Decompositions of Edge-Colored Complete Graphs and Applications to 2-Designs and Universal Algebra

Richard M. Wilson
Department of Mathematics
California Institute of Technology
USA

Let $K_n^{(r)}$ be the complete directed graph on n vertices with exactly one edge of color i joining any vertex x to any other vertex y for every color i in a set of r colors. A family $\mathcal{F}$ of subgraphs of a graph $K_n^{(r)}$ will be called a *decomposition* of $K_n^{(r)}$ provided that each of the $rn(n-1)$ edges belongs to exactly one member of $\mathcal{F}$.

We give examples of how certain decompositions of $K_n^{(r)}$ are connected to 2-designs and related structures. One is usually interested in decompositions into 'copies' of a certain graph, or a family of allowed graphs.

Decompositions of $K_p^{(r)}$ that are cyclic (i.e. admit an automporphism which permutes the p vertices in a single cycle) can be obtained for large primes p using 'cyclotomy'. We show how this result leads, for every fixed block size k, to the construction of infinitely many cyclic Steiner systems $S(2,k,v)$ with $v \equiv 1 \pmod{k(k-1)}$, infinitely many cyclic Steiner systems $S(2,k,v)$ with $v \equiv k \pmod{k(k-1)}$, and also infinitely many 1-rotational Steiner systems $S(2,k,v)$ (with an automorphism that fixes one point and permutes the remaining $v-1$ points in a single cycle). We can construct the 1-rotational Steiner systems so that they are also resolvable.

In most applications, we are interested in decompositions of $K_n^{(r)}$ into *simple* digraphs (where there are no loops and at most one edge directed from x to y for any ordered pair of distinct vertices x and y). We give one result about decompositions into not necessarily simple graphs and illustrate with an application in universal algebra: we show that certain classes of quasigroups cannot be 'equationally defined'.