Abstract

Let $\mathfrak{a}, \mathfrak{b}$ be ideals of a commutative Noetherian ring R and let X be an arbitrary R–module. In this thesis, for an $\mathfrak{a}$–torsion R–module N, the derived functors of $\operatorname{Hom}_R(N, -)$, $N \otimes_R -$ and $\Gamma_{\mathfrak{b}}(-)$ applied to the local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(X)$ are studied.

In Chapter 2, for fixed integers s and t, we discuss the properties of $\operatorname{Ext}^{s+t}_R(N, X)$ and $\operatorname{Ext}^s_R(N, \mathrm{H}^t_{\mathfrak{a}}(X))$ and, for an additive function λ, we find the inequalities:

$$\lambda(\operatorname{Ext}^t_R(N, X)) \leq \sum_{i=0}^{t} \lambda(\operatorname{Ext}^{t-i}_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)))$$

and

$$\lambda(\operatorname{Ext}^s_R(N, \mathrm{H}^t_{\mathfrak{a}}(X))) \leq$$
$$\sum_{i=0}^{t-1} \lambda(\operatorname{Ext}^{s+t+1-i}_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))) + \lambda(\operatorname{Ext}^{s+t}_R(N, X)) + \sum_{i=t+1}^{t+s-1} \lambda(\operatorname{Ext}^{s+t-1-i}_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))).$$

Our main theorems of this chapter enable us to reprove some older facts and present some new facts about the extension functors, cofiniteness and Bass numbers of local cohomology modules. For example, in the local case $(R, \mathfrak{m})$ where $\mu^i(X) = \dim_{R/\mathfrak{m}}(\operatorname{Ext}^i_R(R/\mathfrak{m}, X))$ is the ith Bass number of X with respect to $\mathfrak{m}$, we can deduce the inequalities of Dibaei and Yassemi:

$$\mu^t(X) \leq \sum_{i=0}^{t} \mu^{t-i}(\mathrm{H}^i_{\mathfrak{a}}(X)) \tag{0.0.1}$$

and

$$\mu^s(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq \sum_{i=0}^{t-1} \mu^{s+t+1-i}(\mathrm{H}^i_{\mathfrak{a}}(X)) + \mu^{s+t}(X) + \sum_{i=t+1}^{t+s-1} \mu^{s+t-1-i}(\mathrm{H}^i_{\mathfrak{a}}(X)). \tag{0.0.2}$$

Chapter 3 is devoted to study of torsion functors of local cohomology modules. We find the first quadrant spectral sequence

$$E^2_{p,q} := \operatorname{Tor}^R_p(N, \mathrm{H}^{\operatorname{ara}(\mathfrak{a})-q}_{\mathfrak{a}}(X)) \underset{p}{\Longrightarrow} \operatorname{Tor}^R_{p+q-\operatorname{ara}(\mathfrak{a})}(N, X)$$

and use it to present some properties of torsion R–modules $\operatorname{Tor}^R_s(N, \mathrm{H}^t_{\mathfrak{a}}(X))$ and $\operatorname{Tor}^R_{s-t}(N, X)$, where s, t are fixed integers. We conclude some inequalities about the Betti numbers and projective dimensions of X and its local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(X)$. Although one may consider the inequalities (0.0.1) and (0.0.2), and expect some consistency for the Betti numbers, the similarities are far from obvious. In fact, in the local case $(R, \mathfrak{m})$ where $\beta_i(X) = \dim_{R/\mathfrak{m}}(\operatorname{Tor}^R_i(R/\mathfrak{m}, X))$ is the ith Betti number of X with respect to $\mathfrak{m}$, we get:

$$\beta_t(X) \leq \sum_{i=0}^{\operatorname{ara}(\mathfrak{a})} \beta_{t+i}(\mathrm{H}^i_{\mathfrak{a}}(X)) \tag{0.0.3}$$

and

$$\beta_s(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq \sum_{i=0}^{t-1} \beta_{s-t+i-1}(\mathrm{H}^i_{\mathfrak{a}}(X)) + \beta_{s-t}(X) + \sum_{i=t+1}^{\operatorname{ara}(\mathfrak{a})} \beta_{s-t+i+1}(\mathrm{H}^i_{\mathfrak{a}}(X)). \tag{0.0.4}$$

