Chapter 1

Introduction

The main objects of study in this thesis are local cohomology modules. We write $H^i_{\mathfrak{a}}(M)$ for the ith local cohomology of a module M with respect to some ideal $\mathfrak{a}$. We refer the reader to see [Br-Sh], [Gr2], [Sch4], [Hun2] and [Eis] as suitable sources to study local cohomology and related subjects. Let us first introduce the subject and main problems. After this we will present some known related results and finally we will give a summary of the results obtained in this work.

Local cohomology was introduced by Grothendieck [Gr], in the early 1960s, in part to answer the following conjecture of Pierre Samuel:

Conjecture 1.0.1. *Let R be a Noetherian local ring and $\widehat{R}$ its completion with respect to the maximal ideal. If $\widehat{R}$ is a complete intersection and for each prime ideal P of R of height ≤ 3, R_P is a UFD, then R is a UFD.*

Among many other attributes, local cohomology allows one to answer many seemingly difficult questions. A good example of such a problem, where local cohomology provides a partial answer, is the question of how many generators ideals have up to radical. In general, if $\mathfrak{b}$ is an ideal of a ring R, the radical of $\mathfrak{b}$ is the ideal

$$\operatorname{Rad}\mathfrak{b} = \{x \in R : x^m \in \mathfrak{b} \text{ for some } m\}.$$

We say an ideal $\mathfrak{b}$ is generated up to radical by n elements if there exist $x_1, ..., x_n \in \mathfrak{b}$ such that $\operatorname{Rad}(\mathfrak{b}) = \operatorname{Rad}(x_1, ..., x_n)$. For example, the ideal $\mathfrak{b} \subseteq k[x, y]$ generated by x^2, xy, y^2 is generated up to radical by the two elements x, y. Recall that the radical of an ideal $\mathfrak{a}$ is the intersection of all primes ideals which contain $\mathfrak{a}$.

Given an ideal $\mathfrak{a}$ what is the least number of elements needed to generate it up to radical? A particular example of this problem is the following: let $R = k[x,y,u,v]$ be a polynomial ring in four variables over the field k. Consider the ideal $\mathfrak{a} = (xu, xv, yu, yv)$. This ideal is its own radical, i.e. $\mathfrak{a} = \text{Rad}(\mathfrak{a})$. The four given generators of $\mathfrak{a}$ are minimal. On the other hand, it can be generated up to radical, by the three elements $xu, yv, xv + yu$. This holds since $(xv)^2 = xv(xv + yu) - (xu)(yv) \in (xu, yv, xv + yu)$. Are there two elements which generate it up to radical? Could there even be one element which generates $\mathfrak{a}$ up to radical?

The answer to the last question is no, there cannot be just one element generating the ideal $\mathfrak{a}$ up to radical, due to an obstruction first proved by Krull, namely the height of the ideal. Krull's famous height theorem states:

Theorem 1.0.2. *(Generalized principal ideal Theorem) Let R be a Noetherian ring and $\mathfrak{a} = (x_1, ..., x_n)$ be an ideal generated by n elements. If $\mathfrak{p}$ is a minimal prime over $\mathfrak{a}$, then the height of $\mathfrak{p}$ is at most n. In particular, if an ideal $\mathfrak{a}$ is generated up to radical by n elements, then the height of $\mathfrak{a}$ is at most n.*

In the example we are considering, the height of $\mathfrak{a}$ is two as it is the intersection of the two height two ideals (x,y) and (u,v). Krull's height theorem implies that two is the smallest number of polynomials which could generate $\mathfrak{a}$ up to radical. This still begs the question, are there two polynomials $F, G \in \mathfrak{a}$ such that $\text{Rad}(F,G) = \mathfrak{a}$?

Trying to find two such polynomials F, G by some type of random search would be hard, if not impossible. Of course if there are no such polynomials, no search would find them, but even if two such polynomials do exist, it is likely no random search would find them. The problem is that these polynomials would normally be extremely special, so that writing down general polynomials in $\mathfrak{a}$ would not work. Instead, we would like to find, in some cohomology theory, an obstruction to being generated up to radical by two elements. Local cohomology provides such an obstruction. To a ring R and ideal $\mathfrak{b}$, we will associate for $i \geq 0$ modules $H^i_{\mathfrak{b}}(R)$ with the properties that

(1) $H^i_{\mathfrak{b}}(R) = H^i_{\text{Rad}(\mathfrak{b})}(R)$, and

(2) if $\mathfrak{b}$ is generated by k-elements, then $H^i_{\mathfrak{b}}(R) = 0$ for all $i > k$.

