Abstract

In this thesis, the class of modules whose Cousin complexes have finitely generated cohomologies are studied as a subclass of modules which have uniform local cohomological annihilators and it is shown that these two classes coincide over local rings with Cohen-Macaulay formal fibres. This point of view enables us to obtain some properties of modules with finite Cousin complexes and find some characterizations of them.

In this connection we discuss attached prime ideals of certain local cohomology modules in terms of cohomologies of Cousin complexes. In continuation, we study the top local cohomology modules with specified set of attached primes.

Our approach to study Cousin complexes leads us to characterization of generalized Cohen-Macaulay modules in terms of uniform annihilators of local cohomology. We use these results to study the Cohen-Macaulay loci of modules and find two classes of rings over which the Cohen-Macaulay locus of any finitely generated module is a Zariski–open subset of the spectrum of the ring.

Key words and phrases. Cousin complexes, uniform local cohomological annihilator, Cohen-Macaulay locus, local cohomology, attached primes.

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Introduction

Many concepts in commutative algebra are inspired by algebraic geometric objects. Of particular interest and effective tool in this thesis, is the Cousin complex of a module which is algebraic analogue of the Cousin complex introduced in 1963/64 by A. Grothendieck and R. Hartshorne [17, Chapter IV]. They used this notion to prove a duality theorem for cohomology of quasi–coherent sheaves, with respect to a proper morphism of locally noetherian preschemes.

In 1969, R. Y. Sharp presented the commutative algebraic analogue of the Cousin complex (see section 1.3) and approved it as a powerful tool by characterizing Cohen-Macaulay and Gorenstein rings in terms of Cousin complexes [29]. This concept is developed in [35] by Sharp and is discussed by S. Goto and K. Watanabe in the $\mathbb{Z}$–graded context [15]. In [29], Sharp shows that a commutative noetherian ring R is Cohen-Macaulay if and only if the Cousin complex $\mathcal{C}_R(R)$ of R is exact, which is improved to modules by himself in [30], while R is Gorenstein if and only if $\mathcal{C}_R(R)$ provides the minimal injective resolution of R. He also introduced Gorenstein modules and characterized them by using Cousin complexes in [30].

From the Cousin complex definition is apparent that it terms are very much look like non–finitely generated, and despite of it, R is Cohen-Macaulay if and only if $\mathcal{C}_R(R)$ is exact, i.e. its cohomologies are zero and so finitely generated. Now, one may ask what rings or modules admit finitely generated Cousin complex cohomologies and what properties these rings or modules have.

In 2001, M. T. Dibaei and M. Tousi, while studying the structure of dualizing complexes, found a class of modules whose Cousin complexes have finitely generated cohomologies. The theory of dualizing complexes comes also from algebraic geometry which was discussed firstly by Grothendieck and Hartshorne in 1963/64 and used to prove their duality theorem [17, Chapter V]. Afterwards Sharp and a number of authors studied its commutative algebraic analogue and found it as a useful tool.

For the rest of this section, R is a commutative noetherian ring and M is a finitely generated R–module.

A dualizing complex for a ring R is a bounded injective complex $I^\bullet$, where all cohomology modules $\mathrm{H}^i(I^\bullet)$ are finitely generated R–modules and the natural map $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, I^\bullet), I^\bullet)$ is quasi isomorphism for any finitely generated R–module M.

A dualizing complex $I^\bullet$ is said to be fundamental whenever $\oplus_{i\in\mathbb{Z}}I^i \cong \oplus_{\mathfrak{p}\in\operatorname{Spec} R}E(R/\mathfrak{p})$, where $E(R/\mathfrak{p})$ is the injective envelope of $R/\mathfrak{p}$ as R–module, i.e. each prime ideal of R occurs in exactly one term of $I^\bullet$ and exactly once [34, 1.1]. It is known that a ring R possesses a dualizing complex if and only if it possesses a fundamental dualizing complex (see [16, 3.6] and [34, 1.2]), which is unique up to isomorphism of complexes and shifting (see [32, 4.5] and [16, 4.2]).

