Abstract

Assume that $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ is a standard graded Noetherian ring, and that M is a $\mathbb{Z}$-graded R-module. The set of all homogeneous attached prime ideals of the top non–vanishing local cohomology module of a finitely generated graded module M, $\mathrm{H}^c_{R_+}(M)$, with respect to the irrelevant ideal $R_+ := \bigoplus_{i \geq 1} R_i$ and the set of associated primes of $\mathrm{H}^i_{R_+}(M)$ is studied. The asymptotic behavior of $\mathrm{Hom}_R(R/R_+, \mathrm{H}^s_{R_+}(M))$ for $s \geq f(M)$ is discussed, where $f(M)$ is the finiteness dimension of M. Moreover we show that $\mathrm{H}^h_{R_+}(M)$ is tame if $\mathrm{H}^i_{R_+}(M)$ is Artinian for all $i > h$.

Assume that the base ring R_0 is local. Let K be the residue field of R_0. It is shown that the Tor–regularity $\mathrm{reg}^T_R(M)$ of any finitely generated graded R–module M is less than or equal to the sum of the Castelnuovo–Mumford regularity $\mathrm{reg}_R(M)$ and the Tor–regularity $\mathrm{reg}^T_R(k)$.

Finally, it is shown that for any finitely generated graded R–module M, the Ext–regularity $\mathrm{reg}^E_R(M)$ is less than or equal to the sum of the Castelnuovo-Mumford regularity $\mathrm{reg}_R(M)$, the Ext-regularity $\mathrm{reg}^E_R(K)$ and the Krull dimension $\dim(\frac{M}{R_+M})$. This inequality shows that if the residue field K has finite Ext–regularity, then so do all finitely generated graded modules. In this thesis, section 2, 3 and 4 are presented as original.

Introduction

Throughout this thesis $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ is a standard (also called homogeneous) graded Noetherian ring, so that R_0 is a Noetherian ring, $R = R_0[x_1, \ldots, x_t]$ for some elements $x_1, \ldots, x_t \in R_1$, and $R_+ := \bigoplus_{i \geq 1} R_i$ is the irrelevant ideal of R. Moreover $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a graded R-module which is finitely generated whenever it is explicitly stated. Denote by $\mathrm{H}^i_{R_+}(M)$ the i-th local cohomology module of M with respect to R_+. Keep in mind that the R-modules $\mathrm{H}^i_{R_+}(M)$ carry a natural grading (cf [11, Chapter 12]). If $n \in \mathbb{Z}$ and $i \in \mathbb{N}_0$, we use $\mathrm{H}^i_{R_+}(M)_n$ to denote the n-th graded component of $\mathrm{H}^i_{R_+}(M)$. For any M as above and any integer a we let $M[a]$ denote the a-shift of M, defined as the graded R-module with $M[a]_i = M_{i+a}$.

Now, according to [11, Proposition 15.1.5] we can say that

(a) The R_0-module $\mathrm{H}^i_{R_+}(M)_n$ is finitely generated for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$.

(b) For all $i \in \mathbb{N}_0$ and all $n \gg 0$ we have $\mathrm{H}^i_{R_+}(M)_n = 0$.

The main objects of study in this thesis are local cohomology modules and regularities over standard graded rings.

In Chapter 1, we briefly review the preliminary material needed to understand and prove the results of the later chapters.

In chapter 2, the set of all homogeneous attached prime ideals of the top non–vanishing local cohomology module of a finitely generated graded module M, $\mathrm{H}^c_{R_+}(M)$, with respect to the irrelevant ideal $R_+ := \bigoplus_{i\geq 1} R_i$ and the set of associated primes of $\mathrm{H}^i_{R_+}(M)$ are studied. The asymptotic behavior of $\mathrm{Hom}_R(R/R_+, \mathrm{H}^s_{R_+}(M))$ for $s \geq f(M)$ is discussed, where $f(M)$ is the finiteness dimension of M. It is shown that $\mathrm{H}^h_{R_+}(M)$ is tame if $\mathrm{H}^i_{R_+}(M)$ is Artinian for all $i > h$.

This chapter is divided into three sections. In section 1 we study ${}^*\mathrm{Att}_R(\mathrm{H}^c_{R_+}(M))$, the set of all homogeneous attached prime ideals of $\mathrm{H}^c_{R_+}(M)$ and we will show that if $(R_0, \mathfrak{m}_0)$ is local, then the set of maximal elements of it, $\mathrm{Max}\,({}^*\mathrm{Att}_R(\mathrm{H}^c_{R_+}(M)))$, is a finite set which is equal to $\mathrm{Att}_R(\mathrm{H}^c_{R_+}(M/\mathfrak{m}_0 M))$.

