Abstract

This thesis concerns two topics. The first topic, that is related to the Dedekind-Mertens Lemma, the notion of the so-called content algebra, is discussed in chapter 2. Let R be a commutative ring with identity and M be a unitary R-module and c the function from M to the ideals of R defined by $c(x) = \cap\{I\colon I \text{ is an ideal of } R \text{ and } x \in IM\}$. M is said to be a *content* R-module if $x \in c(x)M$, for all $x \in M$. The R-algebra B is called a *content* R-algebra, if it is a faithfully flat and content R-module and it satisfies the Dedekind-Mertens content formula. In chapter 2, it is proved that in content extensions, minimal primes extend to minimal primes, and zero-divisors of a content algebra over a ring which has Property (A) or whose set of zero-divisors is a finite union of prime ideals are discussed. The preservation of diameter of zero-divisor graph under content extensions is also examined. Gaussian and Armendariz algebras and localization of content algebras at the multiplicatively closed set $S' = \{f \in B\colon c(f) = R\}$ are considered as well.

In chapter 3, the second topic of the thesis, that is about the grade of the zero-divisor modules, is discussed. Let R be a commutative ring, I a finitely generated ideal of R, and M a zero-divisor R-module. It is shown that the M-grade of I defined by the Koszul complex is consistent with the definition of M-grade of I defined by the length of maximal M-sequences in I.

Chapter 1 is a preliminarily chapter and dedicated to the introduction of content modules and also locally Nakayama modules.

Introduction

Throughout this thesis all rings are commutative with unit and all modules are assumed to be unitary. Let R be a commutative ring with identity and X be an indeterminate over R. The *content* $c(f)$ of a polynomial $f \in R[X]$ is the ideal of R generated by the coefficients of f. In other words, if $f = a_0 + a_1X + \cdots + a_nX^n$, where $a_i \in R$ for all $0 \leq i \leq n$ and $n \geq 0$, then $c(f) = (a_0, a_1, \cdots, a_n)$. From the definition of the content of a polynomial, it is obvious that $c(f) = (0) \Leftrightarrow f = 0$ for all $f \in R[X]$. Let, for the moment, $f \in R[X]$ and I be an ideal of R. Consider $f + I[X] \in R[X]/I[X]$ and note that $R[X]/I[X] \cong (R/I)[X]$. It is easy to check that $c(f + I[X]) = c(f) + I$. Also we know that $c(f + I[X]) = (I)$ iff $f + I[X] = I[X]$ (Note that *iff* always stands for if and only if). This means that $c(f) \subseteq I$ iff $f \in I[X]$ that obviously implies that $f \in c(f)[X]$ for all $f \in R[X]$.

Now let M be a unitary R-module and define d a function from M to the ideals of R with this property that $x \in IM$ iff $d(x) \subseteq I$ for all $x \in M$ and ideals I of R. From the definition of d, it is obvious that $d(x)$ is a subset of the intersection of all ideals I of R such that $x \in IM$. Let us define the *content function*, c from M to the ideals of R defined by

$$c(x) = \bigcap\{I \colon I \text{ is an ideal of } R \text{ and } x \in IM\}.$$

Therefore $d(x) \subseteq c(x)$. On the other hand, $x \in d(x)M$ and according to the definition of the content function c, we have $c(x) \subseteq d(x)$ and therefore $d = c$. Note that $x \in c(x)M$ for all $x \in M$. This can be the inspiration of the definition of content modules in [**OR**]: The unitary R-module M is called a *content R-module* if $x \in c(x)M$, for all $x \in M$. The class of content modules are themselves considerable and interesting in the field of module theory. In chapter 1, we introduce content modules and mention some of their basic properties that we need in the rest of the thesis for the convenience of the reader.

From the definition of the content of a polynomial, it is obvious that $c(rf) = rc(f)$ for all $f \in R[X]$ and $r \in R$. Note that if M is a content R-module and $x \in M$, then $x \in c(x)M$ and if $r \in R$, we have $rx \in rc(x)M$ and therefore $c(rx) \subseteq rc(x)$. The question arises when the equality holds. The answer to this question will be given in Theorem 1.1.4.

For $f, g \in R[X]$, it is easy to check that $c(fg) \subseteq c(f)c(g)$ and it is of interest to know when the equality holds. For $f, g \in R[X]$, the Dedekind-Mertens Lemma states that $c(fg)c(g)^m = c(f)c(g)c(g)^m$ for some $m \geq 0$ depending on f [**AG**]. A polynomial f is said to be Gaussian if $c(fg) = c(f)c(g)$ for all $g \in R[X]$. For example if the $c(f)$ is a cancellation ideal of R, then f is Gaussian (An ideal I of R is said to be cancellation if $IJ = IK$ causes $J = K$ for all ideals J and K of R). Also note that we may not have the equality as the following example shows: Let $R = \mathbb{Z} + 2i\mathbb{Z}$ and $f = g = 2i + 2X$. Then it is easy to check that $c(fg) = (4)$, while $c(f)c(g) = (4, 4i)$. However $c(fg)c(g) = c(f)c(g)c(g) = (8, 8i)$.