The study of section functors of local cohomology modules is the aim of Chapter 4 which has some results about Artinian and non-Artinian local cohomology modules. The relation between the cohomological dimensions of M with respect to different ideals is presented. When R is local, it is shown that M is generalized Cohen-Macaulay if there exists an ideal $\mathfrak{a}$ such that all local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M)$ have finite lengths for all $i < \dim_R(M)$. Also, when r is an integer such that $0 \leq r < \dim_R(M)$, any maximal element $\mathfrak{q}$ of the non-empty set of ideals

$$\{\mathfrak{a} : \mathrm{H}^i_{\mathfrak{a}}(M) \text{ is not Artinian for some } i, i \geq r\}$$

is a prime ideal and that all Bass numbers of $\mathrm{H}^i_{\mathfrak{q}}(M)$ are finite for all $i \geq r$. We observe that this is a generalization to a theorem of Delfino and Marley [14] which states $\mathrm{H}^i_{\mathfrak{a}}(M)$ has finite Bass numbers whenever $\dim_R(R/\mathfrak{a}) = 1$.

Introduction

Throughout R is a commutative Noetherian ring with non-zero identity. We use symbols $\mathfrak{a}$, $\mathfrak{b}$, X, M, and N as follows: $\mathfrak{a}$ and $\mathfrak{b}$ denote ideals of R, X to be an arbitrary R–module which is not necessarily finite (i.e. finitely generated), M is used for a finite R–module, and N denotes an $\mathfrak{a}$–torsion R–module.

Let t be a non-negative integer. In [26], Grothendieck introduced the local cohomology modules $\mathrm{H}^t_{\mathfrak{a}}(X)$ of X with respect to the ideal $\mathfrak{a}$. He proved their basic properties. For example, for a finite R–module M, he proved that $\mathrm{H}^t_{\mathfrak{m}}(M)$ is Artinian for all t, whenever R is local with maximal ideal $\mathfrak{m}$. In particular, it is shown that $\operatorname{Hom}_R(R/\mathfrak{m}, \mathrm{H}^t_{\mathfrak{m}}(M))$ is finite. Later Grothendieck asked, in [27], whether a similar statement is valid if $\mathfrak{m}$ is replaced by an arbitrary ideal of R. Hartshorne gave a counterexample in [29], where he defined that an R–module X is $\mathfrak{a}$–*cofinite* if $\operatorname{Supp}_R(X) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is finite for each i. He also asked the following question.

Question 0.0.1. Let $\mathfrak{a}$ be an ideal of R and let M be a finite R–module. When is $\operatorname{Ext}^i_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ finite for every i and j?

Hartshorne proved that this question is true over a complete regular local ring R, when $\mathfrak{a}$ is a prime ideal of R with $\dim_R(R/\mathfrak{a}) = 1$ or a principle ideal.

This result was later extended to more general rings by Delfino and Marley ([14, Theorem 1]), and Melkersson ([37, Corollary 3.14]).

Theorem 0.0.2. *Let M be a finite R–module. Then, for all i and j,* $\operatorname{Ext}_R^i(R/\mathfrak{a}, \operatorname{H}_{\mathfrak{a}}^j(M))$ *is finite in either of the following cases.*

(i) $\mathfrak{a}$ *is an ideal of R with* $\dim_R(R/\mathfrak{a}) = 1$.

(ii) $\operatorname{cd}_R(\mathfrak{a}, R) = 1$.

The following theorem is also from Melkersson and shows that the finiteness of extension modules is equal to the finiteness of torsion modules.

Theorem 0.0.3. ([37, Theorem 2.1]) *Let $\mathfrak{a}$ be an ideal in R and let X be an R–module. Then the following conditions are equivalent:*

(i) $\operatorname{Ext}_R^i(R/\mathfrak{a}, X)$ *is a finite R–module for all i.*

(ii) $\operatorname{Tor}_i^R(R/\mathfrak{a}, X)$ *is a finite R–module for all i.*

This theorem shows that the study of finiteness of torsion functors of local cohomology modules is as important as the study of finiteness of extension functors of local cohomology modules.