An important problem in commutative algebra is to determine when the set of associated primes of the i-th local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(M)$ of M with support in $\mathfrak{a}$ is finite. This question, which was raised by Huneke [23], has been studied by many authors. We review their works. If M is finitely generated then it is well-known that $\mathrm{Ass}_R\mathrm{H}^i_{\mathfrak{a}}(M)$ is a finite set for $i = 0$. If M is a non-zero finitely generated R-module of finite dimension n, then $\mathrm{H}^n_{\mathfrak{a}}(M)$ is Artinian R-module and so $\mid \mathrm{Ass}_R\mathrm{H}^n_{\mathfrak{a}}(M) \mid < \infty$.

The fact that $\mathrm{H}^{\dim R}_{\mathfrak{a}}(M)$ has only finitely many associated primes (in the non-local case) has been observed in [10, Remark 3.11]. In [27], Marley showed that over a local ring R, the module $\mathrm{H}^{\dim R-1}_{\mathfrak{a}}(M)$ has finitely many associated primes. There are several attempts to give some partial answers to this question. In [25] Khashyarmanesh and Salarian used the notions of $\mathfrak{a}$-filter regular sequence and unconditioned strong d-sequence to show the following result.

Theorem 0.0.1. *Let R be a Noetherian ring and let M be a finitely generated R-module. Let t be a non-negative integer. Then* $\mathrm{Ass}_R\mathrm{H}^t_{\mathfrak{a}}(M)$ *is a finite set if one of* *the following holds:*

(a) *For any* $i < t$ *the set* $\operatorname{Supp} \mathrm{H}^i_{\mathfrak{a}}(M)$ *is finite.*

(b) *For any* $i < t$ *the module* $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is finitely generated.*

Let t be a non-negative integer. Assume that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is finitely generated R-module for all $i < t$. In [35] Tajarod and Zakeri showed that the set $\operatorname{Ass}_R \mathrm{H}^t_{\mathfrak{a}}(M)$ has an explicit presentation by using an unconditioned $\mathfrak{a}$-filter regular M-sequence. In [9] Brodmann, Rotthaus and Sharp gave a simple proof (without using the notions of $\mathfrak{a}$-filter regular sequence and unconditioned strong d-sequence) for the following result (note that this result is a consequence of Theorem 0.0.1(b)).

Theorem 0.0.2. *Let* R *be a Noetherian ring and let* M *be a finitely generated* R*-module. Then the following hold:*

(a) *If* $M \neq \mathfrak{a}M$ *then* $\operatorname{Ass}_R \mathrm{H}^{\operatorname{grade}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$ *is a finite set.*

(b) *The set* $\operatorname{Ass}_R \mathrm{H}^1_{\mathfrak{a}}(M)$ *is finite.*

(c) *If* $\mathrm{H}^1_{\mathfrak{a}}(M)$ *is finitely generated then* $\operatorname{Ass}_R \mathrm{H}^2_{\mathfrak{a}}(M)$ *is a finite set.*

Finally, Brodmann and Lashgari proved a generalization of Theorem 0.0.1(b) and Theorem 0.0.2 with a nice and simple proof (without using filter regular sequences), cf. [8]:

Theorem 0.0.3. *Let* M *be a finitely generated* R*-module. Let* t *be a non-negative integer such that for each* $i < t$, $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is finitely generated* R*-module. Then for any finitely generated submodule* N *of* $\mathrm{H}^t_{\mathfrak{a}}(M)$, *the set* $\operatorname{Ass}_R(\mathrm{H}^t_{\mathfrak{a}}(M)/N)$ *has finitely many elements.*

Note that the set

$$\operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(M)) = \operatorname{Ass}_R \mathrm{H}^t_{\mathfrak{a}}(M).$$

So if $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M))$ is finitely generated then $\operatorname{Ass}_R \mathrm{H}^t_\mathfrak{a}(M)$ is a finite set. Thus to give an answer to this question, we can consider the following conjecture which is proposed by Grothendieck ([21]; Expose XIII, 1.1]).

Let M be a finitely generated R-module and let $\mathfrak{a}$ be an ideal of R. Then $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^j_\mathfrak{a}(M))$ is finitely generated R-module for all $j \geq 0$. Although this conjecture is not true in general, cf. [22]; Example 1, we have the following result that is a generalization of Theorem 0.0.1(b), cf. [2].

Theorem 0.0.4. *Let M be a finitely generated R-module and let t be a non-negative integer such that $\mathrm{H}^i_\mathfrak{a}(M)$ is finitely generated R-module for all $i < t$. Then*

$$\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M))$$

is finitely generated.

