THE DISCRETIZATION OF HOMOLOGICAL PROPERTIES

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All the three lectures are based on joint work with Aldo Conca (Genova).

Let $R = K[X_1, \dots, X_n]$ be a polynomial ring over a field K. A monomial order on R is a total order on the set of monomials satisfying the following properties: (i) if $M \leq N$, then $MP \leq NP$ for all $M, N, P \in \mathcal{M}$; (ii) $1 \leq M$; (iii) every strictly descending chain in $\mathcal{M}$ terminates. The *initial monomial* $\mathrm{in}(f)$ of a polynomial f is the largest (with respect to $\leq$) monomial that appears in the representation of f as a linear combination of monomials.

For each vector subspace V of R we can form the vector space $\mathrm{in}(V)$ generated by the monomials $\mathrm{in}(f)$, $f \in V$. If I is an ideal, then $\mathrm{in}(I)$ is an ideal generated by monomials, and $\mathrm{in}(A)$ is a subalgebra generated by monomials if A is a K-subalgebra. More generally we can consider the *initial ideal* $\mathrm{in}(J) \subset \mathrm{in}(A)$ for an ideal $J \subset A$.

We consider $\mathrm{in}(I)$, $\mathrm{in}(A)$, and $\mathrm{in}(J)$ as *discrete objects* since they are "linearizations" of sets with a combinatorial description. Therefore one can try to employ combinatorial methods in the investigation of initial ideals and algebras. However, one is usually interested in properties of R/I or A, and the main goal of the lecture is a description of the "flat deformation" along which homological properties ascend from the initial $R/\mathrm{in}(I)$, $\mathrm{in}(A)$, or, more generally, from $\mathrm{in}(A)/\mathrm{in}(J)$ to R/I, A, and A/J.

INVARIANTS OF THE KNUTH-ROBINSON-SCHENSTED CORRESPONDENCE

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Let X be a matrix of indeterminates over the field K. The determinantal ideals $I_t = I_t(X)$ generated by the t-minors of X in the polynomial rings are classical objects of commutative algebra and algebraic geometry. The most powerful access to their structure is via standard bitableaux and the straightening law. It yields combinatorially well-described vector space bases of the ideals I_t and many objects derived from them. Standard bitableaux correspond (or are, depending on ones viewpoint) products of minors satisfying certain inequalities for the row and column indices involved.

However, it is not possible, or at least very difficult, to compute initial ideals and algebras directly from the standard basis, since the map $\Sigma \mapsto \mathrm{in}(\Sigma)$ is not well behaved for standard monomials Σ. There is a much more powerful map that assigns a monomial to each standard bitableau, the *Knuth–Robinson–Schensted correspondence* KRS. In order to make it useful for the computation of initial ideals one has to rewrite KRS(Σ) in the form in(Δ) where Δ is a product of minors. The connection between Σ and Δ is established via numerical invariants of KRS.

The determination of these invariants is a purely combinatorial problem. We will use them for the description of the initial ideals of the powers and symbolic powers of the ideals I_t.

ALGEBRAS DEFINED BY MINORS

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We are mainly interested in two types of algebras, namely the Rees algebra $\mathcal{R}_t = \mathcal{R}(I_t)$ of the ideal of t-minors and the subalgebra A_t of $R = K[X]$ generated by the t-minors of an $m \times n$-matrix X of indeterminates. These are closely connected, since A_t can be viewed as a retract of $\mathcal{R}_t$.

There are easy cases for A_t: namely $t = 1$ and $t + 1 = m = n$. These are exactly the cases in which A_t is isomorphic to a polynomial algebra. Also the case $t = \min(m, n)$ is well-known: then A_t is the homogeneous coordinate ring of a Grassmann variety; it is a factorial Cohen–Macaulay ring. The structure of A_t is much more complicated in all the other cases.

We will use the initial ideals $\mathrm{in}(I_t^k)$ in order to describe the initial algebras $\mathrm{in}(\mathcal{R}_t)$ and $\mathrm{in}(A_t)$. The discrete objects turn out to be normal semigroup rings. Moreover, one can compute their canonical modules. This allows us to show that the algebras $\mathcal{R}_t$ and A_t are Cohen–Macaulay (if char$K$ is 0 or big enough) and to compute the divisor classes of their canonical modules. It turns out that A_t is Gorenstein if and only if one of the following condition holds: (i) $t = 1$ or $t + 1 = m = n$ (the regular cases), $t = \min(m, n)$ (the classical case of Grassmannians), or (iii) $1/t = 1/m+1/n$.