There are some attempts to show that under some conditions, for fixed integers s and t, the R–modules $\operatorname{Ext}_R^s(R/\mathfrak{a}, \operatorname{H}_{\mathfrak{a}}^t(X))$ and $\operatorname{Tor}_s^R(R/\mathfrak{a}, \operatorname{H}_{\mathfrak{a}}^t(X))$ are finite modules, for example see [4, Theorem 3.3], [23, Theorems A and B], [22, Theorem 6.3.9], [33, Theorems 3.3 and Theorem 4.1] and [6, Theorem 3.1]. In this regard, we study extension and torsion functors of local cohomology modules.

After Chapter 1 which is preliminary, we study some properties of extension R–modules $\operatorname{Ext}_R^s(R/\mathfrak{a}, \operatorname{H}_{\mathfrak{a}}^t(X))$ and $\operatorname{Ext}_R^{s+t}(R/\mathfrak{a}, X)$ in the second chapter, which gives us some results on the Bass numbers and cofiniteness of local cohomology modules, and is divided in three sections.

Theorems 2.1.3, 2.1.5 and 2.1.7 are the crucial points of Section 2.1 which enable us to demonstrate some new facts and improve some older facts about the extension functors of local cohomology modules. We employ these theorems to reprove the results of Dibaei and Yassemi about the inequalities between the Bass numbers of an R–module X and those of $\mathrm{H}^i_{\mathfrak{a}}(X)$, $i = 0, 1, \cdots$ (see [25, Theorems 2.1 and 2.6]). More precisely, for two integers s and t in the local case, we prove the followings:

$$\mu^t(X) \leq \sum_{i=0}^{t} \mu^{t-i}(\mathrm{H}^i_{\mathfrak{a}}(X)) \tag{0.0.5}$$

and

$$\mu^s(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq \sum_{i=0}^{t-1} \mu^{s+t+1-i}(\mathrm{H}^i_{\mathfrak{a}}(X)) + \mu^{s+t}(X) + \sum_{i=t+1}^{t+s-1} \mu^{s+t-1-i}(\mathrm{H}^i_{\mathfrak{a}}(X)). \tag{0.0.6}$$

In Section 2.2, we present a generalization of the concept of cofiniteness with respect to an ideal to Serre subcategories of the category of R–modules (i.e. a class of R–modules which is closed under taking submodules, quotients and extensions). This section recovers and improves most of the results about the cofiniteness of local cohomology modules and is such a survey of cofinite modules on an arbitrary ring. For example, Theorems 2.2.2, 2.2.6 and 2.2.8 generalize [36, Proposition 2.5], [37, Proposition 3.11], [16, Theorem 3.1], [23, Theorems A and B] and [15, Corollary 2.7]. We also study $\mathfrak{a}$–cofinite minimax local cohomology modules in Proposition 2.2.11. Recall that an R–module X is a *minimax module* when it has a finite submodule X' such that X/X' is Artinian. Corollaries 2.2.12 and 2.2.13 are immediate results of this proposition where Corollary 2.2.12 improves [7, Theorem 2.3] and Corollary 2.2.13 characterizes Artinian $\mathfrak{a}$–cofinite local cohomology modules. We close this section with the change of ring principle for Serre cofiniteness.

In Section 2.3, we continue our study of extension functors of local cohomology modules and bring more applications. First, we find the weakest possible conditions for finiteness of associated primes of local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(X)$ and extension module $\operatorname{Ext}^i_R(N,X)$ in Corollaries 2.3.2 and 2.3.3. Then, in Corollary 2.3.4, we improve [33, Theorem 3.3] for any arbitrary R–module X with no conditions on Krull dimension of X. Finally, we study the extension functors of local cohomology modules in some special Serre subcategory of the category of R–module and reprove [1, Theorems 2.9 and 2.13] (see Propositions 2.3.7 and 2.3.8).

