HOMOLOGICAL DIMENSIONS OF COMPLEXES OF MODULES

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The classical homological dimensions of modules can be extended to complexes of modules. For example, there is a general Intersection Theorem

$$\dim_R M \leq \dim(M \otimes_R^{\mathbf{L}} N) + \operatorname{flat\,dim}_R N \qquad (*)$$

whenever M and N are non-trivial bounded complexes of finitely generated modules over a local ring R; if the ring is equicharacteristic, the modules in N need not be finitely generated. If $M = R$ and $\mathrm{H}(N)$ has finite length, then $(*)$ is the New Intersection Theorem. Furthermore, work with Iyengar has resulted in an amplitude inequality for unbounded complexes; this has applications for local ring homomorphisms. ——————

RESTRICTED HOMOLOGICAL DIMENSIONS

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In addition to the classical homological dimensions there are restricted variants. For example, for any module M over a commutative Noetherian ring R, the restricted flat dimension $\mathsf{Rfd}_R\, M$ is given as the supremum of $n \in \mathbb{N}_0$ such that $\operatorname{Tor}_n^R(M, T) \neq 0$ for some R-module T with $\operatorname{flat\,dim}_R T$ finite. $\mathsf{Rfd}_R\, M \leq \dim R$ holds always. Rfd_R is a refinement of $\operatorname{flat\,dim}_R$ in the sense that for any R-module M there is an inequality $\mathsf{Rfd}_R\, M \leq \operatorname{flat\,dim}_R M$ with equality if the latter is finite; we write $\mathsf{Rfd}_R \preceq \operatorname{flat\,dim}_R$. Actually, Holm has shown that $\mathsf{Rfd}_R \preceq \text{G-flat}\dim_R \preceq \operatorname{flat\,dim}_R$. Furthermore, over finite modules we have $\mathsf{Rfd}_R \preceq \text{CM-dim}_R \preceq \text{G-dim}_R \preceq \text{CI-dim}_R \preceq \operatorname{proj\,dim}_R$. In general, there is the Ultimate Auslander–Buchsbaum Formula:

$$\mathsf{Rfd}_R\, M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\,.$$

LOCAL RING HOMOMORPHISMS

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A local ring homomorphism $\varphi\colon R \to S$ is said to be quasi-Gorenstein, if $D \otimes_R^{\mathbf{L}} S$ is a dualizing complex for S when D one for R. These homomorphisms have excellent ability of transferring the Gorenstein property from the source R to the target S, and vice versa: R is Gorenstein and φ is quasi-Gorenstein if and only if S is Gorenstein and φ is of finite G–dimension. This part is work with Avramov.

Furthermore, when $\varphi\colon R \to S$ is a local ring homomorphism and M is an almost finite non-trivial S-module with $\operatorname{flat\,dim}_S M$ finite, then there is an inequality $\operatorname{flat\,dim}_R S \leq \operatorname{flat\,dim}_R M$ (in addition to the well-known $\operatorname{flat\,dim}_R M \leq \operatorname{flat\,dim}_R S + \operatorname{flat\,dim}_S M$); in particular, $\operatorname{flat\,dim}_R S$ and $\operatorname{flat\,dim}_R M$ are finite simultaneously, and it follows that R is regular if S is so. This part is work with Iyengar.