Numerical Analysis of a Convection–Diffusion Problem: the Vlasov–Poisson–Fokker–Planck System

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In this note we investigate the global convergence of numerical approximations for the solution of a deterministic Vlasov–Poisson–Fokker–Planck system: a degenerate type convection dominated convection-diffusion equation formulated viz: Given the initial distribution of particles $f_0(x,v) \geq 0$, in the phase–space $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, $d = 1,2,3$, and the physical parameter $\sigma \geq 0$, find the distribution function $f(x,v,t)$ for $t > 0$, satisfying the nonlinear system of evolution equations

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f - \sigma \Delta_v f = 0, & f(x,v,0) = f_0(x,v), \\ E(x,t) = C_d \displaystyle\int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} \rho(y,t)\, dy, & \rho(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t)\, dv. \end{cases} \tag{1}$$

where $\rho(x,t)$ is the spatial density. The macroscopic force field E can also be assumed to be of the form

$$E(x,t) = -\nabla_x \phi(x,t), \qquad \text{with} \quad \Delta_x \phi(x,t) = -\int_{\mathbb{R}^d} f(x,v,t) dv = -\rho(x,t). \tag{2}$$

The common procedure in solving this system is to split the problem to solving the Poisson equation (2) for ϕ in order to determine E, and then continue with the linear Fokker–Planck equation:

$$f_t + v \cdot \nabla_x f + E \cdot \nabla_v f - \sigma \Delta_v f = g, \quad f(x,v,0) = f_0(x,v), \tag{3}$$

where $E(x,v,t) = \Big(E_i(x,v,t)\Big)_{i=1}^{d}$, is a given vector field and $f_0(x,v)$ and $g(x,v,t)$ are given functions. Equation (1) arises in the kinetic description of a plasma of Columb particles. Existence, uniqueness and stability properties for (1) are based on results on degenerate type problems due to J. L. Lions.

The mathematical study of the VPFP system has been considered by several authors in various settings. Compared to the analytical studies the numerical analysis of the VPFP system is much less developed. In the deterministic approaches the dominant part of numerical studies are using method of characteristics: basically particle methods developed for the Vlasov–Poisson equation. Our goal is to construct and analyse finite element schemes for the VPFP and give numerical results for a Fermi pencil beam problem.