega_R, -)$.

3. Since we have $\mathrm{Tor}_j^R(\omega_R, M) = 0$ for all $j \ge t+1$, similar to (1) we get that $\beta_{j+t}^R(\omega_R) = (n/m)^{j-1} \cdot \beta_{t+1}^R(\omega_R)$ for all $j \ge 1$. Now note that if $n > m$, then we get the exponential growth of $\{\beta_i^R(\omega_R)\}$ with the base n/m. When $n < m$, R will be Gorenstein because then $\beta_i^R(\omega_R) = 0$ for $i \gg 0$. Furthermore, in the case $n = m$ we would obtain constant Betti numbers. This would give the polynomial growth, assuming of course that each $\beta_i^R(\omega_R) \neq 0$.

4. The same interpretations as that of (3).

□

In the following some examples are provided.

Example 3.1.2. Let $(R, \mathfrak{m}, k)$ be a local CM ring which has a canonical module ω_R. Let M be an R-module such that $\mathrm{Tor}_i^R(\omega_R, M) = 0$, $\forall i \gg 0$ (*e.g.*, proj.dim$M < \infty$).

1. If the sequence $0 \to k^2 \to M \to k^3 \to 0$ is exact, then $\{\beta_i^R(\omega_R)\}$ has exponential growth, provided of course that $\beta_R^i(\omega_R) \neq 0$ for $i \gg 0$

2. If the sequence $0 \to k^3 \to M \to k^2 \to 0$ is exact, then $\beta^i_R(\omega_R) = 0$ for $i \gg 0$ and so R is Gorenstein.

3. If the sequence $0 \to k^2 \to M \to k^2 \to 0$ is exact, then the Betti numbers of ω_R are eventually constant.

   This would give a negative answer to Question 3.0.1, assuming of course that each $\beta_i^R(\omega_R) \neq 0$ in case $(2), (3)$ and a positive answer in case (1).

Thanks to the rich literature on the background we could consider the following example where the situation of Theorem 4.1.1 holds. This example deals with the case $\mathfrak{m}^3 = 0$, where we get the exponential growth from the work of [41, Example 2.10] but we could add some extra information to the background.

Example 3.1.3. Let $X = \{X_1, X_2, X_3, X_4\}$ be a set of indeterminacies over a field k with char $\neq 2$ and set $A = k[X]_{(X)}$. Let I be the ideal of A generated by

$$X_1^2, X_1X_2 - X_3X_4, X_1X_2 - X_4^2, X_1X_3 - X_2X_4, X_1X_4 - X_2^2, X_1X_4 - X_2X_3, X_1X_4 - X_3^2$$

and set $R = A/I$. Then R is zero dimensional local ring and $\mathfrak{m}^3 = 0$. Let x_i denote the image of X_i in R for $i = 1, \ldots, 4$ and consider the sequence of homomorphisms of free R-modules:

$$\mathbb{F} : \cdots \xrightarrow{\psi} R^2 \xrightarrow{\varphi} R^2 \xrightarrow{\psi} R^2 \xrightarrow{\varphi} \cdots$$

where

$$\varphi = \begin{pmatrix} x_3 & x_1 \\ x_4 & x_2 \end{pmatrix} \qquad \psi = \begin{pmatrix} x_2 & -x_1 \\ -x_4 & x_3 \end{pmatrix}$$

It is known that F and F^* are exact sequences; see [46, 68] for instance. Then for $M := \operatorname{coker} \varphi$ it is known that $\operatorname{Ext}^i_R(M, R) = 0$ for all $i > 0$ or equivalently $\operatorname{Tor}_i^R(M, \omega_R) = 0$ for all $i > 0$.

Put $N := \mathrm{Im}\varphi$. Indeed N is the first syzygy module of M. Hence $N \subseteq \mathfrak{m}R$, and so $\mathfrak{m}^2 N = 0$. Since $\mathrm{Tor}_i^R(M, \omega_R) = 0$ for all $i > 0$, it is easy to deduce that $\mathrm{Tor}_i^R(N, \omega_R) = 0$ for all $i > 0$.

Now let $a = \dim_k(\mathfrak{m}N)$ and $b = \mu(N)$. Then the above resolution shows that ψ is a minimal presentation matrix for N and so it follows that $b = 2$. In Section 3 of [46], the authors showed that $\mathrm{rank}_k(R^2) = 16$ and $\mathrm{rank}_k(\psi) = 8$, that is, the image of ψ has length 8. It follows that N has length $16 - 8 = 8$. Since $\mathfrak{m}^2 N = 0$, we conclude that $a = \mathrm{length}(N) - b = 6$.

Finally Theorem 4.1.1 implies that $\beta_i(\mathrm{E}(k)) = (a/b)^{i-1}\beta_1(\mathrm{E}(k)) = 3^{i-1}\beta_1(\mathrm{E}(k))$, for all $i > 0$. Explicit computations in Macaulay 2 shows that $\beta_1(\mathrm{E}(k)) = 8$ and so $\beta_i(\mathrm{E}(k)) = 8.3^{i-1}$ which confirms that $\{\beta_i^R(\mathrm{E}(k))\}$ has exponential growth. □

3.2 Constructing rings satisfying our conditions

Let C be a semidualizing R-module. Recall that an R-module M is in the *Auslander class* $\mathcal{A}_C(R)$ if

$$\delta_M^C : M \longrightarrow \mathrm{Hom}_R(C, C \otimes_R M) \text{ is an isomorphism, and}$$

$$\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_R M)\ , \forall i \geq 1.$$

Furthermore, M is said to be in the *Bass class* $\mathcal{B}_C(R)$ if

$$\xi_M^C : C \otimes_R \mathrm{Hom}_R(C, M) \longrightarrow M \text{ is an isomorphism, and}$$

$$\mathrm{Ext}_R^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, M))\ , \forall i \geq 1.$$

Now let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with $\mathrm{E} = \mathrm{E}_R(k)$. Assume that R has a canonical module ω_R. Theorem 4.1.1 applies whenever R has one of the following:

1. a finitely generated module $M \neq 0$ in the Auslander class $\mathcal{A}(R) := \mathcal{A}_{\omega_R}(R)$ (equivalently, such that $\text{G-dim}_R(M) < \infty$) with $\mathfrak{m}^2 M = 0$, or

2. a finitely generated module $N \neq 0$ in the Bass class $\mathcal{B}(R) := \mathcal{B}_{\omega_R}(R)$ (equivalently, such that $\text{Gid}_R(M) < \infty$) with $\mathfrak{m}^2 N = 0$.

Indeed, if $M \in A(R)$, then $\text{Tor}^R_{i\geq 1}(\omega_R, M) = 0$ by definition, and if $N \in \mathcal{B}(R)$, then $\text{Ext}^{i\geq 1}_R(\omega_R, N) = 0$ by definition; see [19] for more details.

Note that Matlis duality tells us the following: If M satisfies (1), then $N = M^V$ satisfies (2); and if N satisfies (2), then $M = N^V$ satisfies (1). In particular, the ability to construct a module as in (1) is equivalent to the ability to construct a module as in (2).

Two cases where such modules exist is expressed in the following: when R is Gorenstein or when R has minimal multiplicity, that is $\text{e}(R) = \mu(\mathfrak{m}) - \dim R + 1$, and infinite residue field. The Gorenstein case is easy since $\omega_R = R$ in this case. When R has minimal multiplicity and infinite residue field, then R has a maximal R-regular sequence $\mathbf{y} \in \mathfrak{m}$ such that the module $M = R/(\mathbf{y})$ satisfies $\mathfrak{m}^2 M = 0$. Of course, since $\text{proj.dim}_R(M) = \dim(R) < \infty$, we have $M \in \mathcal{A}(R)$. Note that non of these cases are interesting in the context of Betti numbers of ω_R because it is known that the Betti numbers of a ring of this form grow exponentially.

The Bass class condition may seem a bit esoteric, so here are some examples. Note that they do not give any new information about Betti numbers of canonical modules. Assume that (Q, τ) is a local Cohen-Macaulay ring with a finitely generated module M such that $\text{proj.dim}_Q(M) < \infty$ and $\tau^2 M = 0$. Set $N = \text{Hom}_Q(M, \text{E})$, and note that $\text{inj.dim}_Q(N) < \infty$ and $\tau^2 N = 0$. Let $\mathbf{x} \in \text{Ann}_Q(M) \subseteq \tau$ be a Q-regular sequence, and set $A = Q/(x)$. Then M and N are naturally A-modules. Furthermore, by construction, we have $\text{CI-dim}(M) < \infty$ and $\text{CI-id}(N) < \infty$: use the quasideformation $A \longleftarrow A \longrightarrow Q$ and see [60]. It follows from Theorems 5.1 and 5.2 in [60] that M is in the Auslander class $\mathcal{A}(A)$ and N is in the Bass class $\mathcal{B}(A)$.

These observations made us think of the following question.

Question 3.2.1. *If R is not Gorenstein and $M \neq 0$ is a finitely generated R-module such that $\mathfrak{m}^2 M = 0$ and either CI-dim$_R(M) < \infty$ or CI-id$_R(M) < \infty$, must R have minimal multiplicity?*

Note that Question 3.2.1 is inherently a question for Cohen-Macaulay rings: (1) If R is a local ring and $M \neq 0$ is a finite length R-module of finite complete intersection dimension, then R is Cohen-Macaulay; (2) If R is a local ring and $N \neq 0$ is a finitely generated R-module of finite complete intersection injective dimension, then R is Cohen-Macaulay.

3.3 Applications to flat dimension

In [3] André showed that for any finitely generated module M with finite flat dimension, the following formula for flat dimension of M holds:

$$\text{flatdim}(M) = \sup\,\{i \mid \text{Tor}_i^R(k, M) \neq 0\}.$$

Routine proofs of this result using Artin-Rees lemma and Nakayama's lemma could also be derived; see [61]. We present a measure of the flat dimension of finitely generated modules. Indeed we show that

$$\text{flatdim}(M) = \sup\,\{i \mid \text{Tor}_i^R(\text{E}(k), M) \neq 0\}.$$

On the other hand, a simple example shows that some special assumptions such as being finitely generated is necessary in order to obtain the above formula.

Lemma 3.3.1. *Let M be a non zero R-module with finite flat dimension. Then*

$$\operatorname{flatdim}(M) = \sup\{i \mid \operatorname{Tor}_i^R(E(k), M) \neq 0\}.$$

Proof. It is clear that $\operatorname{flatdim}(M) \geq \sup\{i \mid \operatorname{Tor}_i^R(\mathrm{E}(k), M) \neq 0\}$. Set $t = \operatorname{flatdim}(M)$. Applying the functor $-\otimes_R M$ to the exact sequence $0 \to k \to \mathrm{E}(k) \to C \to 0$, for some R-module C, we get the following exact sequence,

$$0 = \operatorname{Tor}_{t+1}^R(C, M) \to \operatorname{Tor}_t^R(k, M) \to \operatorname{Tor}_t^R(\mathrm{E}(k), M).$$

Since $\operatorname{Tor}_t^R(k, M) \neq 0$, it yields that $\operatorname{Tor}_t^R(\mathrm{E}(k), M) \neq 0$. Hence

$$t \leq \sup\{i \mid \operatorname{Tor}_i^R(\mathrm{E}(k), M) \neq 0\}.$$

□

We continue with an example of an R-module N with finite flat dimension which is not finitely generated, but $\operatorname{Tor}_i^R(\mathrm{E}(k), N) = 0$ for all i.

Example 3.3.2. Let $(R, \mathfrak{m})$ be a local regular ring of dimension at least 2. Let $d := \dim R_\mathfrak{p} > 0$ for some prime ideal $\mathfrak{p} \neq \mathfrak{m}$. Set $N := k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Clearly N is not f.g. and one can see that

$$\begin{aligned}\operatorname{flatdim}(N) &= \sup\{\operatorname{flatdim}_{R_\mathfrak{p}}(N_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \\ &\geq \operatorname{flatdim}_{R_\mathfrak{p}} k(\mathfrak{p}) = \dim R_\mathfrak{p} > 0,\end{aligned}$$

whereas for all i,

$$\begin{aligned}\operatorname{Tor}_i^R(\mathrm{E}(k), N) &= \operatorname{Tor}_i^R(\mathrm{E}(k), k(\mathfrak{p})) = \operatorname{Tor}_i^R(\mathrm{E}(k), R/\mathfrak{p} \otimes_R R_\mathfrak{p}) \\ &= \operatorname{Tor}_i^R(\mathrm{E}(k), R/\mathfrak{p}) \otimes_R R_\mathfrak{p} = 0, \quad (\text{since } \mathfrak{p} \neq \mathfrak{m})\end{aligned}$$

and so $\sup\{i \mid \operatorname{Tor}_i^R(\mathrm{E}(k), N) \neq 0\} = -\infty$.