Finally, for $\mathfrak{a} = (xu, xv, yu, yv)$, we will prove that $H^3_{\mathfrak{a}}(R) \neq 0$, and therefore $\mathfrak{a}$ cannot be generated up to radical by two elements.

Item (2), is one of the most powerful tools in local cohomology. In a view of above notes, we would like to extend the description to the above question to this question that how many equations it takes to define an algebraic set X set-theoretically over an algebraically closed field. Of course X can be defined by n equations if and only if there is an ideal $\mathfrak{c}$ with n generators, having the same radical as $I(X)$, the ideal of X. Since the local cohomology $H^i_{\mathfrak{a}}(M)$ depends only on the radical of $\mathfrak{a}$, we would have $H^i_{I(X)}(M) = H^i_{\mathfrak{c}}(M) = 0$ for all $i > n$ and all modules M. See [Schm-Vog] and [St-Vog] for some examples where this technique is used, and [Lyu] for a recent survey including many pointers to the literature.

For an R-module M and an ideal $\mathfrak{a}$, consider the family of local cohomology modules $\{H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)\}_{n\in\mathbb{N}}$. For every n there is a natural homomorphism $H^i_{\mathfrak{m}}(M/\mathfrak{a}^{n+1}M) \to H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ such that the family forms a projective system. The projective limit $\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of M with respect to $\mathfrak{a}$. Formal local cohomolgy modules used by Peskine and Szpiro in [Pes-Szp] when R is a regular ring. Recently Schenzel [Sch] has defined formal local cohomology modules for a local ring $(R, \mathfrak{m})$ and a finitely generated R-module M. For more information see chapter three.

1.1 Objectives and conclusions

In the sequel, we are going to introduce the considered problems and results in this work:

• In Chapter 2 we introduce the definitions and notations will be used throughout this work.

At first we give the definition of local cohomology modules in conjunction with some of their properties. Next the concept of colocalization which is introduced by A. Richardson will be considered. Richardson's definition has a great advantage in contrast to the previous definitions, i.e. it preserves Artinian modules through the colocalization.

• Important problems concerning local cohomology modules are vanishing, finiteness and Artinianness results (e.g. [Hun]). Not so much is known about the Artinianness of formal local cohomology modules. In [Asgh-Divan] Asgharzadeh and Divani-Aazar have investigated some properties of these kind of modules. For instance they showed that $\mathfrak{F}_{\mathfrak{a}}^{\dim M}(M)$ is Artinian ([Asgh-Divan, Lemma 2.2]), but $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is not Artinian in general, at $i = \operatorname{fgrade}(\mathfrak{a}, M)$ and $i = \dim M/\mathfrak{a}M$ where they are the first respectively last non-zero amount of formal local cohomology modules (cf. [Asgh-Divan, Theorem 2.7]). We pursue this line to find out conditions for Artinianness of formal local cohomology modules. As a main result in section 3.2 we have following Theorem:

Theorem 1.1.1. *(cf. Theorem 3.2.4) Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. For given integers i and $t > 0$, the following statements are equivalent:*

(1) $\operatorname{Supp}_{\widehat{R}}(\mathfrak{F}_{\mathfrak{a}}^{i}(M)) \subseteq V(\mathfrak{m}\widehat{R})$ *for all $i < t$;*

(2) $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ *is Artinian for all $i < t$;*

(3) $\operatorname{Supp}_{\widehat{R}}(\mathfrak{F}_{\mathfrak{a}}^{i}(M)) \subseteq V(\mathfrak{a}\widehat{R})$ *for all $i < t$.*

(4) $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^{i}(M)))$ *for all $i < t$;*

Suppose that $t \leq \operatorname{depth} M$, then the above conditions are equivalent to

(5) $\mathfrak{F}_{\mathfrak{a}}^{i}(M) = 0$ *for all $i < t$;*

where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R.

This Theorem can be considered as the dual to the Faltings' finiteness Theorem (cf. [Br-Sh, Theorem 9.1.2]) for formal local cohomology modules.