Chapter 3 is devoted to study of some properties of torsion functors of local cohomology modules and is divided in two sections.

The main ideas of Section 3.1 come from the inequalities (0.0.5) and (0.0.6). Although one may expect some consistency for the Betti numbers, the similarities are far from obvious. For two integers s and t, in Corollaries 3.1.3 and 3.1.6, we prove the following inequalities:

$$\beta_t(X) \leq \sum_{i=0}^{\operatorname{ara}(\mathfrak{a})} \beta_{t+i}(\mathrm{H}^i_{\mathfrak{a}}(X)) \tag{0.0.7}$$

and

$$\beta_s(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq \sum_{i=0}^{t-1} \beta_{s-t+i-1}(\mathrm{H}^i_{\mathfrak{a}}(X)) + \beta_{s-t}(X) + \sum_{i=t+1}^{\operatorname{ara}(\mathfrak{a})} \beta_{s-t+i+1}(\mathrm{H}^i_{\mathfrak{a}}(X)), \tag{0.0.8}$$

and use them to present some upper bounds for the projective dimension of a finite R–module M in terms of flat dimensions of local cohomology modules of M with respect to the ideal $\mathfrak{a}$ (Corollaries 3.1.4 and 3.1.7).

In Section 3.2, we continue the study of torsion functors of local cohomology modules and present more applications of our main theorems of Section 3.1 (Theorems 3.1.2, 3.1.5 and 3.1.8). First, in Corollary 3.2.1, we improve [33, Theorem 4.1] for any arbitrary R–module X with no conditions on Krull dimension of X. Then we bring Corollary 3.2.2 as an immediate consequence of Theorem 3.1.5 and use it to show $\mathrm{H}^i_{\mathfrak{a}}(X)$ may not be finite, coatomic, or minimax (Corollary 3.2.3) for certain integer i. Recall that, an R–module X is said to be *coatomic* if any submodule of X is contained in a maximal submodule of X. Finally, we show that, for a positive integer n, the statement "$\mathrm{H}^i_{\mathfrak{a}}(X)$ is coatomic for all $i \geq n$" is equivalent to each of the statements "$\mathrm{H}^i_{\mathfrak{a}}(X)$ is finite for all $i \geq n$" and "$\mathrm{H}^i_{\mathfrak{a}}(X) = 0$ for all $i \geq n$"; also the statement "$\mathrm{H}^i_{\mathfrak{a}}(X)$ is minimax for all $i \geq n$" is equivalent to the statement "$\mathrm{H}^i_{\mathfrak{a}}(X)$ is Artinian for all $i \geq n$" (Corollaries 3.2.4 and 3.2.5).

Chapter 4 is devoted to study of section functors of local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(X)$ and gives us some results about the Artinianness and non-Artinianness of local cohomology modules, which is related to the third of Huneke's four problems in local cohomology [31], and is divided in three sections.

In Section 4.1, we discuss the arithmetic of cohomological dimensions. We show that, in Corollaries 4.1.2 and 4.1.4, the inequalities:

$$\mathrm{cd}\,(\mathfrak{a} + \mathfrak{b}, M) \leq \mathrm{cd}\,(\mathfrak{a}, M) + \mathrm{cd}\,(\mathfrak{b}, M) \qquad (0.0.9)$$

and

$$\mathrm{cd}\,(\mathfrak{a} + \mathfrak{b}, X) \leq \mathrm{ara}(\mathfrak{a}) + \mathrm{cd}\,(\mathfrak{b}, X) \qquad (0.0.10)$$

hold true and we find, in Corollaries 4.1.6 and 4.1.7, some equivalent conditions for which each inequality becomes equality.