Now, a natural and interesting treatment is to determine this unique complex. In 1998, Dibaei and Tousi described that if a local ring R which satisfies the condition (S_2), possesses a dualizing complex, then the fundamental dualizing complex for R is isomorphic to the Cousin complex of the canonical module of R with respect to the height filtration (which is equal to the dimension filtration in this case)[8, 2.4]. As an application they proved that if a local ring R satisfies the condition (S_2) and has a canonical module K, then finiteness of cohomologies of the Cousin complex of K with respect to a certain filtration is necessary and sufficient condition for R to possess a dualizing complex [8, 3.4]. In 2001, they generalized their structural property of dualizing complex of [8] and showed that the Cousin cohomologies of M over a local ring R, are finitely generated if R has a dualizing complex and M is equidimensional which satisfies the condition (S_2), in [9]. In continuation of [8] and [9], Dibaei studied some properties of Cousin complexes through the dualizing complexes in 2005, and proved the following result.

Theorem 1. [4, Theorem 2.1] Assume that all formal fibres of R are Cohen-Macaulay and M satisfies (S_2). If $\widehat{M}$ is equidimensional, then $\mathcal{C}_R(M)$ has finitely generated cohomology modules.

These ideas have been pursued in algebraic geometry by J. Lipman, S. Nayak and P. Sastry in [21]. Taking motivation from [8] and [9], Kawasaki studied Cousin complex of a module over a noetherian ring and improved results, independently from [4], in [20]. More precisely, he proved the following results.

Theorem 2. [20, Theorem 1.1] Assume that M is equidimensional and

(i) R is universally catenary,

(ii) all the formal fibers of all the localizations of R are Cohen-Macaulay,

(iii) the Cohen-Macaulay locus of each finitely generated R–algebra is open,

Then all the cohomology modules of the Cousin complex of M are finitely generated and only finitely many of them are non–zero.

The assumptions of the above result are also necessary in a sense.

Theorem 3. [20, Theorem 1.4] Let R be a catenary ring. Then the following statements are equivalent.

(i) R satisfies the conditions (i), (ii) and (iii) of Theorem 2.

(ii) for any finitely generated equidimensional R–module M, all the cohomology modules of the Cousin complex of M are finitely generated and only finitely many of them are non–zero.

In special case when R is local, Kawasaki obtains a more simple but interesting version of his result, Theorem 2, as the following.

Theorem 4. [20, Theorem 5.5] Assume that R is a universally catenary local ring and M is an equidimensional R–module. If all formal fibres of R are Cohen-Macaulay, then all the cohomology modules of $\mathcal{C}_R(M)$ are finitely generated.

Note that if a local ring R is universally catenary, then $\widehat{M}$ is equidimensional for each finitely generated equidimensional R–module M. In the proof of the above theorem, the assumption that R is universally catenary, is used to show that $\widehat{M}$ is equidimensional. So one may consider this theorem as a generalization of Theorem 1.

After reviewing some well known results and basic concepts which we need throughout the thesis in Chapter 1, we start our study on Cousin complexes by discussing some useful techniques and essential properties of cohomology modules of Cousin complexes in the first section of Chapter 2. As a consequence we remove the condition (S_2) from Theorem 1 and recover Theorem 4, in Corollary 2.1.6 and Proposition 2.3.2.

In all above results about finiteness of the Cousin complex of an R–module M, there are some crucial common conditions on R and M:

(a) M is equidimensional;

(b) R is universally catenary;

(c) all formal fibres of R are Cohen-Macaulay.

When R is a local ring, these conditions are sufficient for finiteness of cohomology modules of $\mathcal{C}_R(M)$ by Theorem 4, and conditions (b) and (c) are necessary for finiteness of cohomology modules of Cousin complexes of all equidimensional R–modules by Theorem 3. It is now natural to ask that which of these conditions are satisfied if $\mathcal{C}_R(M)$ has finitely generated cohomology modules for an R–module M.

In 2006, C. Zhou studied the properties of noetherian rings containing uniform local cohomological annihilators and showed that all such rings are universally catenary and locally equidimensional[37]. Recall that an element $x \in R$ is called a uniform local cohomological annihilator of M, if $x \in R \setminus \cup_{\mathfrak{p} \in \operatorname{Min} M} \mathfrak{p}$ and for each maximal ideal $\mathfrak{m}$ of R, $x\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for all $i < \dim M_{\mathfrak{m}}$.

We continue Chapter 2, improving some results of Zhou for modules which have uniform local cohomological annihilators and find some characterizations of these modules in Section 2.2. We show that if a finitely generated R–module M has a uniform local cohomological annihilator, then M is locally equidimensional and $R/0:_R M$ is universally catenary in Proposition 2.2.2 and Corollary 2.2.6. We also obtain that the property that M has a uniform local cohomological annihilator is independent of the module structure and depends only on the support of M in Corollary 2.2.5. Finally we investigate our main result in this section by proving that if $\mathcal{C}_R(M)$ has finitely generated cohomology modules, then M has a uniform local cohomological annihilator in Theorem 2.2.13 and so M is equidimensional and $R/0:_R M$ is universally catenary.