Since the canonical module is maximal Cohen-Macaulay over a Cohen-Macaulay ring R, we may, and often do, reduce both R and ω_R modulo a maximal regular sequence and assume that R has dimension zero and so $\omega_R \cong \mathrm{E}(k)$. In particular, the Betti numbers of ω_R are equal to the Bass numbers of R. Now Theorem 4.1.1 immediately provides a criterion to check finiteness of flatdim($\mathrm{E}(k)$) for artinian rings. Furthermore, in the following we study the flat dimension of $\mathrm{E}(k)$.

Theorem 3.3.3. *Let $(R, \mathfrak{m}, k)$ be an artinian local ring and let $0 \to k^n \to R \to k^m \to 0$ be an exact sequence for some integers n, m. Then the minimal free cover of $E(k)$ is as follows:*

$$0 \to k^{n^2-1} \to R^n \to E(k) \to 0.$$

Furthermore, if $n > 1$, then $\mathrm{Tor}_i^R(E(k), E(k)) \neq 0$, for all $i > 0$, and R is not Gorenstein. For $n = 1$, R is even complete intersection.

Proof. Note that since $\mathfrak{m}.k^m = \mathfrak{m}.k^n = 0$, we have $\mathfrak{m}^2.R = 0$ and so $\mathfrak{m}^2 = 0$. Let $\mathfrak{m} = (x_1, \cdots, x_d)$. Then $\mathfrak{m} \cong k^d$, and so the given exact sequence is in fact $0 \to k^d \to R \to k \to 0$. Furthermore,

- $x_i x_j = 0$,
- $\ell(R) = \ell(R/\mathfrak{m}) + \ell(\mathfrak{m}/\mathfrak{m}^2) + 0 = 1 + \mu(\mathfrak{m}) = 1 + d$.

That is (if R contains a filed k),

$$R \cong \frac{k[x_1, \cdots, x_d]}{(x_1, \cdots, x_d)^2}$$

If $d = 1$, then R is Gorenstein, even a complete intersection. So from now on assume that $d > 1$. Then it follows that the type of R is exactly d and so R is not Gorenstein.

Let $\mathrm{E} = \mathrm{E}(k)$ and $\Omega\mathrm{E}$ denote the first syzygy module of E. Then minimality implies that $\Omega\mathrm{E} \subseteq \mathfrak{m}.R^d$ and so $\mathfrak{m}.\Omega\mathrm{E} = 0$. That is $\Omega\mathrm{E}$ is a k-vector space of length

$$\ell(\Omega\mathrm{E}) = \ell(R^d) - \ell(\mathrm{E}(k)) = d\ell(R) - \ell(R) = (d-1)\ell(R) = (d-1)(d+1) = d^2 - 1.$$

Hence for all $i > 0$, $\mathrm{Tor}_{i+2}^R(\mathrm{E},\mathrm{E}) \cong \mathrm{Tor}_i^R(k,k)^{(d^2-1)^2}$. We claim that $\mathrm{Tor}_i^R(k,k) \cong k^{d^i}$. For this note that since $R \supseteq k^d$, continuing the construction of the given exact sequence, we end up with the minimal free resolution of k over R as

$$\cdots \to R^{d^3} \to R^{d^2} \to R^d \to R \to k \to 0.$$

Thus $\mathrm{Tor}_{i+2}^R(\mathrm{E},\mathrm{E}) \cong k^{d^i(d^2-1)^2}$ for all $i > 0$, which is of course non-zero. It remains to check the non-vanishing of $\mathrm{Tor}_2^R(\mathrm{E},\mathrm{E})$ and $\mathrm{Tor}_1^R(\mathrm{E},\mathrm{E})$.

Note that since $\beta_R^1(\mathrm{E}) = d^2 - 1$,

$$\mathrm{Tor}_2^R(\mathrm{E},\mathrm{E}) \cong \mathrm{Tor}_1^R(k^{d^2-1},\mathrm{E}) \cong \mathrm{Tor}_1^R(k,\mathrm{E})^{d^2-1} \cong k^{(d^2-1)^2}$$

which is again non-zero.

Finally, the sequence

$$0 \to \mathrm{Tor}_1^R(\mathrm{E},\mathrm{E}) \to \mathrm{E} \otimes_R k^{d^2-1} \to \mathrm{E}^d \to \mathrm{E} \otimes_R \mathrm{E} \to 0$$

is exact. Since $\mathrm{E} \otimes_R k^{d^2-1} \cong (\mathrm{E}/\mathfrak{m}\mathrm{E})^{d^2-1}$ and E^d is of length $(d+1)d$, a simple length comparison shows that

$$d(d^2-1) > (d+1)d \Longleftrightarrow d > 2,$$

hence for $d > 2$, the homomorphism $\mathrm{E} \otimes_R k^{d^2-1} \to \mathrm{E}^d$ has non-zero kernel, that is $\mathrm{Tor}_1^R(\mathrm{E},\mathrm{E}) \neq 0$ if $d > 2$.

But since d can only take values larger than 1, we just need to consider the case $d = 2$. Then $d(d^2-1) = d^2 + d = 6$ and so $\mathrm{Tor}_1^R(\mathrm{E},\mathrm{E}) \neq 0 \Longleftrightarrow \mathrm{E} \otimes_R \mathrm{E} \neq 0$. In fact, $\ell(\mathrm{Tor}_1^R(\mathrm{E},\mathrm{E})) = \ell(\mathrm{E}\otimes_R \mathrm{E})$. Now the non-vanishing of $\mathrm{E}\otimes_R \mathrm{E}$ comes from the surjection

$$\mathrm{E}\otimes_R \mathrm{E} \twoheadrightarrow \mathrm{E}/\mathfrak{m}\mathrm{E}\otimes_R \mathrm{E}/\mathfrak{m}\mathrm{E}$$

and the fact that the right hand side is an $(d^2 =)$ 4-dimensional k-vector space. □

3.4 Further analysis of vanishing of Ext and Tor

In this section we perform a homological analysis similar to Theorem 4.1.1, but for different situations. In a sense, our situation is more general because we consider sequences like (‡) and (‡†) where the ideals are possibly different from $\mathfrak{m}$. On the other hand, we have to make different assumptions about the ideals, namely that J in (‡) and the J_t in (‡†) are generated by regular sequences:

$$0 \to \bigoplus_{s=1}^{a}(R/I) \to M \to \bigoplus_{t=1}^{b}(R/J) \to 0 \quad (‡†)$$
$$0 \to \bigoplus_{s=1}^{a}(R/I_s) \to M \to \bigoplus_{t=1}^{b}(R/J_t) \to 0 \quad (‡‡)$$

As it was mentioned by Hochster and Richert in [36, Remark 2.1] and as we discovered we need to add some extra assumptions to situations (‡†) and (‡‡) which is stated in the following:

1. $J_1, \cdots, J_b$ are ideals generated by R-sequences.

2. $I_s \supseteq J_t$ for all $s = 1, \cdots, a$ and $t = 1, \cdots, b$.

The reason to add such extra assumptions as it will be demonstrated in Lemma 3.4.1 is that when they hold the calculation of the relevant Ext and Tor modules are greatly simplified, that is:

Lemma 3.4.1. *Let J be an ideal of R generated by a regular sequence $f_1, \cdots, f_n$ and let I be an ideal containing J. Then for $j \geq 0$*

1. $\operatorname{Ext}^j_R(R/J, R/I) \cong (R/I)^{\binom{n}{j}}$.
2. $\operatorname{Tor}^R_j(R/J, R/I) \cong (R/I)^{\binom{n}{j}}$.
3. *More generally,* $\operatorname{Ext}^j_R(R/J, N) \cong N^{\binom{n}{j}}$ *and* $\operatorname{Tor}^R_j(R/J, N) \cong N^{\binom{n}{j}}$ *for each R-module N which is killed by J.*

Proof. (1) & (2) Since J is generated by a regular sequence we may use the Koszul complex $K_\bullet(f_1, \cdots, f_n; R)$ as a free resolution of R/J : if $U_1, \cdots, U_n$ is a basis for the free module in degree one such that U_j maps to f_j, then the elements $U_{i_1} \wedge \cdots \wedge U_{i_j}$ such that $i_1 < \cdots < i_j$ are a free basis for the free R-module $K_j = K_j(f_1, \cdots, f_n; R)$. Since I contains J, when we apply $\operatorname{Hom}_R(-, R/I)$ (resp. $- \otimes_R R/I$) to $K_\bullet$, the maps in the complex all become 0, and the jth module may be identified with $\operatorname{Hom}_R(K_j, R/I) \cong \operatorname{Hom}_R(K_j, R/J) \otimes_R R/I$ (resp. $K_j \otimes_R R/I \cong (K_j \otimes_R R/J) \otimes_R R/I$). We may take as an (R/I)-free basis the image of the basis for $\operatorname{Hom}_R(K_j, R/I)$ (resp. $(K_j \otimes_R R/I)$). In particular, $\operatorname{Ext}^1_R(R/J, R/I) \cong (R/I)^n$ (resp. $\operatorname{Tor}^R_1(R/J, R/I) \cong (R/I)^n$). The same idea works for higher Ext and Tor modules.

(3) The idea is essentially the same. Replace R/I by any R-module N which is killed by J. Then we get $\operatorname{Ext}^j_R(R/J, N) \cong N^{\binom{n}{j}} \cong \operatorname{Ext}^j_R(R/J, R/J) \otimes_R N$ and $\operatorname{Tor}^R_j(R/J, N) \cong N^{\binom{n}{j}} \cong \operatorname{Tor}^R_j(R/J, R/J) \otimes_R N$ as well. □

The formulation of case (‡†) is given in the following:

Theorem 3.4.2. *Let R be a local ring and let M be a non zero R-module for which $0 \to (R/I)^a \to M \to (R/J)^b \to 0$ is an exact sequence where J is an ideal of R generated by a regular sequence $f_1, \cdots, f_n$ and let $I \supseteq J$. Let t be an integer.*

1. *Let* $\operatorname{Ext}_R^i(M, R/I) = 0$ *for* $i = t, t+1$. *Then we have*

$$\operatorname{Ext}_R^t(R/I, R/I) \cong (R/I)^{\frac{b}{a}\binom{n}{t+1}}.$$

2. *Let* $\operatorname{Tor}_i^R(M, R/I) = 0$ *for* $i = t, t+1$. *Then we have*

$$\operatorname{Tor}_t^R(R/I, R/I) \cong (R/I)^{\frac{b}{a}\binom{n}{t+1}}.$$

3. *Let* $\operatorname{Ext}_R^i(R/J, M) = 0$ *for* $i = t, t+1$. *Then we have*

$$b\binom{n}{t} = a\binom{n}{t+1}.$$

4. *Let* $\operatorname{Tor}_i^R(R/J, M) = 0$ *for* $i = t, t+1$. *Then we have*

$$b\binom{n}{t+1} = a\binom{n}{t}.$$

Proof. 1. Applying the contravariant functor $\operatorname{Hom}_R(-, R/I)$ to the exact sequence $0 \to (R/I)^a \to M \to (R/J)^b \to 0$, for each $i \geq 0$ we obtain the following exact sequence,

$$\operatorname{Ext}_R^i(M, R/I) \to \operatorname{Ext}_R^i((R/I)^a, R/I) \to \operatorname{Ext}_R^{i+1}((R/J)^b, R/I) \to \operatorname{Ext}_R^{i+1}(M, R/I).$$

Since easily $\operatorname{Hom}_R(R/I, R/I) \cong \operatorname{Hom}_{R/I}(R/I, R/I) \cong R/I$ and since by Lemma 3.4.1, $\operatorname{Ext}_R^j(R/J, R/I) \cong (R/I)^{\binom{n}{j}}$ for each $j \geq 0$, we get the following exact sequence

$$\operatorname{Ext}_R^i(M, R/I) \to (\operatorname{Ext}_R^i(R/I, R/I))^a \to (R/I)^{b\binom{n}{i+1}} \to \operatorname{Ext}_R^{i+1}(M, R/I).$$

Now if $\operatorname{Ext}_R^t(M, R/I) = \operatorname{Ext}_R^{t+1}(M, R/I) = 0$, then we get

$$\operatorname{Ext}_R^t(R/I, R/I))^a \cong (R/I)^{b\binom{n}{t+1}},$$

as required.