- For an R-module M, $\operatorname{Cosupp}_R(M) \subseteq V(\operatorname{Ann}_R M)$, for definition of cosupport of a module, see chapter three. When M is representable, then $\operatorname{Cosupp}_R(M) = V(\operatorname{Ann}_R M)$ (cf. Theorem 3.3.2). Of a particular interest is to see when the cosupport of formal local cohomology module is a closed subset of $\operatorname{Spec} R$ in the Zariski topology. More precisely in order to show that $\operatorname{Cosupp}(\mathfrak{F}_{\mathfrak{a}}^{i}(M))$ being closed, it is enough to show that $\operatorname{Coass}(\mathfrak{F}_{\mathfrak{a}}^{i}(M))$ is finite (cf. Lemma 3.4.5), so it has encouraged us to consider the $\operatorname{Coass}(\mathfrak{F}_{\mathfrak{a}}^{i}(M))$ extensively.
- Of a particular interest are the first non-vanishing (resp. the last non-vanishing) cohomological degree of the local cohomology modules $H_{\mathfrak{a}}^{i}(M)$,

known as the grade $\operatorname{grade}(\mathfrak{a}, M)$ (resp. cohomological dimension $\operatorname{cd}(\mathfrak{a}, M)$). It is a well-known fact that

$$\operatorname{grade}(\mathfrak{a}, M) \leq \operatorname{cd}(\mathfrak{a}, M) \leq \dim M.$$

In the case of $\mathfrak{a} = \mathfrak{m}$ it follows that $\operatorname{cd}(\mathfrak{m}, M) = \dim M$. While for an arbitrary ideal $\mathfrak{a} \subset R$ the Hartshorne-Lichtenbaum Vanishing Theorem says that

$$H^d_{\mathfrak{a}}(M) = 0 \Longleftrightarrow \dim \widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{p} > 0 \text{ for all } \mathfrak{p} \in \operatorname{Ass}_{\widehat{R}} \widehat{M} \text{ such that } \dim \widehat{R}/\mathfrak{p} = d,$$

$d = \dim M$ (see [Hart] and [Br-Sh]). Here $\widehat{M}$ resp. $\widehat{R}$ denotes the completion of M resp. R. When $H^d_{\mathfrak{a}}(M) \neq 0$ one of the most important views concerning this is to express $H^d_{\mathfrak{a}}(M)$ via $H^d_{\mathfrak{m}}(M)$. More precisely the kernel of the natural epimorphism $H^{\dim M}_{\mathfrak{m}}(M) \to H^{\dim M}_{\mathfrak{a}}(M)$ has been calculated explicitly.

Theorem 1.1.2. *(cf. Theorem 4.1.5 and Corollary 4.1.7) Let $\mathfrak{a}$ denote an ideal of a local ring $(R, \mathfrak{m})$. Let M be a finitely generated R-module and $d = \dim M$. Then there is a natural isomorphism*

$$H^d_{\mathfrak{a}}(M) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{M}/Q_{\mathfrak{a}\widehat{R}}(\widehat{M})) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{M}/P_{\mathfrak{a}}(\widehat{M})\widehat{M}),$$

where $Q_{\mathfrak{a}\widehat{R}}(\widehat{M})$ is a certain submodule of $\widehat{M}$ (cf. 4.1.3) and $P_{\mathfrak{a}}(\widehat{M}) \subseteq \widehat{R}$ is the ideal as defined in 4.1.6.

The above results lead us to establish some properties of $\operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$. First of all a brief about endomorphism rings could be instrumental for understanding the content.

One can often translate properties of an object into properties of its endomorphism ring. For instance, a module is indecomposable if and only if its endomorphism ring does not contain any non-trivial idempotents (cf. [Jacob]). Note that a module M is decomposable if $M = M_1 \oplus M_2$ where $M_i \neq 0$ for $i = 1, 2$ are submodules of M. Otherwise M is indecomposable.

Not so much is known about the ring $\operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ and its relation to a given ring R. In Theorem 4.2.2, for a local ring $(R, \mathfrak{m})$ and its $\mathfrak{m}$-adic completion $\widehat{R}$, we show that the map

$$\Phi : \widehat{R} \to \operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$$

is an isomorphism if and only if $Q_{\mathfrak{a}\widehat{R}}(\widehat{R}) = 0$ and $\widehat{R}/Q_{\mathfrak{a}\widehat{R}}(\widehat{R})$ satisfies Serre's condition S_2 (for more details see section two of chapter 4). Furthermore we show that $\operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ is a finitely generated $\widehat{R}$-module and it is a commutative semi-local Noetherian ring (cf. Theorem 4.2.2(3),(4)).