Our approach in studying Cousin complexes is also useful for discussing about uniform local cohomological annihilators and helps us to recover some results in this context by a different and may be simple method, for instance see Corollary 2.2.9 and Proposition 2.2.11.

This point of view, also enables us to characterize modules with finite Cousin cohomologies over a local ring R with Cohen-Macaulay formal fibres. The last section of Chapter 2 is devoted to some applications of our approach. In Theorem 2.3.3, we show that over these rings, $\mathcal{C}_R(M)$ has finitely generated cohomologies if and only if M has a uniform local cohomological annihilator, if and only if $\widehat{M}$ is equidimensional $\widehat{R}$–module. Our results about the annihilators of cohomologies of Cousin complexes in Section 2.1, lead us to present the height of an ideal of R in terms of Cousin complex in Theorem 2.3.5.

Another important subject which is strongly related to the uniform annihilators of local cohomology, is the notion $\mathrm{a}(M)$ which is defined for a finitely generated R–module M over a local ring $(R,\mathfrak{m})$ as $\mathrm{a}(M) = \bigcap_{i<\dim M}(0:_R \mathrm{H}^i_{\mathfrak{m}}(M))$. Note that by definition, an R–module M has a uniform local cohomological annihilator if and only if $\mathrm{a}(M) \nsubseteq \cup_{\mathfrak{p}\in \mathrm{Min}\, M}\mathfrak{p}$. On the other hand if $\mathfrak{p} \in \mathrm{Min}\, M$, then $\mathrm{a}(M) \subseteq \mathfrak{p}$ if and only if $\mathfrak{p} \in \mathrm{Att}\,\mathrm{H}^i_{\mathfrak{m}}(M)$ for some $i < \dim M$ (see Lemma 2.2.7). These facts motivate us to study the relations between local cohomology modules and Cousin complexes.

It is well known that for a finitely generated R–module M with finite dimension $d = \dim M$, $\mathrm{Att}\,\mathrm{H}^d_{\mathfrak{m}}(M) = \mathrm{Assh}\, M$ (see Theorem 1.2.4), we start Chapter 3 by discussing $\mathrm{Att}\,\mathrm{H}^t_{\mathfrak{m}}(M)$ for certain t, in particular $\mathrm{Att}\,\mathrm{H}^{d-1}_{\mathfrak{m}}(M)$, in terms of cohomologies of $\mathcal{C}_R(M)$. As a consequence we find a non–vanishing criterion of $\mathrm{H}^{d-1}_{\mathfrak{m}}(M)$ when $\mathcal{C}_R(M)$ has finitely generated cohomologies in Corollary 3.1.6.

The main object of Section 3.2 is the following question which is raised by Dibaei and S. Yassemi in [10]. They investigate the set $\mathrm{Att}\,\mathrm{H}^d_{\mathfrak{a}}(M)$ for a finitely generated R–module M and an ideal $\mathfrak{a}$ of R and show that $\mathrm{Att}\,\mathrm{H}^d_{\mathfrak{a}}(M) \subseteq \mathrm{Assh}\, M$. Now it is natural to ask,

Question 5. [10, Question 2.9] For any subset T of $\mathrm{Assh}\, M$, is there an ideal $\mathfrak{a}$ of R such that $\mathrm{Att}\,\mathrm{H}^d_{\mathfrak{a}}(M) = T$?

Theorem 3.2.11 presents a positive answer to this question in the case where $(R,\mathfrak{m})$

is a complete local ring. In [11, Theorem 1.6], it is proved that if $(R, \mathfrak{m})$ is a complete local ring, then for any pair of ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, if $\operatorname{Att} \mathrm{H}_{\mathfrak{a}}^d(M) = \operatorname{Att} \mathrm{H}_{\mathfrak{b}}^d(M)$, then $\mathrm{H}_{\mathfrak{a}}^d(M) \cong \mathrm{H}_{\mathfrak{b}}^d(M)$. As a consequence we show that the number of non–isomorphic top local cohomology modules of M with respect to all ideals of R is equal to $2^{|\operatorname{Assh} M|}$ in Corollary 3.2.12.