2. Apply the covariant functor $- \otimes_R R/I$ to the given exact sequence and use Lemma 3.4.1 (2) in the middle of the derived exact sequence to obtain the following exact sequence for each $i \geq 0$:

$$\mathrm{Tor}_{i+1}^R(M, R/I) \to (R/I)^{b\binom{n}{i+1}} \to (\mathrm{Tor}_i^R(R/I, R/I))^a \to \mathrm{Tor}_i^R(M, R/I).$$

3. We have the following exact sequence

$$\mathrm{Ext}_R^i(R/J, M) \to (\mathrm{Ext}_R^i(R/J, R/J))^b \to (\mathrm{Ext}_R^{i+1}(R/J, R/I))^a \to \mathrm{Ext}_R^{i+1}(R/J, M),$$

which is a part of the derived long exact sequence of the given exact sequence after applying the covariant functor $\mathrm{Hom}_R(R/J, -)$. Now note that the vanishing $\mathrm{Ext}_R^i(R/J, M)$ for $i = t, t+1$ and using the results provided by Lemma 3.4.1 we end up with

$$b\binom{n}{t} = a\binom{n}{t+1}.$$

4. This part is essentially the same as that of (3) with a slight change on the direction of the derived exact sequence.

□

Finally we provide a proof for the general situation in (‡‡).

Theorem 3.4.3. *Let R be a local ring and let M be a non zero R-module for which $0 \to \bigoplus_{s=1}^{a} R/I_s \to M \to \bigoplus_{t=1}^{b} R/J_t \to 0$ is an exact sequence where $J_1, \cdots, J_b$ are ideals generated by R-sequences of length $n_1, \cdots, n_b$ respectively. Then we have the following:*

1. *Let $\mathrm{Ext}_R^i(M, R/I_u) = \mathrm{Ext}_R^{i+1}(M, R/I_u) = 0$ for some $1 \leq u \leq a$ and for some i, and let $I_u \supseteq J_t$ for each t. Then*

$$\bigoplus_{s=1}^{a} \mathrm{Ext}_R^i(R/I_s, R/I_u) \cong (R/I_u)^{\binom{n_1}{i+1}} \bigoplus \cdots \bigoplus (R/I_u)^{\binom{n_t}{i+1}}.$$

2. *Let* $\operatorname{Ext}_R^i(M, R/I_u) = \operatorname{Ext}_R^{i+1}(M, R/I_u) = 0$ *for each* $1 \leq u \leq a$ *and for some* i*, and let* $I_s \supseteq J_t$ *for each* s*,* t*. Then*

$$\bigoplus_{u=1}^{a} \bigoplus_{s=1}^{a} \operatorname{Ext}_R^i(R/I_s, R/I_u) \cong \bigoplus_{u=1}^{a} (R/I_u)^{\binom{n_1}{i+1}} \bigoplus \cdots \bigoplus \bigoplus_{u=1}^{a} (R/I_u)^{\binom{n_t}{i+1}}.$$

Proof. 1. Since for each $1 \leq u \leq a$ we have the following exact sequence

$$\operatorname{Ext}_R^i(M, R/I_u) \longrightarrow \operatorname{Ext}_R^i(\bigoplus_{s=1}^{a} R/I_s, R/I_u) \longrightarrow \operatorname{Ext}_R^{i+1}(\bigoplus_{t=1}^{b} R/J_t, R/I_u) \longrightarrow \operatorname{Ext}_R^{i+1}(M, R/I_u)$$

the vanishing of the the outer sides will implies that

$$\bigoplus_{s=1}^{a} \operatorname{Ext}_R^i(R/I_s, R/I_u) \cong \bigoplus_{t=1}^{b} \operatorname{Ext}_R^{i+1}(R/J_t, R/I_u).$$

After b-times applying Lemma 3.4.1 (1) to the right hand side we deduce that

$$\bigoplus_{s=1}^{a} \operatorname{Ext}_R^i(R/I_s, R/I_u) \cong (R/I_u)^{\binom{n_1}{i+1}} \bigoplus \cdots \bigoplus (R/I_u)^{\binom{n_t}{i+1}}.$$

2. Since for each $1 \leq u \leq a$ and for each $1 \leq t \leq b$, $I_u \supseteq J_t$ the result of (1) is applicable and so taking direct sums will give the result.

□

Chapter 4

Graded minimal free resolution of ideals

In [1, 2] Alwis considered the general n-gon with vertices at the points $1, 2, \cdots, n$. For its suspension, the simplicial complex that involves two more vertices, say at $n+1$ and $n+2$, he found the minimal free resolution and the Betti numbers of the S-module S/I where I is the associated ideal to the suspension in the Stanley-Reisner sense. In this paper, we generalize this result to the following form. Let J_1 be an ideal of S and

$$0 \to S^{\beta_c^S} \xrightarrow{f_c} S^{\beta_{c-1}^S} \to \ldots \to S^{\beta_1^S} \xrightarrow{f_1} S^{\beta_0^S} \xrightarrow{f_0} \frac{S}{J_1} \to 0$$

be the minimal free resolution of the S-module S/J_1. Let $x_{n+1}, \ldots, x_{n+r}$ be r indeterminates over S, for some non-negative integer r, and $R = K[x_1, \ldots, x_{n+r}]$. We construct the minimal free resolution of the R-module R/I where $I = J_1R + (y)$ and y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$. More precisely, in Theorem 4.1.1 we show that the following is the minimal free resolution for R/I:

$$0 \to R^{\beta_c^S} \xrightarrow{\delta_{c+1}} R^{\beta_c^S} \oplus R^{\beta_{c-1}^S} \to \ldots \to R^{\beta_1^S} \oplus R^{\beta_0^S} \xrightarrow{\delta_1} R^{\beta_0^S} \to R/I \to 0.$$

Our approach in order to reach this resolution is based on mapping cone. By an inductive argument we may generalize the ideal (y) to $(f_1, \cdots, f_n)$ where f_i is any homogenous polynomial in $K[x_{r_i}, \cdots, x_{r_{i+1}}]$ for $r_1 < r_2 < \cdots < r_t$.

In Section 4.2 the graded version of our main theorem is considered. As an application, let I be a graded ideal of S such that S/I is Cohen-Macaulay with a pure resolution where its Betti numbers are given in [16, Theorem 4.1.15]. Then in Corollary 4.2.3 we have the Betti numbers of ideal $J := I + (y)$ where y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$.

Section 4.3 is devoted to further analysis of a special class of Stanley-Reisner ideals. Our interest to study the Betti numbers of In fact we assume that $I = (z_1, \cdots, z_t)$, where $z_i = \prod_{j=1}^{k_i} x_{i_j}$ and that each x_{i_j} occurs only once in I. Now the Betti numbers of R/I can be easily obtained from our main theorem. It can also be seen from the fact that I is generated by a regular sequence and using Koszul complex. We analysis this certain family of ideals in terms of simplicial complexes. Let Δ be the simplicial complex corresponding to I. From the primary decomposition of I we see that Δ is pure of dimension $n-t-1$. In fact it is consisting of $k_1 \cdots k_t$ facets all of dimension $n-t-1$. Furthermore, the ideal I is perfect unmixed and R/I is a Cohen-Macaulay ring. By a result of Eagon, Reiner and a result of Terai we deduced that the regularity of R/I_{Δ^*} is $\mathrm{reg}(R/I_{\Delta^*}) = \mathrm{proj.dim} R/I - 1$, where Δ^* is the Alexander dual of Δ. On the other hand, the regularity of R/I is $k_1 + \cdots + k_t - t$. Finally we provide some concrete examples to verify our results.

4.1 Minimal free resolution of sums of ideals

Given a morphism $\alpha : F \longrightarrow G$ of two complexes (F, φ) and (G, ψ) the mapping cone $M := M(\alpha)$ of α is the complex such that $M(\alpha)_i = F_{i-1} \oplus G_i$, with differential

$$F_i \oplus G_{i+1} \stackrel{\sigma_{i+1}}{\to} F_{i-1} \oplus G_i,$$

where $\sigma_{i+1} = \begin{pmatrix} -\varphi_i & \alpha_i \\ \psi_{i+1} & 0 \end{pmatrix}$, that is, on G_{i+1} the map is the differential of G, but on F_i the map is the sum of the differential of F and the given map α of complexes; see [26, pp. 650] for more details.

Our main result of this section is the following:

Theorem 4.1.1. *Let J_1 be a homogenous ideal of the polynomial ring $S = K[x_1, \ldots, x_n]$. Let*

$$0 \to S^{\beta_c^S} \xrightarrow{f_c} S^{\beta_{c-1}^S} \to \ldots \to S^{\beta_1^S} \xrightarrow{f_1} S^{\beta_0^S} \xrightarrow{f_0} \frac{S}{J_1} \to 0 \tag{4.1}$$

be the minimal free resolution of the S-module S/J_1 with appropriate boundary maps. Let $x_{n+1}, \ldots, x_{n+r}$ be r indeterminate over S, for some non-negative integer r, and $R = K[x_1, \ldots, x_{n+r}]$. Then the following is the minimal free resolution of the R-module R/I where $I = J_1R + (y)$ and y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$:

$$0 \to R^{\beta_c^S} \xrightarrow{\delta_{c+1}} R^{\beta_c^S} \oplus R^{\beta_{c-1}^S} \to \ldots \to R^{\beta_1^S} \oplus R^{\beta_0^S} \xrightarrow{\delta_1} R^{\beta_0^S} \to R/I \to 0. \tag{4.2}$$

Proof. Let $J = J_1R$. Tensoring the exact sequence (4.12) with the K-module $K[x_{n+1}, \ldots, x_{n+r}]$ which is a free module over K, we deduce the following exact sequence of R-modules.

$$0 \to R^{\beta_c^S} \xrightarrow{d_c} R^{\beta_{c-1}^S} \to \ldots \to R^{\beta_1^S} \xrightarrow{d_1} R^{\beta_0^S} \xrightarrow{d_0} \frac{R}{J} \to 0,$$

where $d_i = f_i \otimes id$. This means that the following complex is exact at all places except at degree 0 :

$$\begin{array}{cccccc} c & c-1 & & & 1 & 0 \end{array}$$

$$0 \to R^{\beta_c^S} \xrightarrow{d_c} R^{\beta_{c-1}^S} \to \ldots \to R^{\beta_1^S} \xrightarrow{d_1} R^{\beta_0^S} \to 0. \tag{4.3}$$

Consider the following diagram where the two rows are the same as the complex (4.3) and the vertical maps are multiplications by y :

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & R^{\beta_c^S} & \xrightarrow{\mathrm{d_c}} & R^{\beta_{c-1}^S} & \longrightarrow & \ldots & \longrightarrow & R^{\beta_1^S} & \xrightarrow{\mathrm{d_1}} & R^{\beta_0^S} & \longrightarrow & 0 \\ & & \downarrow \mathrm{y} & & \downarrow \mathrm{y} & & \downarrow \mathrm{y} & & \downarrow \mathrm{y} & & \downarrow \mathrm{y} & & \\ 0 & \longrightarrow & R^{\beta_c^S} & \xrightarrow{\mathrm{d_c}} & R^{\beta_{c-1}^S} & \longrightarrow & \ldots & \longrightarrow & R^{\beta_1^S} & \xrightarrow{\mathrm{d_1}} & R^{\beta_0^S} & \longrightarrow & 0 \end{array} \tag{4.4}$$

As our maps in complex (4.4) are R-module homomorphisms and $y \in R$, (4.4) is a double complex and its corresponding mapping cone is given by

$$0 \to R^{\beta_c^S} \xrightarrow{\delta_{c+1}} R^{\beta_c^S} \oplus R^{\beta_{c-1}^S} \to \ldots \to R^{\beta_1^S} \oplus R^{\beta_0^S} \xrightarrow{\delta_1} R^{\beta_0^S} \to 0$$

where $\delta_i : R^{\beta_i^S} \oplus R^{\beta_{i-1}^S} \longrightarrow R^{\beta_{i-1}^S} \oplus R^{\beta_{i-2}^S}, i = 1, 2, \ldots, c+1$ is given by $\delta_i(p,q) = (d_i(p) + (-1)^i yq, d_{i-1}(q))$ for $i = 2, 3, \ldots, c$, and, $\delta_1(p,q) = (d_1(p) + (-1)yq, 0)$, $\delta_{c+1}(p,q) = ((-1)^{c+1}yq, d_c(q))$. Now it is easy to see that $\delta_{i-1} \circ \delta_i = 0$.

