Abstract

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring possessing a normalized dualizing complex D. We denote the Matlis duality functor by $(-)^\vee$. We investigate on connections between Foxby equivalence and local duality through generalized local cohomology modules. Assume that X and Y are two homologically bounded complexes of R-modules with finitely generated homology modules. If Gorenstein projective dimension of X and injective dimension of Y are finite, then we show that

$$\mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}\operatorname{Hom}_R(X,Y)) \simeq (\mathbf{R}\operatorname{Hom}_R(Y, D \otimes^{\mathbf{L}}_R X))^\vee.$$

Also, we prove that if Gorenstein injective dimension of Y and projective dimension of X are finite, then

$$\mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}\operatorname{Hom}_R(X,Y)) \simeq (\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(D,Y),X))^\vee.$$

As some applications, we characterize Cohen-Macaulay modules and Gorenstein modules and also we establish Grothendieck's non-vanishing Theorem in the context of generalized local cohomology modules. It is proved that if there exists a nonzero Cohen-Macaulay R-module with finite Gorenstein projective dimension, then there exists a nonzero finitely generated R-module with finite Gorenstein injective dimension. We prove that if there exists a nonzero finitely generated R-module with finite Gorenstein injective dimension, then there exists a Cohen-Macaulay complex with finite Gorenstein projective dimension. Also, we show that if there exists a nonzero Cohen-Macaulay R-module with finite Gorenstein injective dimension, then there exists a nonzero Cohen-Macaulay R-module with finite Gorenstein projective dimension. Now, assume that S is a commutative Noetherian ring and $\mathfrak{a}$ is an ideal of S. In the sequel, we define weakly equivalence preserving functors and we will prove that the $\mathfrak{a}$-section (resp. $\mathfrak{a}$-adic completion) functor is weakly equivalence preserving on the category of complexes bounded to the left (resp. right) of Gorenstein injective (resp. flat) modules. Also, we prove that for computing the local (co)homology modules of complexes, one can use their Gorenstein (injective) flat resolutions. Next, we show that for any $X \in \mathcal{D}_{\square}(S)$, $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X) \leq \operatorname{Gfd}_S X$, $-\inf \mathbf{R}\Gamma_{\mathfrak{a}}(X) \leq \operatorname{Gid}_S X$ and if S possesses a dualizing complex, then $\operatorname{Gfd}_S \mathbf{L}\Lambda^{\mathfrak{a}}(X) \leq \operatorname{Gfd}_S X$ and $\operatorname{Gid}_S \mathbf{R}\Gamma_{\mathfrak{a}}(X) \leq \operatorname{Gid}_S X$. Finally, we establish a criterion for the regularity of Gorenstein local rings and we deduce several methods for computing $\mathbf{R}\Gamma_{\mathfrak{a}}(-,\sim)$, $\mathbf{L}\Lambda^{\mathfrak{a}}(-,\sim)$ and their homology modules.