• In Chapter 5, we give some connectedness Theorems. Let R be a commutative ring. The spectrum of R, denoted by $\operatorname{Spec}(R)$, is the topological space consisting of all prime ideals of R, with topology defined by the closed sets $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$, for each ideal $\mathfrak{a}$ of R. This topology is called the Zariski topology. Clearly if R is nonzero, then $\operatorname{Spec} R$ is non-empty. $\operatorname{Spec} R$ enjoys very nice properties. For instance it is compact and moreover it is irreducible if and only if its nilradical is a prime ideal (a topological space X is irreducible if it cannot be written as a union of two closed proper subsets A, B of X). However $\operatorname{Spec}(R)$ is not a connected space in general. It is known that for a local ring R, $\operatorname{Spec} R$ is connected. More generally $\operatorname{Spec} R$ is disconnected if and only if R contains a non-trivial idempotents element. The concept of a topological space being connected in codimension k ($\in \mathbb{N} \cup \{0\}$) was made precise by Hartshorne [Hart2]. For definitions and more details see also chapter 5.

Next we recall a definition given by Hochster and Huneke (see [Hoch-Hun, (3.4)]).

Definition 1.1.3. *Let $(R, \mathfrak{m})$ denote a local ring. We denote by $\mathbb{G}(R)$ the undirected graph whose vertices are primes $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R = \dim R/\mathfrak{p}$, and two distinct vertices $\mathfrak{p}, \mathfrak{q}$ are joined by an edge if and only if $(\mathfrak{p}, \mathfrak{q})$ is an ideal of height one.*

We extend a classical result of Hochster-Huneke to an arbitrary ideal $\mathfrak{a}$ of R as follows:

Theorem 1.1.4. *(cf. Theorem 5.2.5) Let $(R, \mathfrak{m})$ denote a complete local ring and $d = \dim R$. For an ideal $\mathfrak{a} \subset R$ the following conditions are equivalent:*

(1) $H^d_{\mathfrak{a}}(R)$ is indecomposable.

(2) $\operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), E(R/\mathfrak{m}))$ is indecomposable.

(3) The endomorphism ring of $H^d_{\mathfrak{a}}(R)$ is a local ring.

(4) The graph $\mathbb{G}(R/Q_{\mathfrak{a}}(R))$ is connected,

for the definition of $Q_{\mathfrak{a}}(R)$, see 4.1.3.

• In chapter 6 at first we give some results on the attached prime ideals of local cohomology via colocalization. Next we give a short simple proof to the Sharp's asymptotic prime divisor. Let R be a commutative ring and $\mathfrak{a}$ an ideal of R. For every Artinian R-module A, $\mathrm{Att}(0 :_A \mathfrak{a}^n)$ and $\mathrm{Att}(0 :_A \mathfrak{a}^n / 0 :_A \mathfrak{a}^{n-1})$ are ultimately constant and $At(\mathfrak{a}, A)$ and $Bt(\mathfrak{a}, A)$ denote their ultimate constant values (cf. [Sh2]). In [Sh1], Sharp showed that

$$At(\mathfrak{a}, A) \setminus Bt(\mathfrak{a}, A) \subseteq \mathrm{Att}_R(A)$$

for every Artinian module A, by generalization of Heinzer-Lantz Theorem. Schenzel [Sch2] has given an alternative proof for mentioned Theorem in case that for a local ring $(R, \mathfrak{m})$, if $\mathfrak{m} \in At_R(\mathfrak{a}, A) \setminus Bt_R(\mathfrak{a}, A)$, then $\mathfrak{m} \in \mathrm{Att}_R A \cap V(\mathfrak{a})$, where $V(\mathfrak{a})$ is the set of prime ideals of R containing ideal $\mathfrak{a}$. Then we give a short simple proof for Sharp's Theorem using the concept of colocalization introduced by Richardson [Rich], (cf. Theorem 6.0.8).

Note on references: Some of the materials in this Thesis have been submitted elsewhere. Some of the results have been appeared in [E].