It remains to prove the exactness and the minimality of (4.2). First we prove that the sequence (4.2) is exact at all places except degree 0 where its homology is R/I. Let $D_i := R^{\beta_i^S}$. For $i > 1$, take

$$D_{i+1} \oplus D_i \xrightarrow{\delta_{i+1}} D_i \oplus D_{i-1} \xrightarrow{\delta_i} D_{i-1} \oplus D_{i-2}.$$

We claim that $\mathrm{Ker}\delta_i = \mathrm{Im}\delta_{i+1}$ for all i. If $(p,q) \in \mathrm{Ker}\delta_i$ then

$$d_i(p) + (-1)^i yq = 0 \quad , \quad d_{i-1}(q) = 0.$$

Therefore, $q \in \mathrm{Ker} d_{i-1} = \mathrm{Im} d_i$, so $q = d_i(q_1)$ for some $q_1 \in D_i$. The equation $d_i(p) + (-1)^i yq = 0$ yields $d_i(p + (-1)^i yq_1) = 0$, which is to say that $p + (-1)^i yq_1 \in \mathrm{Ker} d_i = \mathrm{Im} d_{i+1}$ and so, $p + (-1)^i yq_1 = d_{i+1}(p_1)$ for some $p_1 \in D_{i+1}$. Hence, $\delta_{i+1}(p_1, q_1) = (p, q)$, i.e., $(p, q) \in \mathrm{Im}\delta_{i+1}$. We conclude that $\mathrm{Ker}\delta_i = \mathrm{Im}\delta_{i+1}$ for $i > 1$.

For $i = 1$, let $(p, q) \in \mathrm{Ker}\delta_1$. After simplification it follows that $yq \in J$. Although $y \notin J$, using the primary decomposition of J gives $q \in J = \mathrm{Ker} d_0 = \mathrm{Im} d_1$, and so $q = d_1(q_1')$ for some $q_1{}' \in D_1$. Hence $d_1(p) = yq = yd_1(q_1{}') = d_1(yq_1{}')$, which implies that $p - yq_1{}' = d_2(p_1{}')$ for some $p_1{}' \in D_2$. This yields $\delta_2(p_1{}', q_1{}') = (p, q)$ and hence $\mathrm{Ker}\delta_1 = \mathrm{Im}\delta_2$.

Finally, for $i = 0$, we consider the exact sequence $D_1 \oplus D_0 \xrightarrow{\delta_1} D_0 \to 0$. As $\mathrm{Im}\delta_1 = \{j - yq | j \in J, q \in R\} = J + (y) = I$, we are done. Therefore, (4.2) has homology equal to R/I at the zeroth spot.

Now we show that the resolution (4.2) is minimal. We just need to check that

$$\delta_i \otimes id : (R^{\beta_i^S} \oplus R^{\beta_{i-1}^S}) \otimes_R K \to (R^{\beta_{i-1}^S} \oplus R^{\beta_{i-2}^S}) \otimes_R K$$

is zero for $i = 1, 2, \ldots, c + 1$. Note that the following diagram is easily commutative:

$$\begin{array}{ccc} (R^{\beta_i^S} \oplus R^{\beta_{i-1}^S}) \otimes_R K & \longrightarrow & (R^{\beta_{i-1}^S} \oplus R^{\beta_{i-2}^S}) \otimes_R K \\ \downarrow & & \downarrow \\ (R^{\beta_i^S} \otimes_R K) \oplus (R^{\beta_{i-1}^S} \otimes_R K) & \longrightarrow & (R^{\beta_{i-1}^S} \otimes_R K) \oplus (R^{\beta_{i-2}^S} \otimes_R K) \end{array} \tag{4.5}$$

But since $S^{\beta_i^S} \otimes_S K \to S^{\beta_{i-1}^S} \otimes_S K$ is zero for such i (by the minimality of (4.12))) and since $T = K[x_{n+1}, \ldots, x_{n+r}]$ is a free K-module and $R = S \otimes_K T$, we deduce that $K \otimes_R R^{\beta_i^S} \to K \otimes_R R^{\beta_{i-1}^S}$ is zero. Hence the first row of (4.5) is also zero, and the proof is complete. □

The following consequences are now immediate:

Corollary 4.1.2. *Let $S = K[x_1, \ldots, x_n]$, J_1 any homogenous ideal of S for which proj.dim$(S/J_1) = c$. Then for the ideal $I = J_1R + (y)$ of $R = S[x_1, \ldots, x_{n+r}]$ where y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$. Then the i^{th} Betti number R/I, $\beta_i^R(R/I)$, is given by*

$$\beta_i^R(R/I) = \begin{cases} \beta_0^S(S/J_1), & i=0, \\ \beta_i^S(S/J_1) + \beta_{i-1}^S(S/J_1), & i=1,2,\ldots,c, \\ \beta_c^S(S/J_1), & i=c+1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Example 4.1.3. Let $S = K[x_1, \ldots, x_3]$ and $J_3 = (x_1^3, x_2x_3^3)$ be an ideal of S. Then $I_3 = (x_1^3, x_2x_3^3, x_4^2)$ is an ideal of $R = K[x_1, \ldots, x_4]$. Now (4.6) enables us to compute the Betti numbers of R/I_3 in terms of the Betti numers of S/J_3, i.e.,

$$\begin{cases} \beta_0^R(R/I_3) = \beta_0^S(S/J_3) = 1, \\ \beta_1^R(R/I_3) = \beta_1^S(S/J_3) + \beta_0^S(S/J_3) = 2 + 1 = 3, \\ \beta_2^R(R/I_3) = \beta_2^S(S/J_3) + \beta_1^S(S/J_3) = 1 + 2 = 3, \\ \beta_3^R(R/I_3) = \beta_2^S(S/J_3) = 1. \end{cases}$$

One can note that in order to compute $\beta_i^S(S/J_3)$ for $i = 0, 1, 2$, we apply (4.6) once again to $S = K[x_1]$ and $J_1 = (x_1^3)$.

Remark 4.1.4. By an inductive argument the ideal (y) in Theorem 4.1.1 can be extended to $(f_1, \cdots, f_t)$ where f_i is any homogenous polynomial in $K[x_{r_i}, \cdots, x_{r_{i+1}}]$ for $r_1 < r_2 < \cdots < r_t$. So, our general result as we mentioned in the abstract can be obtained from this observation.

4.2 The graded version

In the following we have the graded version of our Theorem 4.1.1.

Theorem 4.2.1. *Let $S = K[x_1, \cdots, x_n]$ be a polynomial ring over a field K, and let J be a graded ideal of S with the (minimal) graded free resolution*

$$0 \longrightarrow \oplus S(-a_{cj})^{\beta_{cj}} \longrightarrow \cdots \longrightarrow \oplus S(-a_{1j})^{\beta_{1j}} \longrightarrow S \longrightarrow S/J \longrightarrow 0, \tag{4.7}$$

then for the ring $R = K[x_1, \cdots, x_{n+r}]$ and its ideal $I := JR + (y)$, where y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$ of degree $e = \sum_{i=n+1}^{n+r} \alpha_i$, the (minimal) graded free resolution of R/I is as follows:

$$\begin{aligned} 0 &\longrightarrow \oplus R(-a_{cj} - e)^{\beta_{cj}} \longrightarrow \oplus R(-a_{cj})^{\beta_{cj}} \bigoplus \oplus R(-a_{c-1\,j} - e)^{\beta_{c-1\,j}} \longrightarrow \cdots \longrightarrow \\ &\oplus R(-a_{2j})^{\beta_{2j}} \bigoplus \oplus R(-a_{1j} - e)^{\beta_{1j}} \longrightarrow \oplus R(-a_{1j})^{\beta_{1j}} \bigoplus R(-e) \longrightarrow R \longrightarrow R/I \longrightarrow 0. \end{aligned} \tag{4.8}$$

Proof. For a moment ignore the graded settings. Then by Theorem 4.1.1 the desired resolution is obtained provided (4.7) is a free resolution of S/J_1. Furthermore (4.8) is minimal as long as (4.7) is minimal.

It only remains to verify that in (4.8) the maps are zero maps, that is they preserve the degree. But this simple matter of checking holds due to the formulation of differentials in the new resolution:

$$\delta_i(p, q) = (d_i(p) + (-1)^i yq, d_{i-1}(q)) \quad \text{for } i = 2, 3, \ldots, c, \text{ and,}$$

$$\delta_1(p, q) = (d_1(p) + (-1)yq, 0),$$

$$\delta_{c+1}(p, q) = ((-1)^{c+1} yq, d_c(q)).$$

□

As an example of Theorem 4.2.1 we have:

Example 4.2.2. Let $R = Q[x_1, \cdots, x_7]$. It is easy to see that for the ideal $I_1 := (x_1, x_2^2)$, the minimal free resolution of R/I_1 is

$$0 \longrightarrow R(-3) \longrightarrow R(-1) \oplus R(-2) \longrightarrow R \longrightarrow R/I_1 \longrightarrow 0$$

In the following we compute the minimal free resolution of some new ideals:

(i) Let $I_2 := I_1 + (x_3^5)$. Then for R/I_2 we get

$$0 \longrightarrow R(-8) \longrightarrow R(-3) \oplus R(-6) \oplus R(-7) \longrightarrow$$
$$R(-1) \oplus R(-2) \oplus R(-5) \longrightarrow R \longrightarrow R/I_2 \longrightarrow 0.$$

(ii) For $I_3 := I_2 + (x_4^4) = (x_1, x_2^2, x_3^5, x_4^4)$, the minimal free resolution of R/I_3 is

$$0 \longrightarrow R(-12) \longrightarrow R(-7) \oplus R(-8) \oplus R(-10) \oplus R(-11) \longrightarrow$$
$$R(-3) \oplus R(-5) \oplus R^2(-6) \oplus R(-7) \oplus R(-9) \longrightarrow$$
$$R(-1) \oplus R(-2) \oplus R(-4) \oplus R(-5) \longrightarrow R \longrightarrow R/I_3 \longrightarrow 0.$$

(iii) Finally for $I_4 := I_2 + (x_4x_5) = (x_1, x_2^2, x_3^5, x_4x_5)$ we obtain

$$0 \longrightarrow R(-10) \longrightarrow R(-5) \oplus R^2(-8) \oplus R(-9) \longrightarrow$$
$$R^2(-3) \oplus R(-4) \oplus R(-6) \oplus R^2(-7) \longrightarrow$$
$$R(-1) \oplus R^2(-2) \oplus R(-5) \longrightarrow R \longrightarrow R/I_4 \longrightarrow 0.$$

By [16, Theorem 4.1.15] for a graded ideal I of a polynomial ring $S = K[x_1, \cdots, x_n]$ over a field K such that S/I is Cohen-Macaulay with a pure resolution of type $(d_1, \cdots, d_p)$ its Betti numbers are given by this formula

$$\beta_i^S(S/I) = (-1)^{i+1} \prod_{j \neq i} \frac{d_j}{(d_j - d_i)}.$$

Now as an application of Corollary 4.1.2 we can compute the Betti numbers of the ideals in the following form. In fact set $R = K[x_1, \cdots, x_{n+r}]$ and $J := I + (y)$ where y is any homogenous polynomial $f(x_{n+1}, \cdots, x_{n+r})$.

Corollary 4.2.3. *With the notations as above we have* $\beta_0^R(R/J) = \beta_0^S(S/I) = 1$,

$$\beta_i^R(R/J) = (-1)^{i+1} \prod_{j \neq i} \frac{d_j}{(d_j - d_i)} + (-1)^i \prod_{j \neq i-1} \frac{d_j}{(d_j - d_{i-1})}$$

for $i = 1, \cdots, p$, $\beta_{p+1}^R(R/J) = \beta_p^S(S/I)$ *and* $\beta_i^R(R/J) = 0$ *for* $i > p+1$.

