Primes and Irreducibles in Exponential Integer Parts of Ordered Exponential Fields

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An integer part (IP for short) $Z$ of an ordered field $F$ is a discretely ordered subring, with 1 as least positive element, and such that for every $x \in F$, there is a $z \in Z$ such that $z \leq x < z + 1$.

Shepherdson [S] showed that IP's of real closed fields correspond to models of a fragment of Peano Arithmetic called Open Induction (OI for short). He used this observation to investigate the arithmetic properties of these rings. For example, he constructed an IP for the field of algebraic Puiseux series (with coefficients in the field of real algebraic numbers, and exponents in the group of rational numbers) with only standard irreducibles. In particular, the set of primes is not cofinal in this countable recursive model. Thus the "infinity of primes" is not provable from OI.

On the other hand, subsequent to his work, several authors (e.g. [M], [B–O]) constructed such recursive rings with unboundedly many infinite primes.

In [M–R], the authors establish the existence of an IP for any real closed field. In [R], a proof for an exponential analogue is sketched: every exponential field (see [K]) has an *exponential integer part* (EIP for short; an EIP is an IP that satisfies moreover some closure conditions under the exponential function, see [R] for exact definition).

For fields of generalized power series $K = \mathbb{R}((G))$, an integer part is given by the ring $\mathbb{R}((G^{<0})) \oplus \mathbb{Z}$ (cf. [K; Chap. 1 Sect. 5] and [B] for notation and terminology). In [B], the author characterizes irreducible elements in this ring by the order type of their support. In the Concluding Remarks, he asks whether this criterium for irreducibility applies in the presence of exponentiation. More precisely, one should investigate whether *every* EIP of an exponential field has unboundedly many irreducibles (primes).

We will first review the main notions and the background for the problems. We shall then turn to results of [K–K] on EIP's of ordered exponential fields. We shall focus on two particulary interesting cases: EIP's of *countable* exponential fields (for the structure of these fields, see [K; Chap. 1 Sect. 7]), and those of *Exponential- Logarithmic power series fields* (see [K; Chap. 5 Sect. 2] for the construction of these models).

For the countable models (the "Exponential Algebraic Puiseux Series Fields") described in [K; Example 1.45], we construct a "canonical" EIP. Based on generalizations of results of [B], we show that this EIP has cofinally many *irreducibles with finite support.*

Using the above argument and the result of [R], we establish that *every exponential field has an EIP with cofinally many irreducibles.* This also appears in [Bi2; Theorem 18], based on results claimed in [Bi1].

Similarly, we construct a "canonical" EIP for the Exponential- Logarithmic power series fields. Generalizing results of [P], we show that this EIP has cofinally many *primes with infinite support.*

We finish by a discussion of a possible approach to answer Berarducci's question mentioned above.

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