Abstract

In the present thesis, first of all, we will take a look at the structure, basic notions and open problems in the Local cohomology theory. Then we will study the graded local cohomoloy modules $H^i_{\mathfrak{a}}(M)$, in the case where $\mathfrak{a}$ is an ideal of the homogeneous Noetherian ring $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ which contains the irrelevent ideal $R_+ = \bigoplus_{n\in\mathbb{N}} R_n$, M ia a finitely generated graded R-module and $i \in \mathbb{N}_0$. Althought, the graded components $H^i_{\mathfrak{a}}(M)_n$ could be non-finitely generated but, we show that like $H^i_{R_+}(M)_n$, they vanish for sufficiently large values of n. Also, we study the asymptotic behaviuor of the sequences $(\mathrm{Ass}_{R_0}(H^i_{\mathfrak{a}}(M)_n))_{n\in\mathbb{Z}}$, for $n \to -\infty$. Specially, we prove that for all $i \leq f^{R_+}_{\mathfrak{a}}(M)$ there exists a finite set $X \subseteq \mathrm{Spec}(R_0)$ such that $\mathrm{Ass}_{R_0}(H^i_{\mathfrak{a}}(M)_n) = X$ for all $n \ll 0$. In addition some results concerning the Artinianess of certain quotients and submoduls of $H^i_{\mathfrak{a}}(M)$ have been presented.

In chapter three of this thesis, we study the asymptotic behaviour of the sequences $(\mathrm{grade}(\mathfrak{a}_0,\ (H^i_{R_+}(M)_n))_{n\in\mathbb{Z}}$ of integers, for $n \to -\infty$, in the case where $\mathfrak{a}_0$ is an ideal of the bese ring R_0 of the homogenious Noetherian ring R and M is a finitely generated graded R-module. First of all, the case $i = f_{R_+}(M)$ has been considered and a lower bound, independent of n, for the sequence $(\mathrm{grade}(\mathfrak{a}_0, (H^{f_{R_+}(M)}_{R_+}(M)_n))_{n\in\mathbb{Z}}$, when $n \to -\infty$ has been found. Also, in the special case where $f_{R_+}(M) = \mathrm{cd}_{R_+}(M)$ we show that $\mathrm{grade}(\mathfrak{a}_0, (H^{f_{R_+}(M)}_{R_+}(M)_n) = f^{R_+}_{\mathfrak{a}_0+R_+}(M) - f_{R_+}(M)$, for all $n \ll 0$. Then we present some results concerning the above problem in the case where $\dim(R_0) \leq 2$.

Consider the previous assumptions on R and M and assume, in addition, that the local base ring $(R_0, \mathfrak{m}_0)$ is Artinian. For each $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$, let $d^i_M(n) := \mathrm{length}_{R_0}(D^i_{R_+}(M)_n)$, where $D^i_{R_+}(M)$ denotes the i-th right derived functor of the R_+-transform functor $D_{R_+}(M) = \lim\limits_{n\to\infty} \mathrm{Hom}_R(R^n_+ M, M)$. Also, let $K^i(M)$ denotes the i-th deficiency module of M. Our main result in chapter four says that the $\mathrm{reg}(K^i(M))$ is bounded in terms of $\mathrm{beg}(M)$, the beginning of M, and the 'diagonal values' $d^0_M(0), d^1_M(-1), \cdots, d^{\dim(M)-1}_M(1-\dim(M))$. This nice result plays a crucial rule in chapter five. Furthermore, as applications of that result, some other upper bounds for $\mathrm{reg}(K^i(M))$ have been presented.

Let $d \in \mathbb{N}$ and let $\mathfrak{D}^d$ denotes the set of all pairs (R, M) in which R is a homogenious Noetherian ring with Artinian base ring and M is a finitely generated graded R-module with

$\dim(M) \leq d$. In chapter five, we look for the subclasses $\mathcal{C}$ of $\mathfrak{D}^d$ such that the set of sequences $(d^i_M(n))_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}}$, where $(R,M) \in \mathcal{C}$, is finite. Specially, it has been shown that if $x_0, x_1, \cdots, x_{d-1} \in \mathbb{N}_0$ and $n_0, n_1, \cdots, n_{d-1} \in \mathbb{Z}$, such that $n_0 > n_1 > \cdots > n_{d-1}$, then the set of all sequences $(d^i_M(n))_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}}$, where $(R,M) \in \mathfrak{D}^d$ and $d^0_M(n_0) \leq x_0, d^1_M(n_1) \leq x_1, \ldots, d^{d-1}_M(n_{d-1}) \leq x_{d-1}$, is finite. Also, the converse of this result has been proved, too. In other words, we prove that if $\mathbb{S} \subseteq \{0, \cdots, d-1\} \times \mathbb{Z}$ and for any sequences $(h^{(i,n)})_{(i,n)\in\mathbb{S}}$ of integers, the set of sequences $(d^i_M(n))_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}}$, where $(R,M) \in \mathfrak{D}^d$ and $d^i_M(n) \leq h^{(i,n)}$ for each $(i,n) \in \mathbb{S}$, is finite, then $\mathbb{S}$ must contain a set of the form $\{(i,n_i) \mid i \in \{0, \cdots, d-1\}\}$, where $n_0, \cdots, n_{d-1} \in \mathbb{Z}$ and $n_0 > n_1 > \cdots > n_{d-1}$.

In the last chapter, we will present some results concerning the cohomological dimension and finiteness dimension of generalized local cohomology modules.

Keywords: Local cohomology modules, graded modules, Artinian modules, set of associated prime ideals, Castelnouvo-Mamford regularity, Hilbert functions.