4.3 Analysis of a special class of Stanley-Reisner ideals

Let Δ be the following simplicial complex which corresponds to the n-gon with vertices at the points $1, 2, \ldots, n$. Clearly Δ is a pure simplical complex (of dimension 1).

$$\Delta = \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}, \{1,2\}, \{2,3\}, \ldots, \{n,1\}\}. \tag{4.9}$$

Let $S = K[x_1, x_2, \ldots, x_n]$, and let J_1 be the Stanley-Reisner ideal associated to Δ in (4.9), i.e., J_1= the ideal in S generated by all monomials of the form $x_{i_1}x_{i_2}\ldots x_{i_r}$, where $1 \leq i_1 < i_2 < \ldots < i_r \leq n$ and $\{i_1, \ldots, i_r\} \notin \Delta$. Then it easily follows that for each $n \geq 3$ we get:

$$J_1 = \begin{cases} (x_1x_2x_3), & \text{n=3;} \\ (x_1x_3, x_1x_4, \cdots, x_1x_{n-1}, x_2x_4, \cdots, x_2x_n, \cdots, x_{n-2}x_n), & \text{otherwise.} \end{cases} \tag{4.10}$$

In [2] the author showed that the ith Betti number of the S-module S/J_1, denoted by $\beta_i^S(S/J_1)$ or simply β_i^S, which is the ith Betti number of the n-gon, for $n \geq 3$ is given by

$$\beta_i^S = \begin{cases} 1, & \text{i=0,} \\ \binom{n}{i+1}\frac{i(n-i-2)}{n-1}, & \text{i=1,2,...,n-3,} \\ 1, & \text{i=n-2,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.11}$$

As well we have

$$0 \longrightarrow S^{\beta_{n-2}^S} \xrightarrow{f_{n-2}} S^{\beta_{n-3}^S} \longrightarrow \ldots \longrightarrow S^{\beta_1^S} \xrightarrow{f_1} S^{\beta_0^S} \xrightarrow{f_0} \frac{S}{J_1} \longrightarrow 0 \tag{4.12}$$

is the minimal free resolution of the S-module S/J_1 with appropriate boundary maps. As a consequence of our Theorem4.1.1 we compute the Betti numbers of a special class of Stanley-Reisner ideals which can be obtained from the *Koszul complex* as it is shown in the proof of the following theorem.

We analysis this certain family of ideals in terms of simplicial complexes.

Theorem 4.3.1. *Let Δ be a simplicial complex for which $I := I_\Delta = (z_1, \cdots, z_t)$, where $z_i = \prod_{j=1}^{k_i} x_{i_j}$ and that each x_{i_j} occurs only once in I_Δ. Then we have*

(i) the Betti numbers of I are given by the following formula:

$$\beta_i^R(R/I) = \begin{cases} 1, & \textit{i=0,} \\ \binom{t}{i}, & \textit{i=1,2,\dots,t-1,} \\ 1, & \textit{i=t,} \\ 0, & \textit{otherwise.} \end{cases}$$

(ii) I is perfect and unmixed and also R/I is Cohen-Macaulay.

Proof. We prove (i) by induction on t. Since for $t = 1$, I is just of the form $I = (x_1^{\alpha_1} \cdots x_s^{\alpha_s})$ for some s, where $\alpha_i \in \{0, 1\}$. So one has

$$\beta_i^R(R/I) = \begin{cases} 1, & \text{i=0,} \\ 1, & \text{i=1,} \\ 0, & \text{otherwise.} \end{cases}$$

Now let $t > 1$, and assume that the case $t - 1$ is settled. Take $S = k[x_{i_j} : i = 1, \cdots, t-1]$. Consider the ideal $J = (z_1, \cdots, z_{t-1})$ of S. Then by induction hypothesis we have

$$\beta_i^S(S/J) = \begin{cases} 1, & \text{i=0,} \\ \binom{t-1}{i}, & \text{i=1,2,\dots,t-2,} \\ 1, & \text{i=t-1,} \\ 0, & \text{otherwise.} \end{cases}$$

Formula (4.6) implies that

$$\beta_i^R(R/I) = \begin{cases} 1, & \text{i=0,} \\ \binom{t-1}{i} + \binom{t-1}{i-1} = \binom{t}{i}, & \text{i=1,2,\dots,t-1,} \\ 1, & \text{i=t,} \\ 0, & \text{otherwise.} \end{cases}$$

For the proof (ii) we note that by Theorem 1.5 Δ is pure of dimension $n - t - 1$. In fact it is consisting of $k_1 \cdots k_t$ facets all of dimension $n - t - 1$. Hence,

$$\dim R/I = \dim \Delta + 1 = n - t - 1 + 1 = n - t = \dim R - t.$$

Then by [16, Theorem 2.1.2 (c)] it follows that $z_1, \cdots, z_t$ is a regular sequence on R. Furthermore, by the Auslander-Buchsbaum formula we have

$$\text{depth}R/I = \text{depth}R - \text{proj.dim}R/I = n - t,$$

hence the ring R/I is Cohen-Macaulay and so Δ is Cohen-Macaulay.

In addition, I is perfect, i.e., we have

$$\text{grade}\, I = \text{height}\, I = \dim R - \dim R/I = t = \text{proj.dim}R/I$$

see [16, Corollary 2.1.4]. The first equality can also be seen from the primary decomposition of I and [16, Proposition 1.2.10 (c)].

Finally let $p_1, \ldots, p_r$ be the prime ideals in the primary decomposition of I. Since I is generated by $t = \text{height}\, I$ elements over the polynomial ring R, I is unmixed; see

[50]. Hence $p_1, \ldots, p_r$ are the minimal prime ideals of I by [16, Theorem 2.1.6]. Thus $\mathrm{Ass}\,(R/I) = \{p_1, \ldots, p_r\}$. $\square$

Remark 4.3.2. The ideal I is generated by a regular sequence on R. Thus the Castelnuovo-Mumford regularity of R/I is $k_1 + \cdots + k_t - t$.

Let Δ be a simplicial complex and Δ^* denote the Alexander dual of Δ, i.e., the simplicial complex

$$\Delta^* = \{F \subseteq [n] : [n] - F \notin \Delta\}$$

Corollary 4.3.3. *Consider the graded version of Theorem 4.3.1. Then the regularity of R/I_{Δ^*} is*

$$\mathrm{reg}(R/I_{\Delta^*}) = \mathit{proj.dim} R/I - 1.$$

Proof. Using the primary decomposition of I_{Δ^*} we have

$$I_{\Delta^*} = (x_{1,1}, \ldots, x_{1,t_1}) \cap \cdots \cap (x_{s,1}, \ldots, x_{s,t_s}).$$

By a known result of Eagon and Reiner, $K[\Delta]$ is Cohen-Macaulay if and only if I_{Δ^*} has a linear resolution. Furthermore, $\mathrm{proj.dim}(K[\Delta]) = \mathrm{reg}(I_{\Delta^*})$ by a result of Terai. In view of Theorem 4.2.1, R/I is Cohen-Macaulay and $\mathrm{proj.dim}(K[\Delta]) = t$. Therefore, $\mathrm{reg}(I_{\Delta^*}) = t$ and so $\mathrm{reg}(R/I_{\Delta^*}) = t - 1$. $\square$

In the following we have some examples.

Example 4.3.4. Let $S = K[x_1, x_2]$ and $J = (x_1x_2)$ be an ideal of S. Obviously we have $\beta_0^S(S/J) = 1, \beta_1^S(S/J) = 1$. Now let $I_0 = (x_1x_2, x_3x_4)$ be an ideal of $R = K[x_1, \ldots, x_4]$. Then (4.6) implies that

$$\begin{cases} \beta_0^R(R/I_0) = \beta_0^S(S/J) = 1, \\ \beta_1^R(R/I_0) = \beta_1^S(S/J) + \beta_0^S(S/J) = 1 + 1 = 2, \\ \beta_2^R(R/I_0) = \beta_1^S(S/J) = 1. \end{cases}$$

Furthermore, applying [16, Excersice 4.4.16 (b)], it is easy to see that

$$\begin{aligned} I_0 =&(x_1x_2, x_3x_4) = (x_1, x_3x_4) \cap (x_2, x_3x_4) \\ =&(x_1, x_3) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_2, x_4), \end{aligned}$$

Hence I_0 is the Stanley-Reisner ideal of a pure simplicial complex Δ_0 consisting of 4 facets all of dimension 1. As a result

$$\dim K[\Delta_0] = \dim R/I_0 = \dim \Delta_0 + 1 = 1 + 1 = 2. \quad \square$$

Example 4.3.5. Let $S = K[x_1, \dots, x_4]$ and $J_1 = (x_1x_3, x_2x_4)$ be an ideal of S. Then $I_1 = (x_1x_3, x_2x_4, x_5x_6)$ is an ideal of $R = K[x_1, \dots, x_6]$ and using Example 4.3.4 we have

$$\begin{cases} \beta_0^R(R/I_1) = \beta_0^S(S/J_1) = 1, \\ \beta_1^R(R/I_1) = \beta_1^S(S/J_1) + \beta_0^S(S/J_1) = 2 + 1 = 3, \\ \beta_2^R(R/I_1) = \beta_2^S(S/J_1) + \beta_1^S(S/J_1) = 1 + 2 = 3, \\ \beta_3^R(R/I_1) = \beta_2^S(S/J_1) = 1. \end{cases}$$

Furthermore, by the help of [16, Excersice 4.4.16 (b)]

$$\begin{aligned} I_1 =&(x_1x_3, x_2x_4, x_5x_6) = (x_1, x_2x_4, x_5x_6) \cap (x_3, x_2x_4, x_5x_6) \\ =&(x_1, x_2, x_5x_6) \cap (x_1, x_4, x_5x_6) \cap (x_3, x_2, x_5x_6) \cap (x_3, x_4, x_5x_6) \\ =&(x_1, x_2, x_5) \cap (x_1, x_2, x_6) \cap (x_1, x_4, x_5) \cap (x_1, x_4, x_6) \cap (x_3, x_2, x_5) \cap (x_3, x_2, x_6) \\ &\cap (x_3, x_4, x_5) \cap (x_3, x_4, x_6). \end{aligned}$$

Thus I_1 is the Stanley-Reisner ideal of a pure simplicial complex Δ_1 which consists of 8 facets all of dimension 2. One can easily see that

$$\dim K[\Delta_1] = \dim R/I_1 = \dim \Delta_1 + 1 = 2 + 1 = 3. \quad \square$$

Example 4.3.6. Let $S = K[x_1, \ldots, x_4]$ and $J_2 = (x_1x_3, x_2x_4)$ be an ideal of S. Then $I_2 = (x_1x_3, x_2x_4, x_5x_6x_7)$ is an ideal of $R = K[x_1, \ldots, x_7]$ and similar to Example 4.3.5 we have

$$\beta_0^R(R/I_2) = 1, \beta_1^R(R/I_2) = 3, \ \beta_2^R(R/I_2) = 3, \text{ and } \beta_3^R(R/I_2) = 1.$$

Furthermore, using [16, Excersice 4.4.16 (b)]

$$\begin{aligned} I_2 =&(x_1x_3, x_2x_4, x_5x_6x_7) = (x_1, x_2x_4, x_5x_6x_7) \cap (x_3, x_2x_4, x_5x_6x_7) \\ =&(x_1, x_2, x_5x_6x_7) \cap (x_1, x_4, x_5x_6x_7) \cap (x_3, x_2, x_5x_6x_7) \cap (x_3, x_4, x_5x_6x_7) \\ =&(x_1, x_2, x_5) \cap (x_1, x_2, x_6) \cap (x_1, x_2, x_7) \cap (x_1, x_4, x_5) \cap (x_1, x_4, x_6) \cap (x_1, x_4, x_7) \cap \\ &(x_3, x_2, x_5) \cap (x_3, x_2, x_6) \cap (x_3, x_2, x_7) \cap (x_3, x_4, x_5) \cap (x_3, x_4, x_6) \cap (x_3, x_4, x_7). \end{aligned}$$

Hence I_2 is the Stanley-Reisner ideal of a pure simplicial complex Δ_2 which consists of 12 facets all of dimension 3. One can easily see that

$$\dim K[\Delta_2] = \dim R/I_2 = \dim \Delta_2 + 1 = 3 + 1 = 4. \quad \square$$

Chapter 5

Bass numbers of Local Cohomology modules

In this chapter we study the Bass numbers, the dual notion of the Betti numbers, of local cohomology modules. In fact it is an important problem in local cohomology to determine when the set of associated primes of such modules is finite, and so when the Bass numbers of them is finite. Here we are interested in the following contexts:

- Finiteness of the support (and associated primes) of local cohomology modules
- Artinian local cohomology modules

Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module. Let t be a non-negative integer. We proved that

- If $\mathrm{H}^i_{\mathfrak{a}}(M)$ has finite support for all $i < t$, then $\mathrm{Ass}\,(\mathrm{H}^t_{\mathfrak{a}}(M))$ is finite.
- If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is Artinian for all $i < t$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ need not be Artinian but it has a finitely generated submodule N such that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))/N$ is Artinian.