"In [**Ed**, page 2], Edwards states that the theorem which Dedekind proved (See [**De**]) said that if f and g are polynomials whose coefficients are algebraic numbers such that the coefficients of fg are algebraic integers, then the product of any coefficient of f with an arbitrary coefficient of g is also an algebraic integer. In modern language this is just saying that the ideal $c(f)c(g)$ is integral over the ideal $c(fg)$, a theorem which is considerably weaker than what is now known as the Dedekind-Mertens Lemma. Apparently the fact that there exists an N such that $c(f)^N c(f)c(g) = c(f)^N c(fg)$ was known to Mertens, Kronecker [**Kro**], and perhaps Hurwitz [**Hur**]. Mertens' proof in 1892 shows that in characteristic 0, the number N can be chosen approximately equal to $\deg(g)^2$. However, the precise value $N = \deg(g)$ was given by Dedekind in his 1892 paper [**De**, page 10]. Prüfer reproved this theorem with $N = \deg(g)$ [**Pr**, page 24] in 1932. The earliest reference we have found with the name 'Dedekind-Mertens Lemma' is in Krull [**Kru**, page 128]. It is interesting that the theorem Krull states as the Dedekind-Mertens Lemma has the value N equal to the sum of the degrees of f and g. In an article in 1959 [**No**] Northcott says he has not been able to find a reference for the Dedekind-Mertens Lemma and gives a proof he attributes to Emil Artin. Arnold and Gilmer [**AG**] give a proof and a generalization whose idea we use in this paper." [**HH**, p. 1306]

Gilmer, Grams and Parker [**GGP**, Theorem 3.6, 3.7] and Anderson and Kang [**AK**] proved and generalized the Dedekind-Mertens Lemma: if g has $k+1$ nonzero terms, then $c(f)^k c(f)c(g) = c(f)^k c(fg)$ holds for all polynomials $f \in R[X]$. Heinzer and Huneke define an interesting notion called the Dedekind-Mertens number of $\mu(g)$ of a polynomial $g \in R[X]$ that is the smallest positive integer k such that $c(fg)c(g)^{k-1} = c(f)c(g)c(g)^{k-1}$ and sharpen the upper bound for $\mu(g)$ replacing the number of nonzero coefficients by the maximum of the number of the minimal generators of $c(g)R_{\mathbf{m}}$, where $\mathbf{m}$ runs over all maximal ideals of R [**HH**, Theorem 2.1].

There are many works on the Dedekind-Mertens Lemma. One may refer to [**AG**], [**AK**], [**BG**], [**GGP**], [**HH**], [**No**] and [**T**]. However among all these works, a nice generalization of the Dedekind-Mertens Lemma in [**No**] is of our special interest, since it is a good example for a concept related to the Dedekind-Mertens Lemma, the notion of the so-called *content algebra*.

Let S be a commutative semigroup. The canonical form of an element f in the semigroup ring $R[S]$ is of the form $f = a_0X^{s_0} + a_1X^{s_1} + \cdots + a_nX^{s_n}$, where $a_i \in R$ and $s_i \in S$ for all $0 \leqslant i \leqslant n$. The content of f is the ideal of R generated by the coefficients of f, i.e. $c(f) = (a_0, a_1, \cdots, a_n)$.

Northcott in [**No**] proves that if S is a commutative, cancellative and torsion-free semigroup, then for all $f, g \in R[S]$, there exists an $m \geqslant 0$ such that

$$c(fg)c(g)^m = c(f)c(g)c(g)^m.$$

By the Dedekind-Mertens Lemma, one may give a very simple proof for McCoy's Theorem for the zero-divisors of semigroup rings. McCoy's Theorem says that if f is a zero-divisor of $R[S]$, then there exists a nonzero $r \in R$ such that $f.r = 0$ or equivalently $c(f).r = (0)$. The proof is as follows: Assume that $f \in R[S]$ and $g \in R[S] - \{0\}$, such that $fg = 0$. So there exists a natural number k such that $c(f)^k c(g) = c(f)^{k-1} c(fg) = (0)$. Take t the smallest natural number such that $c(f)^t c(g) = (0)$ and choose r a nonzero element of $c(f)^{t-1} c(g)$. It is easy to check that $c(f).r = (0)$ and equivalently $f.r = 0$.