In [22] Hartshorne introduced the notion of a module cofinite with respect to an ideal $\mathfrak{a}$.

Definition 0.0.5. *The R-module M is called $\mathfrak{a}$-cofinite if* $\operatorname{Supp}(M) \subseteq \mathrm{V}(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is a finitely generated R-module for each i.*

Note that any finitely generated R-module is $\mathfrak{a}$-cofinite module for arbitrary ideal $\mathfrak{a}$. Also if $(R, \mathfrak{m})$ is local ring then an R-module M is $\mathfrak{m}$-cofinite if and only if M is Artinian. Recently Divaani-Aazar and Mafi introduced weakly Laskerian modules. An R-module M is called *weakly Laskerian* if for any submodule N of M, the set $\operatorname{Ass}_R(M/N)$ is finite. It is easy to see that finitely generated modules, Artinian modules and the modules with finite support are weakly Laskerian. By using the technique of spectral sequences they proved the following result, cf. [14]. This result is a generalization of 0.0.1 and 0.0.4.

Theorem 0.0.6. *Let M be a weakly Laskerian module. Let t be a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < t$. Then the set of associated primes of* $\mathrm{Hom}_R(R/a, \mathrm{H}^t_{\mathfrak{a}}(M))$ *and* $\mathrm{Ext}^1_R(R/a, \mathrm{H}^t_{\mathfrak{a}}(M))$ *are finite.*

On the other hand Dibaei and Yassemi proved (without using spectral sequences) the following result that is a generalization of 0.0.4.

Theorem 0.0.7. (c.f. [16, Theorem 2.1]) *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Let s be a non-negative integer. Let M be an R-module such that $\mathrm{Ext}^s_R(R/a, M)$ is a finitely generated R-module. If $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < s$, then the module* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ *is finitely generated.*

Then, by a refinement of the proof of [16, Theorem 2.1] Dibaei and Yassemi show that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ is finitely generated provided some certain finitely many conditions satisfied on the local cohomologies $\mathrm{H}^i_{\mathfrak{a}}(M), i < s$.

Theorem 0.0.8. (c.f.[18, Theorem 6.3.9]) *Let $\mathfrak{a}$ be an ideal of R and let M be an R-module. Let s be a non-negative integer. Consider the following statements.*

(a) $\mathrm{Ext}^s_R(R/\mathfrak{a}, M)$ *is a finitely generated R-module.*

(b) $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(M))$ *is a finitely generated R-module. Then the following hold:*

(i) If $\mathrm{Ext}^{s-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ *is finitely generated for all $j < s$ then (b) $\Longrightarrow$ (a).*

(ii) If $\mathrm{Ext}^{s+1-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ *is finitely generated for all $j < s$ then (a) $\Longrightarrow$ (b).*

In particular, if $\mathrm{Ext}^{t-j}_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ *is finitely generated for $t = s, s+1$ and for all $j < s$, then (a) $\Longleftrightarrow$ (b).*

In section 2, we study $\mathrm{Ass}_R(\mathrm{H}^s_{R_+}(M))$, for $s \geq 0$. We first show that if K is a module (not necessarily finite) over a Noetherian ring S, then for each ideal $\mathfrak{a}$ of S and each non-negative integer s, we have

$$\mathrm{Ass}_S(\mathrm{H}^s_{\mathfrak{a}}(K)) \subseteq \bigcup_{0\leq j<s} \mathrm{Ass}_S(\mathrm{Ext}^{s-j+1}_S(S/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(K))/L_j) \cup \mathrm{Ass}_S(\mathrm{Ext}^s_S(S/\mathfrak{a}, K)/L)$$

for some submodules $L, L_0, \cdots, L_{s-1}$ of their appropriate modules. When M is a graded R–module ($R = R_0[R_1]$ is, as usual, a homogeneous Noetherian graded ring) and the modules $\mathrm{Ext}^s_R(R/R_+, M)$, $\mathrm{Ext}^{s-j+1}_R(R/R_+, \mathrm{H}^j_{R_+}(M)), 0 \leq j < s$, are *weakly Laskerian (see Definition 2.2.3), then $\mathrm{Ass}_R(\mathrm{H}^s_{R_+}(M))$ is a finite set.