For the second issue note that it is already known that if the local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < t$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is finitely generated. Thus our response is a negative answer for the similar situation of being Artinian. Under the light of this statement we could prove the following result for *minimax* modules. Recall that an R-module M is called minimax, if there is a finitely generated submodule N of M, such that M/N is Artinian:

- If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is minimax for all $i < t$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is a minimax module and so $\mathrm{H}^t_{\mathfrak{a}}(M)$ has only finite number of associated primes. In addition, if $R/\mathrm{Ann}\,(M)$ is a complete semi-local ring and if for all $i < t$, $\mathrm{H}^i_{\mathfrak{a}}(M)$ is reflexive with respect to the minimal injective cogenerator of the category of R-modules E, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is reflexive with respect to E.

5.1 Finiteness of support and associated primes of local cohomology

Throughout this chapter all rings are commutative Noetherian with nonzero identity and all modules are finitely generated. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal, M a finitely generated R-module. A crucial problem in local cohomology is determining when the set of associated primes of the ith local cohomology module, $\mathrm{H}^i_{\mathfrak{a}}(M)$, with support in $\mathfrak{a}$ is finitely generated. Huneke [37] conjectured that if R is a regular local ring then the local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(R)$ has finitely many associated primes for all i. If R (not necessarily local) contains a field of positive characteristic, the conjecture was proved by Huneke and Sharp [39]. If R contains a field of zero characteristic or is mixed characteristic unramified, the conjecture was proved by G. Lyubeznik [49]. On the other hand Katzman [42] constructed for any field k, a local cohomology module of a local finitely generated k-algebra (which is not regular) with an infinite set of associated primes. If R is not required to be local, Singh [62] made an example showing that these sets of associated primes may be infinite. In 1999 Khashayarmaneh and Salarian [43] proved that $\mathrm{H}^t_{\mathfrak{a}}(M)$ has finite associated primes, when t is an integer for which all the local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M)$) for $i < t$ have one of the following properties:

(a) are finitely generated,

(b) have finite support.

Their approach is mainly by using unconditioned strong d-sequence and filter regular sequences. In [14] Brodmann and Lashgari proved the first case of the above theorem by using induction. Nhan in her paper [54] proved the second case by replacing the condition of being Artinian instead of having finite support when the ring is local. We note that recently Dibaee and Nazari [23] proved a more general result where both (a) and (b) are answered at the same time. In fact they showed that if K is a module (not necessarily finitely generated) over a Noetherian ring S, then for each ideal $\mathfrak{a}$ of S and each non-negative integer s, we have

$$\mathrm{Ass}_S(\mathrm{H}^s_{\mathfrak{a}}(K)) \subseteq \bigcup_{0\le j<s} \mathrm{Ass}_S(\mathrm{Ext}^{s-j+1}_S(S/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(K)/L_j)) \cup \mathrm{Ass}_S(\mathrm{Ext}^s_S(S/\mathfrak{a}, K)/L)$$

for some submodules $L, L_0, \cdots, L_{s-1}$ of their appropriate modules.

Here we provide a simple proof for (b) by induction without using unconditioned strong d-sequence and filter regular sequences.

We begin with the following easy lemma.

Lemma 5.1.1. *Let $\mathfrak{a}$ be an ideal of R and M an $\mathfrak{a}$-torsion R-module, i.e., $M = \bigcup_{n\geq 1}(0 :_M \mathfrak{a}^n)$. Then*

$$Ass\,(M) = Ass\,(0 :_M \mathfrak{a}).$$

Proof. We have the following equalities:

$$\operatorname{Ass}(M) = \bigcup_{n\geq 0} \operatorname{Ass}(0 :_M \mathfrak{a}^n) = \operatorname{Ass}(0 :_M \mathfrak{a}).$$

□

Remark 5.1.2. In Lemma 5.1.1 one can easily see that $\operatorname{Supp}(M) = \operatorname{Supp}(0 :_M \mathfrak{a})$.

Now we prove the following result which is a substantial tool in the inductive step of the next result; see [11, Proposition 2.3].

Proposition 5.1.3. *Let R be a ring and M be an R-module. If N is a submodule of M, then*

$$Ass\,(M/N) \subseteq Ass\,(M) \cup Supp\,(N).$$

In particular, if the set $Supp\,(N)$ is finite, then $Ass\,(M/N)$ is finite if and only if $Ass\,(M)$ is finite.

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(M/N) \setminus \operatorname{Supp}(N)$. So there is a non zero element x of M such that $\mathfrak{p} = (N :_R x)$, so we have $\mathfrak{p}x \subseteq N$. Set $\sqrt{\operatorname{Ann}(\mathfrak{p}x)} = \bigcap \mathfrak{q}_i$. So that there exists a positive integer t such that $(\mathfrak{q}_1 \dots \mathfrak{q}_n)^t \mathfrak{p}x = 0$. Set $\mathfrak{q} = (\mathfrak{q}_1 \dots \mathfrak{q}_n)^t$. So that $\mathfrak{q}\mathfrak{p}x = 0$ and therefore $\mathfrak{p} \subseteq \operatorname{Ann}(\mathfrak{q}x) \subseteq (N :_R \mathfrak{q}x)$. Now let $a \in (N :_R \mathfrak{q}x)$, so that $a\mathfrak{q}x \subseteq N$ and therefore $a\mathfrak{q} \subseteq \mathfrak{p}$. If $a \notin \mathfrak{p}$ this means that $\mathfrak{q} \subseteq \mathfrak{p}$ and thus $\mathfrak{q}_i \subseteq \mathfrak{p}$ for some $1 \leq i \leq n$. Since $\mathfrak{q}_i \in \operatorname{Supp}(\mathfrak{p}x)$ and $\mathfrak{p}x \subseteq N$ we obtain $\mathfrak{p} \in \operatorname{Supp}(N)$ and this is a contradiction. It follows that $a \in \mathfrak{p}$ and $\mathfrak{p} = \operatorname{Ann}(\mathfrak{q}x)$, therefore $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{q}x)$ and hence $\mathfrak{p} \in \operatorname{Ass}(M)$, this ends the proof. □

In [11, Theorem 2.3] we proved the following:

Theorem 5.1.4. *Let M be a finitely generated R-module and $\mathfrak{a}$ be an ideal of R. Suppose that there is a positive integer n such that for each $i < n$ the set Supp $(H^i_{\mathfrak{a}}(M))$ is finite. Then Ass $(H^n_{\mathfrak{a}}(M))$ is finite.*

Proof. We will induct on n. If $n = 0$, there is nothing to prove. If $n = 1$, the statement follows easily by [**BL**, Theorem 2.2]. Assume inductively that $n > 1$ and the result settled for $i < n$. It is harmless to assume that M is $\mathfrak{a}$-torsion free R-module, note that there is an isomorphism $\mathrm{H}^i_{\mathfrak{a}}(M) \cong \mathrm{H}^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ for all $i \geq 1$. So there is an M-regular element $x \in \mathfrak{a}$. Now the exact sequence $0 \to M \xrightarrow{x} M \to M/xM = \overline{M} \to 0$ induces the long exact sequence

$$\cdots \to \mathrm{H}^{n-1}_{\mathfrak{a}}(M) \xrightarrow{x} \mathrm{H}^{n-1}_{\mathfrak{a}}(M) \xrightarrow{g} \mathrm{H}^{n-1}_{\mathfrak{a}}(\overline{M}) \xrightarrow{f} \mathrm{H}^{n}_{\mathfrak{a}}(M) \to \cdots .$$

It can be seen that Supp $(\mathrm{H}^i_{\mathfrak{a}}(\overline{M}))$ is a finite set for all $i < n$. Using induction hypothesis Ass $(\mathrm{H}^{n-1}_{\mathfrak{a}}(\overline{M}))$ is finite. Furthermore note that Supp (Img) is a subset of Supp $(\mathrm{H}^{n-1}_{\mathfrak{a}}(M))$ which is finite by the hypothesis. Applying Proposition 5.1.3 to the exact sequence $0 \to \mathrm{Im}g \to \mathrm{H}^{n-1}_{\mathfrak{a}}(\overline{M}) \to \mathrm{Im}f \to 0$ we deduce that Ass (Imf) is finite. The result is immediately follows by noting that $\mathrm{Im}f = (0 :_{\mathrm{H}^n_{\mathfrak{a}}(M)} x)$, and using Lemma 5.1.1. □

From an example due to Katzman [42] it follows that the condition of finiteness of support cannot be weakened by finiteness of associated primes:

Example 5.1.5. Let K be an arbitrary field and consider the hypersurface

$$S = \frac{K[s,t,u,v,x,y]}{(su^2x^2 - (s+t)uxvy + tv^2y^2)}.$$

Katzman showed that the local cohomology module $\mathrm{H}^2_{(x,y)}(S)$ has infinitely many associated prime ideals. Furthermore, since the defining equation of this hypersurface factors, S is not an integral domain.

The following corollary is now immediate.

Corollary 5.1.6. *Suppose that Supp* $(H^i_{\mathfrak{a}}(M))$ *is finite for all* $i < n$ *and* N *is a submodule of* $H^n_{\mathfrak{a}}(M)$ *such that Ass* $(\mathrm{Ext}^1_R(R/\mathfrak{a}, N))$ *is finite, then Ass* $(H^n_{\mathfrak{a}}(M)/N)$ *is finite.*

Proof. The exact sequence

$$0 \to N \to \mathrm{H}^n_{\mathfrak{a}}(M) \to \mathrm{H}^n_{\mathfrak{a}}(M)/N \to 0$$

induces the long exact sequence

$$\cdots \to \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M)) \to \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M)/N) \to \mathrm{Ext}^1_R(R/\mathfrak{a}, N) \to \cdots .$$

Note that Ass $(\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M)))$ is finite by Lemma 5.1.1 and Ass $(\mathrm{Ext}^1_R(R/\mathfrak{a}, N))$ is finite by hypothesis, hence Ass $(\mathrm{H}^n_{\mathfrak{a}}(M)/N)$ is finite. □

Nhan [54, Proposition 5.5] proved the following result when the ring is local. Since Artinian modules have finite support, the following corollary is an immediate consequence of Theorem 5.1.4.

Corollary 5.1.7. *Let* M *be a finitely generated* R*-module and* $\mathfrak{a}$ *be an ideal of the ring* R*. Suppose that there is a positive integer* n *such that for all* $i < n$*,* $H^i_{\mathfrak{a}}(M)$ *is Artinian. Then Ass* $(H^n_{\mathfrak{a}}(M))$ *is finite.*

5.2 Minimax, Artinian and Reflexive local cohomology modules

In [30] Grothendieck conjectured the following:

For any ideal $\mathfrak{a}$ and any finitely generated R-module M, the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ is finitely generated for all j.

Although this conjecture is not true in general, cf. [33, Example 1], there are some attempts to show that for some non-negative integer t, the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is finite. For example as we mentioned in Section 5.1 in [5] Asadollahi, Khashyarmanesh and Salarian proved the following:

Let $\mathfrak{a}$ be an ideal of R and let M be a finite R-module. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is a finite R-module for all $i < t$, then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is finite. The above result implies that the set of associated primes of the module $\mathrm{H}^t_{\mathfrak{a}}(M)$ is finite; see also [14, 43].

Now it is natural to ask the following question.

Question 5.2.1. *Let $\mathfrak{a}$ be an ideal of R and let M be a finite R-module. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is an Artinian R-module for all $i < t$. Is the module* $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ *Artinian?*

In [12, Proposition 2.4] we gave a negative answer to this question and in fact in [12, Proposition 2.2] it is shown that there is a finite submodule N such that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))/N$ is Artinian. This result implies that the set of associated primes of $\mathrm{H}^t_{\mathfrak{a}}(M)$ is finite.

As we mentioned an R-module M is called minimax, if there is a finite submodule N

of M, such that M/N is Artinian, cf. [71]. The class of minimax modules includes all finite and all Artinian modules. Moreover it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules, cf. [59, 71, 72]. Obviously this class is strictly larger than the class of all finite modules and also Artinian modules, cf. [8, Theorem 12]. Keep in mind that a minimax R-module has only finitely many associated primes.

Lemma 5.2.2. *Let M be a minimax R-module and let $\mathfrak{a}$ be an ideal of R. Then M is $\mathfrak{a}$-torsion free if and only if $\mathfrak{a}$ contains an M-regular element.*

Proof. Follows immediately by the same proof as [15, Lemma 2.1.1]. □

In [12, Theorem 2.2] we proved the following:

Theorem 5.2.3. *Let $\mathfrak{a}$ be an ideal of R and let t be a non-negative integer. Let M be an R-module such that $\operatorname{Ext}^t_R(R/\mathfrak{a}, M)$ is a minimax R-module. If $H^i_{\mathfrak{a}}(M)$ is minimax for all $i < t$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is a minimax module. Furthermore, if L is a finite R-module such that $Supp(L) \subseteq V(\mathfrak{a})$, then $\operatorname{Hom}_R(L, H^t_{\mathfrak{a}}(M))$ is a minimax module.*

Proof. We use induction on t. If $t = 0$, then $\mathrm{H}^0_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(M)$ and $\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is equal to the minimax R-module $\operatorname{Hom}_R(R/\mathfrak{a}, M)$. So, the assertion holds.