This short introduction and some other examples and results in the thesis will show us that it is useful to consider content algebras:

Let R be a commutative ring with identity and R' an R-algebra. R' is defined to be a *content R-algebra*, if the following conditions hold:

(1) R' is a content R-module.
(2) (*Faithful flatness*) For any $r \in R$ and $f \in R'$, the equation $c(rf) = rc(f)$ holds and $c(R') = R$.
(3) (*Dedekind-Mertens content formula*) For each f and g in R', there exists a natural number n such that $c(f)^n c(g) = c(f)^{n-1} c(fg)$.

Chapter 2 begins by the definition of content and weak content algebras and we prove that if R is a ring and S, a commutative monoid, then the monoid ring $B = R[S]$ is a content R-algebra if and only if one of the following conditions satisfies:

(1) For $f, g \in B$, if $c(f) = c(g) = R$, then $c(fg) = R$.
(2) (*McCoy's Property*) For $g \in B$, g is a zero-divisor of B iff there exists $r \in R - \{0\}$ such that $rg = 0$.
(3) S is a cancellative and torsion-free monoid.

In 2.2, we discuss prime ideals of content and weak content algebras and we show that in content extensions, minimal primes extend to minimal primes. More precisely, if B is a content R-algebra, then there is a correspondence between $\operatorname{Min}(R)$ and $\operatorname{Min}(B)$, with the function $\varphi : \operatorname{Min}(R) \longrightarrow \operatorname{Min}(B)$ defined by $\mathbf{p} \longrightarrow \mathbf{p}B$.

In 2.3, we introduce a family of rings which have very few zero-divisors. It is a well-known result that the set of zero-divisors of a Noetherian ring is a finite union of its associated primes [**K**, p. 55]. Rings having few zero-divisors have been introduced in [**Dav**]. We define that a ring R has *very few zero-divisors*, if $Z(R)$ is a finite union of prime ideals in $\mathrm{Ass}(R)$. In this section, we prove that if B is a content R-algebra, then R has very few zero-divisors iff B has very few zero-divisors.

Another celebrated property of Noetherian rings is that every ideal entirely contained in the set of their zero-divisors has a nonzero annihilator. A ring R has *Property (A)*, if each finitely generated ideal $I \subseteq Z(R)$ has a nonzero annihilator [**HK**]. In Section 2.3, we also prove some results for content algebras over rings having Property (A) and then we discuss rings having few zero-divisors in more details. Let us recall that a ring R is said to have few zero-divisors, if the set $Z(R)$ of zero-divisors is a finite union of prime ideals. It is well-known that a ring R has few zero-divisors iff its classical quotient ring $T(R)$ is semi-local [**Dav**]. We may suppose that $Z(R) = \bigcup_{i=1}^n \mathbf{p}_i$ such that $\mathbf{p}_i \nsubseteq \bigcup\cup_{j=1 \wedge j \neq i}^n \mathbf{p}_j$ for all $1 \leq i \leq n$. Then we have $\mathbf{p}_i \nsubseteq \mathbf{p}_j$ for all $i \neq j$ and by the Prime Avoidance Theorem, these prime ideals are uniquely determined. In such a case, it is easy to see that $\mathrm{Max}(T(R)) = \{\mathbf{p}_1 T(R), \ldots, \mathbf{p}_n T(R)\}$, where by $T(R)$ we mean total quotient ring of R. Such prime ideals are called maximal primes in $Z(R)$. We denote the number of maximal primes in $Z(R)$ by $\mathrm{zd}(R)$. As one of the main results of this section, we show that if R has Property (A) and $\mathrm{zd}(R) = n$ and B is a content R-algebra, then $\mathrm{zd}(B) = n$. At the end of this section, we consider the interesting case, when $\mathrm{zd}(R) = 1$, i.e. $Z(R)$ is an ideal of R. Such a ring is called a primal ring [**Dau**].

We let $Z(R)^*$ denote the (nonempty) set of proper zero-divisors of R, where by a proper zero-divisor we mean a zero-divisor different from zero. We consider the graph $\Gamma(R)$, called the zero-divisor graph of R, whose vertices are the elements of $Z(R)^*$ and edges are those pairs of distinct proper zero-divisors $\{a, b\}$ such that $ab = 0$. Section 2.4 is devoted to examine the preservation of diameter of zero-divisor graph under content extensions.

In 2.5, we discuss Gaussian and Armendariz content algebras that are natural generalization of the same concepts in polynomials rings. In this section, we show that if B is a content R-algebra, then B is a Gaussian R-algebra iff for any ideal I of R, B/IB is an Armendariz (R/I)-algebra. This is a generalization of a result in [**AC**].