In last section 3.3, we use results of sections 3.1 and 3.2 and those of Chapter 2 for studying the class of generalized Cohen-Macaulay modules. In Corollary 3.3.4, we find a new characterization of generalized Cohen-Macaulay rings in terms of uniform annihilators of local cohomologies. Our results in this section are useful in the last chapter of thesis to study the Cohen-Macaulay loci of modules.

The Cohen-Macaulay locus of M is denoted by

$$\mathrm{CM}(M) := \{\mathfrak{p} \in \operatorname{Spec} R : M_{\mathfrak{p}} \text{ is Cohen-Macaulay as } R_{\mathfrak{p}}\text{–module}\}.$$

The topological property of Cohen-Macaulay loci of modules and determining when it is a zariski–open subset of $\operatorname{Spec} R$ have been studied by many authors. Grothendieck in [14] states that $\mathrm{CM}(M)$ is a Zariski–open subset of $\operatorname{Spec} R$ whenever R is an excellent ring and in [17], Hartshorne shows that $\mathrm{CM}(R)$ is open when R possesses a dualizing complex. In [26], C. Rotthaus and L. M. Şega study the Cohen-Macaulay loci of graded modules over a noetherian homogeneous graded ring $R = \bigoplus_{i \in \mathbb{N}} R_i$ considered as R_0–modules.

Our aim in the first section of Chapter 4, is to determine when $\mathrm{CM}(M)$ is a Zariski–open subset of $\operatorname{Spec} R$. We find two classes of rings, over which, $\mathrm{CM}(M)$ is open for all R–modules M. The first is the class of rings whose formal fibres are Cohen–Macaulay (see Remark 4.1.7) and the second is the class of catenary local rings R with finite non–$\mathrm{CM}(R)$, where non–$\mathrm{CM}(M) = \operatorname{Spec} R \setminus \mathrm{CM}(M)$ (see Corollary 4.1.9). Finally, we present examples to show that these two classes of rings are significant in 4.1.11 and 4.1.12.

Inspired by the above results, we study rings whose formal fibres are Cohen-Macaulay in Section 4.2. One of our main results in this section is Theorem 4.2.2 which gives a characterization of a finitely generated R–module which admits a uniform local cohomological annihilator in terms of certain set of formal fibres of R. In particular we show that, for a prime ideal $\mathfrak{p}$ of R, $R/\mathfrak{p}$ is universally catenary and the formal fibre of R over $\mathfrak{p}$ is Cohen-Macaulay if and only if $R/\mathfrak{p}$ has a uniform local cohomological annihilator (see Theorem 4.2.2 and Lemma 2.2.10).

Corollary 4.2.3 is a good summary of connections between uniform annihilators of local cohomologies and Cousin complexes which shows that a local ring R is universally catenary and all of it's formal fibres are Cohen-Macaulay if and only if $\mathcal{C}_R(R/\mathfrak{p})$ has finitely generated cohomologies for all $\mathfrak{p} \in \operatorname{Spec} R$, if and only if $R/\mathfrak{p}$ has a uniform local cohomological annihilator for all $\mathfrak{p} \in \operatorname{Spec} R$.

Note that for an R–module M, non–$\mathrm{CM}(M) = \mathrm{V}(\mathrm{a}(M))$ whenever $\mathcal{C}_R(M)$ is finite (see Corollary 4.2.4). We close this section with Theorem which characterizes those modules

M satisfying non–CM(M) = V(a(M)) without assuming that the Cousin complex of M to be finite, which implies also that when $\mathrm{CM}_R(M)$ is finite, then the formal fibres of R over some certain prime ideals are Cohen-Macaulay (see Corollary 4.2.9).

Observe that if $\mathcal{C}_R(M)$ has finitely generated cohomologies, then M is locally equidimensional and $R/0:_R M$ is universally catenary by Corollary 2.3.1. On the other hand if all formal fibres of a universally catenary local ring R are Cohen-Macaulay, then $\mathcal{C}_R(M)$ has finitely generated cohomologies for all equidimensional R–module M (Theorem 4).

These results strengthen our guess that over a local ring R, if $\mathcal{C}_R(R)$ has finitely generated cohomologies, then all formal fibres of R are Cohen-Macaulay (Section 4.4).

Throughout this thesis, all known definitions and statements are quoted with a reference afterward and all others with no references are supposed to be new, most of them have been appeared in [5], [6] and [7].