Suppose that $t > 0$ and that the case $t - 1$ is settled. Since $\Gamma_{\mathfrak{a}}(M)$ is minimax, $\operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is minimax for all i. Now by using the exact sequence $0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M/\Gamma_{\mathfrak{a}}(M) \to 0$ we get that $\operatorname{Ext}^t_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is minimax. On the other hand $\mathrm{H}^0_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$ and $\mathrm{H}^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) \cong \mathrm{H}^i_{\mathfrak{a}}(M)$ for all $i > 0$. Thus we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Let E be an injective hull of M and put $N = E/M$. Then $\Gamma_{\mathfrak{a}}(E) = 0$ and $\operatorname{Hom}_R(R/\mathfrak{a}, E) = 0$. Consequently $\operatorname{Ext}^i_R(R/\mathfrak{a}, N) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, M)$ and $\mathrm{H}^i_{\mathfrak{a}}(N) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(M)$ for all $i \geq 0$.

Now the induction hypothesis yields that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{\mathfrak{a}}(N))$ is minimax and hence $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M))$ is minimax.

For the last assertion, since $\mathrm{Supp}\,(L) \subseteq \mathrm{V}(\mathfrak{a})$ by using Gruson's theorem there is a finite chain $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ such that L_i/L_{i-1} is a homomorphic image of finitely many copies of $R/\mathfrak{a}$ for all $i = 1, 2, \cdots, n$. By induction, we may immediately reduce to the case where $n = 1$. Therefore, there is a short exact sequence $0 \to K \to (R/\mathfrak{a})^m \to L \to 0$ for some positive integer m and R-module K. Now, the exact sequence $0 \to \mathrm{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M)) \to \mathrm{Hom}_R((R/\mathfrak{a})^m, \mathrm{H}^t_{\mathfrak{a}}(M))$ shows that the R-module $\mathrm{Hom}_R(L, \mathrm{H}^t_{\mathfrak{a}}(M))$ is minimax. □

This enable us to reach a local cohomology module which is not minimax.

Example 5.2.4. From the Example 5.1.5, let $S = K[s,t,u,v,x,y]/(su^2x^2 - (s+t)uxvy + tv^2y^2)$. Then $\mathrm{H}^1_{(x,y)}(S)$ is not minimax. Otherwise by Theorem 5.2.3 we conclude that $\mathrm{Ass}\,(\mathrm{Hom}_S(S/(x,y), \mathrm{H}^2_{(x,y)}(S))) = \mathrm{Ass}\,(\mathrm{H}^2_{(x,y)}(S))$ is finite which is impossible.

The following result is a generalization of [14, Proposition 2.1] and [43, Theorem B].

Corollary 5.2.5. *Let $\mathfrak{a}$ be an ideal of R and let M be a minimax R-module. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is minimax for all $i < t$. Let N be a submodule of $H^t_{\mathfrak{a}}(M)$ such that $\mathrm{Ext}^1_R(R/\mathfrak{a}, N)$ is minimax. Then $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)/N)$ is a minimax module. In particular, $H^t_{\mathfrak{a}}(M)/N$ has finitely many associated primes.*

Proof. Let N be a submodule of $\mathrm{H}^t_{\mathfrak{a}}(M)$ such that $\mathrm{Ext}^1_R(R/\mathfrak{a}, N)$ is minimax. The short exact sequence

$$0 \to N \to \mathrm{H}^t_{\mathfrak{a}}(M) \to \mathrm{H}^t_{\mathfrak{a}}(M)/N \to 0$$

induces the following exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)) \to \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N) \to \mathrm{Ext}^1_R(R/\mathfrak{a}, N).$$

Since the left hand (by Theorem 5.2.3) and the right hand are minimax, we have that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ is minimax. On the other hand $\mathrm{Supp}\,\mathrm{H}^t_{\mathfrak{a}}(M)/N \subseteq \mathrm{Supp}\,\mathrm{H}^t_{\mathfrak{a}}(M) \subseteq \mathrm{V}(\mathfrak{a})$ and $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)/N)$ has finitely many associated primes. Therefore the same holds for $\mathrm{H}^t_{\mathfrak{a}}(M)/N$. □

Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is Artinian for all $i < t$ and $\mathrm{H}^t_{\mathfrak{a}}(M)$ is not Artinian. The integer t is equal to the *filter depth*, $\text{f-depth}_{\mathfrak{a}}(M)$, of M in $\mathfrak{a}$, i.e., the length of a maximal filter regular sequence of M in $\mathfrak{a}$, cf. [52, Theorem 3.1]. In the following we show that for $s = \text{f-depth}_{\mathfrak{a}}(M)$ the module $\mathrm{Hom}(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is not Artinian but it is minimax.

Proposition 5.2.6. *Let $(R, \mathfrak{m})$ be a local ring. Let $\mathfrak{a}$ be an ideal of R and let M be a finitely generated R-module such that $\mathrm{Supp}\,(M/\mathfrak{a}M) \nsubseteq \{\mathfrak{m}\}$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M))$, where $s = \text{f-depth}_{\mathfrak{a}}(M)$, is not Artinian but is minimax.*

Proof. By [52, Theorem 3.1] the module $\mathrm{H}^s_{\mathfrak{a}}(M)$ is not Artinian and so by [52, Theorem 1.1] the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is not Artinian. Whereas, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is minimax by Theorem 5.2.3. □

Thanks to [25, 24] now we can state and prove the following interesting result for minimax local cohomology modules:

Theorem 5.2.7. *Let $\mathfrak{a}$ be an ideal of R and let M be an R-module. Let s be a non-negative integer. Then the following statements hold.*

(a) If $\mathrm{Ext}_R^{s-j}(R/\mathfrak{a}, H_{\mathfrak{a}}^j(M))$ *is minimax for all* $j < s$ *and* $\mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^s(M))$ *is a minimax R-module then* $\mathrm{Ext}_R^s(R/\mathfrak{a}, M)$ *is a minimax R-module.*

(b) If $\mathrm{Ext}_R^{s+1-j}(R/\mathfrak{a}, H_{\mathfrak{a}}^j(M))$ *is minimax for all* $j < s$ *and* $\mathrm{Ext}_R^s(R/\mathfrak{a}, M)$ *is a minimax R-module then* $\mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^s(M))$ *is a minimax R-module.*

In particular, if $\mathrm{Ext}_R^{t-j}(R/\mathfrak{a}, H_{\mathfrak{a}}^j(M))$ *is minimax for* $t = s, s+1$ *and for all* $j < s$*, then* $\mathrm{Ext}_R^s(R/\mathfrak{a}, M)$ *is minimax R-module if and only if* $\mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^s(M))$ *is minimax R-module.*

Proof. (a) We prove it by induction on s. For $s = 0$, the result follows from the equality $\mathrm{Hom}_R(R/\mathfrak{a}, M) = \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^0(M))$. Assume $s > 0$ and $s-1$ is settled. Assume that E is an injective hull of $M/\Gamma_{\mathfrak{a}}(M)$, and set $N = E/(M/\Gamma_{\mathfrak{a}}(M))$. For all $i \geq 0$, as $\mathrm{H}_{\mathfrak{a}}^i(E) = 0 = \mathrm{Ext}_R^i(R/\mathfrak{a}, E)$, we get the isomorphisms $\mathrm{Ext}_R^i(R/\mathfrak{a}, N) \cong \mathrm{Ext}_R^{i+1}(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ and

$$\mathrm{H}_{\mathfrak{a}}^i(N) \cong \mathrm{H}_{\mathfrak{a}}^{i+1}(M/\Gamma_{\mathfrak{a}}(M)) \cong \mathrm{H}_{\mathfrak{a}}^{i+1}(M).$$

Therefore

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{s-1}(N)) \cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^s(M))$$

is a minimax R-module. In addition, for all $j < s-1$ the modules

$$\mathrm{Ext}_R^{s-1-j}(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^j(N)) \cong \mathrm{Ext}_R^{s-1-j}(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{j+1}(M))$$

are minimax. Now, by induction hypothesis, $\mathrm{Ext}_R^{s-1}(R/\mathfrak{a}, N)$ is minimax. Thus $\mathrm{Ext}_R^s(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is minimax too. Consider the following exact sequence

$$\mathrm{Ext}_R^s(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \to \mathrm{Ext}_R^s(R/\mathfrak{a}, M) \to \mathrm{Ext}_R^s(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)).$$

Since $\mathrm{Ext}_R^s(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is minimax, we have that $\mathrm{Ext}_R^s(R/\mathfrak{a}, M)$ is minimax.

(b) We will again induct on s. For $s = 0$, the result is clear. Let $s > 0$ and $s-1$ is settled. Assume that E and N are as in the proof of (i). For any $0 \leq j < s-1$, we have

$$\operatorname{Ext}_R^{s-j}(R/\mathfrak{a}, \operatorname{H}_{\mathfrak{a}}^{j}(N)) \cong \operatorname{Ext}_R^{s-j}(R/\mathfrak{a}, \operatorname{H}_{\mathfrak{a}}^{j+1}(M)).$$

Consider the following exact sequence

$$\operatorname{Ext}_R^{s}(R/\mathfrak{a}, M) \to \operatorname{Ext}_R^{s}(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \to \operatorname{Ext}_R^{s+1}(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)).$$

Since $\operatorname{Ext}_R^{s}(R/\mathfrak{a}, M)$ is minimax (by assumption) and $\operatorname{Ext}_R^{s+1}(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ is minimax (by hypothesis), we have $\operatorname{Ext}_R^{s}(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is minimax and hence $\operatorname{Ext}_R^{s-1}(R/\mathfrak{a}, N)$ is minimax. This shows that $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{H}_{\mathfrak{a}}^{s-1}(N))$ is minimax by induction hypothesis. Now the assertion holds. □

Finally we study the local cohomology of generalized reflexive modules introduced by [8]. Suppose that E is the minimal injective cogenerator of the category of R-modules. An R-module M is called reflexive with respect to E if the canonical injection

$$M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, E), E))$$

is an isomorphism. It is well-known that an R-module M is reflexive (with respect to E) if and only if M is minimax and $R/\operatorname{Ann}(M)$ is a complete semilocal ring, cf. [8, Theorem 2]. Recall that if N is an arbitrary submodule of a module M, then M is reflexive if and only if both N and M/N are reflexive, cf. [8, Lemma 5]. Consequently, a finite direct sum of modules is reflexive if and only if each direct summand is reflexive. In [12, Theorem 2.5] we proved the following:

Theorem 5.2.8. *Let M be a finitely generated R-module such that M is reflexive with respect to a minimal injective cogenerator E in the category of R-modules.*

Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is a reflexive R-module for $i < t$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is reflexive. This implies that not only the set of associated primes of $H^t_{\mathfrak{a}}(M)$ is finite but also that the Bass numbers of $H^t_{\mathfrak{a}}(M)$ are finite.

Proof. We argue by induction on t. Set $t = 0$. By [70, Theorem 1.6], M is reflexive and so $\mathrm{H}^0_{\mathfrak{a}}(M)$ is reflexive. Suppose inductively $t > 0$. Inspired by the ideas of Lemma 5.2.2, we see that there exists $x \in \mathfrak{a} \setminus \mathbb{Z}(M)$. Consider the exact sequence $0 \to M \xrightarrow{.x} M \to M/xM \to 0$. From the induced exact sequence

$$\cdots \to \mathrm{H}^{i-1}_{\mathfrak{a}}(M) \to \mathrm{H}^{i-1}_{\mathfrak{a}}(M/xM) \to \mathrm{H}^i_{\mathfrak{a}}(M) \to \cdots ,$$

it follows that $\mathrm{H}^i_{\mathfrak{a}}(M/xM)$ is reflexive for all $i \le t-2$. Note that $R/\operatorname{Ann}(M/xM)$ is a quotient of $R/\operatorname{Ann}(M)$ and so is reflexive. Thus by induction $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{\mathfrak{a}}(M/xM))$ is reflexive. Now the exact sequence

$$\mathrm{H}^{t-1}_{\mathfrak{a}}(M) \xrightarrow{g} \mathrm{H}^{t-1}_{\mathfrak{a}}(M/xM) \xrightarrow{f} \mathrm{H}^t_{\mathfrak{a}}(M) \xrightarrow{.x} \mathrm{H}^t_{\mathfrak{a}}(M),$$

induces the following exact sequence

$$0 \to \operatorname{Im} g \to \mathrm{H}^{t-1}_{\mathfrak{a}}(M/xM) \to Imf \to 0.$$

So we have the following exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{\mathfrak{a}}(M/xM)) \xrightarrow{h} \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} f) \xrightarrow{k} \operatorname{Ext}^1_R(R/\mathfrak{a}, \operatorname{Im} g).$$

By using the facts that any subquotient of a reflexive module is again reflexive, and any finite direct sum of reflexive modules is reflexive, we obtain that $\operatorname{Ext}^1_R(R/\mathfrak{a}, \operatorname{Im} g)$ is reflexive. On the other hand $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{t-1}_{\mathfrak{a}}(M/xM))$ is reflexive. Thus $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} f)$ is reflexive. Now the assertion follows from the fact that

$$\operatorname{Hom}(R/\mathfrak{a}, \operatorname{Im} f) = \operatorname{Hom}(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)).$$

□

Corollary 5.2.9. *With the same assumption as Theorem 5.2.8, the Bass numbers of the R-module* $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ *are all finite.*

Proof. The assertion follows from [7, Lemma2]. □

We end this chapter, with the following question.