In 2.6, we prove some results about a special content algebra, i.e. $l(R, B)$, a generalization of the algebra $R(X) = R[X]_S$, where $S = \{f \in R[X] \colon c(f) = R\}$.

Some of the results for zero-divisors of content algebras can be proved for semigroup modules. Section 2.7 is devoted to zero-divisors of semigroup modules. Let S be a commutative semigroup and M be an R-module. One can define the semigroup module $M[S]$ as an $R[S]$-module constructed from the semigroup S and the R-module M simply similar to standard definition of semigroup rings. Obviously similar to semigroup rings, the zero-divisors of the semigroup module $M[S]$ are interesting to investigate ([**G1**, p. 82] and [**J**]).

Chapter 3 is devoted to the grade of zero-divisor modules. For doing that we need to know about *locally Nakayama module* and investigate modules having very few zero-divisors with another approach.

In 1.2, locally Nakayama modules have been introduced and we have obtained a necessary and sufficient condition for $\mathrm{Hom}_{R_p}(N_p, M_p) \neq 0$. An R-module M is said to be *locally Nakayama* if $M_p \neq 0$ implies that $M_p/pM_p \neq 0$, for all $p \in \mathrm{Spec}(R)$.

In 3.1, we discuss modules having very few zero-divisors with another view and construct enough materials to generalize some interesting results about the grade of a module. Let I be a finitely generated ideal of R. We can define $\mathrm{grade}(I,M)$ in two different ways: either by the notion of the longest M-sequence in I, when R is a Noetherian ring and M a finitely generated R-module or by the homological grade which is defined by using the Koszul complex. In [**BH**, Theorem 1.6.17], it has been shown that for a finitely generated module over a Noetherian ring the second definition of grade is consistent with the first one. The question arises whether these two definitions are consistent for a larger class of modules as well.

In section 3.2, we give a large class of modules, containing finitely generated modules, for which these two definitions coincide. An R-module M has *very few zero-divisors*, if $Z_R(M)$, the set of zero divisors of M over R, is a finite union of prime ideals in $\mathrm{Ass}_R(M)$, where by $\mathrm{Ass}_R(M)$ we mean the set of associated prime ideals of the R-module M. Let I be a finitely generated ideal of R and M an R-module such that $M \neq IM$. We show that if $M/\mathbf{x}M$ has very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an M-sequence in I, then the length of a maximal M-sequence in I is equal to $\mathrm{grade}(I,M)$ defined by the concept of Koszul complex. Also, we prove that if $M/\mathbf{x}M$ has very few zero-divisors for any ideal $\mathbf{x}$ of R generated by an M-sequence in I, then $\mathrm{grade}(I,M)$ defined by the concept of Koszul complex is the least integer i such that $H^i_I(M) \neq 0$, where $H^i_I(M) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/I^n, M)$ is i-th local cohomology module of M with respect to I. The reader can refer to [**BS**], for the basic properties of local cohomology modules. Note that this is a generalization of a well-known theorem in theory of local cohomology of finitely generated modules over Noetherian rings [**BS**, Theorem 6.2.7].

An R-module M is said to be a *zero-divisor module* (ZD-module), if M/N has very few zero-divisors, for any submodule N of M. According to [**DE**, Example 2.2], the class of ZD-modules is much larger than that of finitely generated modules. As one of the interesting results of this section, we prove that if R is Noetherian, and M a ZD-module, then for all submodules N of M, the following statement are equivalent:

(1) $\mathrm{Ass}_R(H^0_I(M/N)/L)$ is finite for all finitely generated submodules L of $H^0_I(M/N)$;

(2) if $H^0_I(M/N), \cdots, H^{i-1}_I(M/N)$ are finitely generated, then $\mathrm{Ass}_R(H^i_I(M/N)/L)$ is finite for all finitely generated submodules L of $H^i_I(M/N)$.

Some parts of the section 3.1 and the sections 3.2 and 1.2 come from the joint paper of the author of this thesis and Payrovi with the title "Modules having very few zero-divisors", published at Communications in Algebra, Volume 38, Issue 9, Pages 3154–3162 in 2010. The main content of the sections 2.1, 2.2, 2.3 and 2.4 and the whole section 2.7 come from the paper "Zero-divisors of content algebras" published at Archivum Mathematicum (Brno), Volume 46, Issue 4, Pages 237–249 in 2010 and the paper "Zero-divisors of semigroup modules" to appear in Kyungpook Mathematical Journal, respectively, both composed by the author of this thesis.

Unless otherwise stated, our notation and terminology will follow as closely as possible that of Gilmer [**G1**].