Section 3 is devoted to study the asymptotic behavior of the modules $\mathrm{H}^i_{R_+}(M)$ and $\mathrm{Hom}_R(R/R_+, \mathrm{H}^i_{R_+}(M))$. We examine the asymptotic stability of associated primes, the asymptotic stability of supports, and tameness of $\mathrm{Hom}_R(R/R_+, \mathrm{H}^i_{R_+}(M))$ for graded module M. This gives a generalization of the result that $\mathrm{H}^f_{R_+}(M)$ is tame, where $f := f(M) = \inf\{i \mid \mathrm{H}^i_{R_+}(M) \text{is not finitely generated}\}$ (cf. [9, Theorem 3.6(a)]). Let $\mathrm{cd}\,(M) = \mathrm{Max}\,\{i \in \mathbb{Z} \mid \mathrm{H}^i_{R_+}(M) \neq 0\}$. It is shown in [6, Theorem 4.8 (e)] that $\mathrm{H}^{\mathrm{cd}\,(M)}_{R_+}(M)$ is tame. We show that $\mathrm{H}^h_{R_+}(M)$ is tame if $\mathrm{H}^i_{R_+}(M)$ is Artinian R–module for all $i > h$. This, in particular, implies that $\mathrm{H}^{\dim_R(M)-1}_{R_+}(M)$ is tame.

In chapter 3 and 4, we also assume that the base ring R_0 is local. Let K be the residue field of R_0. For a graded module M over R, the Tor-, Castelnuovo–Mumford and Ext-regularities of M are defined, respectively, as

$$\mathrm{reg}^T_R(M) = \mathrm{Max}\,\{\mathrm{end}\,\mathrm{Tor}^R_i(M, R/R_+) - i \mid i \geq 0\},$$

$$\mathrm{reg}_R(M) = \mathrm{Max}\,\{\mathrm{end}\,\mathrm{H}^i_{R_+}(M) + i \mid i \geq 0\},$$

$$\mathrm{reg}^E_R(M) = \mathrm{Max}\,\{j \mid \mathrm{Ext}^i_R(M, R)_{-j-i} \neq 0 \text{ for some } i \geq 0\}.$$

Recently Römer [32, Theorem 3.1] and Chardin [12, Proposition 2.4] proved that for a finitely generated graded module M, over standard graded K–algebra R, the inequality

$$\mathrm{reg}^T_R(M) \leq \mathrm{reg}_R(M) + \mathrm{reg}^T_R(K) \qquad (0.0.1)$$

holds true.

In Theorem 3.2.4 we give a generalization of this result. In chapter 3, after some preliminary results in section 1, in section 2, we first show that if Q is a polynomial ring with local base ring, then $\operatorname{reg}^T_Q(M) = \operatorname{reg}_Q(M)$ (see Lemma 3.2.3). Next we bring our main result which states that the Tor–regularity of M over R is less than or equal to the sum of the Castelnuovo–Mumford regularity $\operatorname{reg}_R(M)$ and the Tor–regularity $\operatorname{reg}^T_R(K)$ (see Theorem 3.2.4).

In chapter 4, we show that for any finitely generated graded R–module M, the Ext–regularity $\operatorname{reg}^E_R(M)$ is less than or equal to the sum of the Castelnuovo-Mumford regularity $\operatorname{reg}_R(M)$, the Ext-regularity $\operatorname{reg}^E_R(K)$ and the Krull dimension $\dim(\frac{M}{R_+M})$. This inequality shows that if the residue field K has finite Ext–regularity, then so do all finitely generated graded modules.

This chapter is divided into three sections. In Section 1, first we prove the inequality $\operatorname{reg}^E(\frac{R}{\mathfrak{p}_0+R_+}) \leq \operatorname{reg}^E(K) + \dim \frac{R}{\mathfrak{p}_0+R_+}$ for all $\mathfrak{p}_0 \in \operatorname{Spec} R_0$ (Lemma 4.1.3) and then present, in the main result of this chapter (Theorem 4.1.5), the inequality

$$\operatorname{reg}^E(M) \leq \operatorname{reg}(M) + \operatorname{reg}^E(K) + \dim \frac{M}{R_+M} \qquad (0.0.2)$$

holds true for any finitely generated graded R-module M.

In section 2 we study those R_+-torsion modules for which the equality holds in (0.0.2), and in Theorem 4.2.2, we derive equivalent conditions for validity of the equality

$$\operatorname{reg}^E(\frac{R}{\mathfrak{p}_0 + R_+}) = \operatorname{reg}^E(K) + \dim \frac{R}{\mathfrak{p}_0 + R_+}.$$

In section 3, we compare the invariants involved in results of Römer [32, Theorem 3.1] and Chardin [12, Proposition 2.4] that is in the inequality $\operatorname{reg}^T_R(M) \leq \operatorname{reg}_R(M) + \operatorname{reg}^T_R(K)$ with the invariants involved in the inequality (0.0.2) above.