Question 5.2.10. *Let M be a finitely generated R-module. Grothendieck proved that, when R is a homomorphic image of a regular local ring, the least integer t such that $H^t_{\mathfrak{a}}(M)$ is not finitely generated, is*

$$\mathrm{Min}\{\mathrm{depth} M_{\mathfrak{p}} + \mathrm{ht}((\mathfrak{a}+\mathfrak{p})/\mathfrak{p}) | \mathfrak{p} \not\supseteq \mathfrak{a}\}.$$

In [52] Melkersson showed that, when $Supp\, M/\mathfrak{a}M \nsubseteq \{\mathfrak{m}\}$, the least integer t such that $H^t_{\mathfrak{a}}(M)$ is not Artinian is

$$\mathrm{Min}\{\mathrm{depth}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) | \mathfrak{p} \in Supp\, M/\mathfrak{a}M \setminus \{\mathfrak{m}\}\}.$$

Now it is natural to ask "What is the least integer t such that $H^t_{\mathfrak{a}}(M)$ is not minimax (resp. reflexive)?"

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Appendix A

Algorithms for our criterion for linear resolutions

In this appendix we develop some algorithms which support the theoretical aspects of Chapter 2. The programming in CoCoA (or any other Computer Algebra System) is then straightforward.

Let K be a field, $I = (f_1, \ldots, f_m)$ be a graded ideal of $S = K[x_1, \cdots, x_r]$ generated in a single degree, say d. Consider the Rees algebra of I

$$R(I) = \bigoplus_{j \geq 0} I^j t^j = S[f_1 t, \ldots, f_m t] \subseteq S[t].$$

Let $T = S[t_1, \cdots, t_m]$. Consider the natural surjective homomorphism of bigraded K-algebras $\varphi : T \longrightarrow R(I)$ with $\varphi(x_i) = x_i$ for $i = 1, \ldots, r$ and $\varphi(t_j) = f_j t$ for $j = 1, \ldots, m$. Furthermore, let $\deg(x_i) = (0, 1)$ and $\deg(t_j) = (1, 0)$. So we can easily write

- $R(I) = T/P$,
- $R(I)_{(k,j)} = (I^k)_{kd+j}$

One can simply note that P is a bigraded ideal of T. It is easy to see that based on the definition of φ

$$P = (t_i - uf_i : i = 1, \cdots, m) \cap K[\underline{\mathrm{x}}, \underline{\mathrm{t}}].$$

Note that this elimination is evidently very costly; see also [67, Proposition 1.5]. The algorithm for calculating P is given in the following:

Data: an equigenerated ideal I of S

Result: The associated ideal of Rees ring I, i.e., P

begin

$R \longleftarrow k[x_1, \ldots, x_r, t_1, \cdots, t_m, u]$

$I \longleftarrow IR$

`/*Gens, The standard function of CoCoA to give the generators of an ideal.*/`

$G \longleftarrow \mathrm{Gens}(I)$

`/*Elim, The standard function of CoCoA to elimination` u `in order to compute Ker(`φ`).*/`

`/*Len, The standard function of CoCoA to calculate length of an ideal.*/`

$P \longleftarrow \texttt{Elim}(u, \mathrm{Ideal}([t[i] - u * G[i] \mid i = 1, \cdots \texttt{Len}(G)]))$

return P

end

Algorithm 1: Algorithm for calculating P

Then we need some functions to calculate the Good and Bad parts (items having $\deg_x \leq$ or > 1) of the initial ideal of P with respect to some term order. For example start with the term order $\underline{\mathrm{x}} < \underline{\mathrm{t}}$.

```
Data: a monomial ideal U of T
Result: The Good and Bad parts of U, i.e., G and B
begin
    G ⟵ Ideal(m ∈ Gens(U) | deg_x(m) ≤ 1)
    B ⟵ Ideal(m ∈ Gens(U) | deg_x(m) > 1)
    return G, B
end
```

Algorithm 2: Algorithm for calculating Good and Bad parts of $\mathrm{in}(P)$

Obviously deg_x is dependent on the term ordering we start with. For example if we choose term order $\underline{\mathrm{x}} < \underline{\mathrm{t}}$, then $deg_x(p)$ of a term p, is the sum of the first r exponents of x_i, $1 \leq i \leq r$. The following simple algorithm will take care of the general situation:

Data: a term p of T, and a fixed term order $<$ on T

Result: $\deg_x(p), \deg_t(p)$

begin

$M \longleftarrow$ `Len`$(\underline{x}) = r$

$N \longleftarrow$ `Len`$(\underline{t}) = m$

$L \longleftarrow$ `Log`(p)

`/*Log, The standard function of CoCoA which gives the list of exponents of the leading term of a polynomial. For example, let` $F := x^3y^2z^5 + x^2y + xz^4$ `then Log(`F`)` $= [3, 2, 5]$`.*/`

if $\underline{x} < \underline{t}$ **then**

$$\deg_x(p) = \sum_{i=1}^{M} L[i]$$

$$\deg_t(p) = \sum_{i=M+1}^{M+N} L[i]$$

else

$$\deg_x(p) = \sum_{i=N+1}^{N+M} L[i]$$

$$\deg_t(p) = \sum_{i=1}^{N} L[i]$$

return $\deg_x(p), \deg_t(p)$

end

Algorithm 3: Algorithm for calculating x-degree and t-degree

With the assumptions and notation introduced in Theorem 2.1.5 assume that $B = (m_1, \cdots, m_p)$ and $\mathrm{bideg}(m_i) = (t_i, \geq 2)$. By $(t_i, \geq 2)$ we mean that the $\deg_x(m_i) \geq 2$. It is harmless to assume that $t_1 \leq \cdots \leq t_p$. If for all $i = 1, \cdots, p$ and all $\alpha \in \mathbb{N}^m$ with

$\mid \alpha \mid = t_p + 1 - t_i$ we have $\underline{t}^{\alpha} m_i \subseteq G$ then $I_{(k,\star)} = G_{(k,\star)}$ for all $k > t_p + 1$.

So we need to know the count of elements of B which does not satisfy the conditions of Theorem 2.1.5 (simply the condition $x - gen = 1$). For such reason we first provide an algorithm (Algorithm 4) to calculate the maximum t-degree of any subset of T. Now let $M := \max\{deg_t(b) \mid b \in B\}$ and

Data: a subset X of T

Result: $\max \deg_t(X)$

begin
    $MaxTDeg \longleftarrow 0$
    foreach x *in* X **do**
        if $deg_t(x) > MaxTDeg$ **then**
            $MaxTDeg := deg_t(x)$
        endif
    end
    return $MaxTDeg$
end

Algorithm 4: Algorithm for calculating maximum t-degree of a subset of T

$$N := \mid \{b \in B \mid \text{Ideal}(b)(\underline{t})^{M+1-deg_t(b)} \nsubseteq G\} \mid . \tag{A.1}$$

Then algorithm 5 will compute N for us.

If fortunately $N = 0$, we are done and from Theorem 2.1.5 we deduce the linear resolution of I^k for $k > M$. Otherwise having N in hand, we suggest the following two approaches; even most of the time, we use a combination of the two:

- Change order

```
Data: a monomial ideal U of T
Result: N in (A.1)
begin
    Counter ⟵ 0
                                        /*see Algorithm 2*/
    B ⟵ BadParts(U)
    G ⟵ GoodParts(U)
                                        /*see Algorithm 4*/
    M ⟵ MaxTDeg(B)
    foreach b in B do
        P_b ⟵ M + 1 − deg_t(b)
        W_b ⟵ Ideal(b)(t^{P_b})
        if W_b ⊈ G then
            Counter=Counter+1
        endif
    end
    return Counter
end
```

Algorithm 5: Algorithm for the condition $x - gen = 1$.

- Switch to a sparse upper triangular bi-change of coordinates

Note that if $\mid N \mid$ is large enough or more precisely when $\frac{|N|}{|B|}$ is almost 1, we are advised to do the change ordering. Of course, this is better recognized by experience since it is quite dependent on the degree of elements of P, the ideal associated to Rees ring of I and its initial ideal, in(P). That is if the large powers of P are more concentrating on t's than x's, it is a good idea to choose the term order $\underline{\text{t}} < \underline{\text{x}}$. But if it didn't help yet (which is true most of the time), we use the trick mentioned in Theorem 2.1.5. That is start the whole story for $g(P)$ instead of P, where g is a bi-homogenous isomorphism on $K[\underline{\text{x}}, \underline{\text{t}}]$. One could try to take g "generic", as in (A.2).

$$\begin{aligned} g &:= g_1 \times g_2, \\ g_1 &:= x_i \longmapsto \text{Random}(\text{Sum}(x_1, \cdots, x_6)), \\ g_2 &:= t_j \longmapsto \text{Random}(\text{Sum}(t_1, \cdots, t_{10})), \end{aligned} \tag{A.2}$$

for all $i = 1, \cdots, 6$ and all $j = 1, \cdots, 10$, where by Random(Sum$(x_1, \cdots, x_6)$) we mean a linear combination of $x_1, \cdots, x_6$ with random coefficients and the same interpretation for $t_1, \cdots, t_{10}$. But we realized that a properly chosen sparse random upper triangular g does the job as well. Hence we suggest to use the following algorithm to generate **a sparse random upper triangular bi-change of coordinates**. We choose the term order based on our experiences that we gained so far.

`/*Rand(x,y) returns a random integer between x and y*/`

Data: P of Rees ring $R(I)$

Result: a sparse upper triangular random bi-change of coordinates g

begin

`/*the degree of sparsity, here 5 times product of random 0 and 1*/`

$DS \longleftarrow 5$

for $i \leftarrow 1$ **to** r **do**

$$X_i \longleftarrow x_i + \sum_{j=1}^{i-1} (\prod_{k=1}^{DS} \texttt{Rand}(0,1) x_j)$$

end

for $i \leftarrow 1$ **to** m **do**

$$T_i \longleftarrow t_i + \sum_{j=1}^{i-1} (\prod_{k=1}^{DS} \texttt{Rand}(0,1) t_j)$$

end

if $(X_1, \cdots, X_r, T_1, \cdots, T_m) = (x_1, \cdots, x_r, t_1, \cdots, t_m)$ **then**

$g := x_1 \mapsto X_1, \cdots, x_r \mapsto X_r, t_1 \mapsto T_1, \cdots, t_m \mapsto T_m$

return g

else

Generate again

endif

end

Algorithm 6: Algorithm for generating a sparse random upper triangular bi-change of coordinates.

After finding such g, we apply the standard CoCoA function Subst, which substitutes values for indeterminates. Then we apply our algorithm for the condition $x - gen = 1$ (Algorithm 5) to in($g(P)$). The following simple algorithm is seeking for such g until we obtain one for which N in (A.1) is zero:

`/*MainFnc(); the function in algorithm 5*/`

`/*CalcP(); the function in algorithm 1*/`

`/*Randgen(); the function in algorithm 6*/`

Data: I an equigenerated ideal I of S

Result: a bi-transformation g for which Theorem 2.1.5 is true

begin

  $P \longleftarrow$ `CalcP`(I)

  $C \longleftarrow$ `MainFnc`(in(P))

  repeat

    $g \longleftarrow$ `Randgen`()

    $C \longleftarrow$ `MainFnc`(in($g(P)$))

  until $C = 0$

end

Algorithm 7: Algorithm for seeking a desired g.