In Section 4.2, we study Artinian local cohomology modules. We first, for ideals $\mathfrak{a} \subseteq \mathfrak{b}$, introduce the notation $\mathrm{cd}\,(\mathfrak{b}/\mathfrak{a}, X)$ to be the infimum of the set

$$\{\mathrm{cd}\,(\mathfrak{c}, X) : \mathfrak{c} \text{ is an ideal of } R \text{ and } \sqrt{\mathfrak{b}} = \sqrt{\mathfrak{c} + \mathfrak{a}}\}.$$

Then, we observe that over a local ring $(R, \mathfrak{m})$ if there is an integer n such that $\dim_R(\mathrm{H}^i_{\mathfrak{a}}(X)) \leq 0$ for all $i \leq n$ (resp. for all $i \geq n$), then $\mathrm{H}^i_{\mathfrak{a}}(X) \cong \mathrm{H}^i_{\mathfrak{m}}(X)$ for all $i \leq n$ (resp. for all $i \geq n + \mathrm{ara}(\mathfrak{m}/\mathfrak{a})$). In this situation, if X is finite then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is Artinian for all $i \leq n$ (resp. for all $i \geq n + \mathrm{cd}\,(\mathfrak{m}/\mathfrak{a}, X)$) (Theorem 4.2.2 and

Corollary 4.2.3). It is deduced that M is generalized Cohen-Macaulay if there exists an ideal $\mathfrak{a}$ such that all local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M)$ have finite lengths for all $i < \dim_R(M)$ (Corollary 4.2.4). Finally, in Theorem 4.2.7, we prove that if $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{a}$, then $\mathrm{H}^i_{\mathfrak{m}}(M)$ is Artinian for all $i \leq n$ (resp. for all $i \geq n + \mathrm{cd}\,(\mathfrak{m}/\mathfrak{a}, M))$ whenever $\mathrm{H}^i_{\mathfrak{a}}(M)$ is minimax for all $i \leq n$ (resp. for all $i \geq n$).

Section 4.3 is devoted to study non-Artinian-ness of local cohomology modules. We show that if there exist $x_1, ..., x_n \in R$ such that $\mathrm{cd}\,(\mathfrak{a}+(x_1, ..., x_n), X) = \mathrm{cd}\,(\mathfrak{a}, X)+ n$, then $\dim_R(\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},X)}_{\mathfrak{a}}(X)) \geq n$ and so $\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},X)}_{\mathfrak{a}}(X)$ is not Artinian (Corollary 4.3.1). In Corollary 4.3.2, we improve [8, Proposition 3.2] by showing that over local ring $(R, \mathfrak{m})$ if $\mathfrak{a}$ is generated by a subset of system of parameters $x_1, ..., x_n$ of M, then $\dim_R(\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},M)}_{\mathfrak{a}}(M)) = \dim_R(M)-n$. In particular, if $n < \dim_R(M)$, then $\mathrm{H}^{\mathrm{cd}\,(\mathfrak{a},M)}_{\mathfrak{a}}(M)$ is not Artinian. For each integer r, $0 \leq r < \dim_R(M))$, we introduce $\mathcal{L}^r(M)$, the set of all ideals $\mathfrak{a}$ for which $\mathrm{H}^i_{\mathfrak{a}}(M)$ is not Artinian for some $i \geq r$. It is evident that if $\dim_R(M) > 0$ then $\mathcal{L}^r(M)$ is not empty. We show that any maximal element $\mathfrak{q}$ of $\mathcal{L}^r(M)$ is a prime ideal and that all Bass numbers of $\mathrm{H}^i_{\mathfrak{q}}(M)$ are finite for all $i \geq r$. We conclude that this statement generalizes [14, Corollary 2] (see Theorem 4.3.7 and Comment 4.3.8).

Some of the results of this thesis are contained in the following.

1) M. Aghapournahr, A. J. Taherizadeh, A. Vahidi, *Extension functors of local cohomology modules*, arXiv: 0903.2093v1 [math.AC].

2) M. T. Dibaei, A. Vahidi, *Artinian and non-Artinian local cohomology modules*, To appear in Canadian Mathematical Bulletin.

3) M. T. Dibaei, A. Vahidi, *Betti numbers of local cohomology modules with respect to an ideal*, Submitted.

4) S. H. Hassanzadeh, A. Vahidi, *On vanishing and cofiniteness of generalized local cohomology modules*, Comm. Algebra, **37** (2009), 2290–2299.