No Escape from Vardanyan's Theorem

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The idea of assigning structures of kind Y to structures of kind X in order to obtain information is a useful methodology in mathematics. The structures of kind Y are often simpler and better understood. Provability logic is a case in point: we assign to arithmetical theories certain propositional modal logics that are in many respects simpler than the original theories. These modal logics give us full information about arithmetical reasoning of a certain restricted sort. A disadvantage of this approach is that provability logic does not yield *differential* information: the modal logic assigned is the same for a wide class of reasonable theories.

The situation becomes a bit better when we expand the modal language. One possible expansion is to a predicate logical modal language. Here we certainly do have differential information about the original theories. Vardanyan's theorem concerns this situation. Vardanyan's Theorem tells us that the set of PA-valid principles of Quantified Modal Logic, QML, is complete Π^0_2. Thus, the PA-valid modal principles are more complicated than the theory PA itself. Thus, the result can be viewed as a negative result for the case of PA.

One might hope that for other theories than PA the situation would be a bit better. No such luck: we will show that Vardanyan's Theorem extends to a wide range of theories. The main idea of the proof is to avoid the use of Tennenbaum's Theorem.

FAITH & FALSITY

Lecture for the Workshop

Logic, Algebra & Arithmetic

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Albert Visser

Comparing Theories

Interpretability:
$M : T \rhd U :\Leftrightarrow$
$\forall C \in \mathcal{S}_U \;\; U \vdash C \Rightarrow T \vdash C^M$.

Notations:

- $M : U \lhd T :\Leftrightarrow M : T \rhd U$,
- $U \stackrel{M}{\rightarrow} T :\Leftrightarrow M : T \rhd U$,
- $T \rhd U :\Leftrightarrow \exists M \;\; M : T \rhd U$,
- $T \equiv U :\Leftrightarrow T \rhd U$ and $U \rhd T$.

Interpretability is good if you are talking about consistency:

- $T \rhd U \Rightarrow (\mathsf{con}(T) \Rightarrow \mathsf{con}(U))$.
- $(T + (\mathsf{Q} \wedge \mathsf{con}(U))^K) \rhd U$.
  (Interpretation Existence Lemma)
- $T \rhd (T + (\mathsf{Q} \rightarrow \mathsf{incon}(T + \mathsf{Q}^K))^K)$.
  (Second Incompleteness Theorem)
- $\equiv$ preserves reflexivity.
- $\mathsf{GB} \equiv (\mathsf{Q} + \mathsf{con}(\mathsf{ZF}))$.

About decidability:

- $\lhd$ preserves (left to right) essential undecidability.
- $\mathsf{AG} \equiv \mathsf{FOL}_2$.
  Hence $\equiv$ does not preserve decidability.

Faithful Interpretability

$M : T \rhd_{\mathsf{f}} U :\Leftrightarrow$
$\forall C \in \mathcal{S}_U \;\; U \vdash C \Leftrightarrow T \vdash C^M.$

Suppose T is decidable and $T \rhd_{\mathsf{f}} U$, then U is decidable.

Examples

We have:

$\mathsf{G}^c \equiv_{\mathsf{f}} \mathsf{FOL}_2$.

$\mathsf{Q}^- \equiv_{\mathsf{f}} \mathsf{FOL}_2$.

(But Q^- is not recursively boolean isomorphic to FOL_2.)

Comparing Interpretability and Faithful Interpretability

The embedding functor of the ppo of faithful interpretability into the ppo of interpretability has a right adjoint.

$$U \lhd_{\mathsf{f}} \widetilde{V} \Leftrightarrow \mathsf{emb}(U) \lhd V.$$

W is *trustworthy* iff
whenever $W \rhd U$ then $W \rhd_{\mathsf{f}} U$.

V is trustworthy iff $V \equiv_{\mathsf{f}} \widetilde{V}$.

Examples:

theory	trustworthy?
PA	*yes*
$\mathsf{PA} + \mathsf{incon}(\mathsf{PA})$	*no*
$I\Sigma_1 + \mathsf{incon}(I\Sigma_1)$	*yes*
G^c	*yes*
AG	*no*

Theorem (Friedman)
Suppose W is consistent, finitely axiomatized and sequential, then W is trustworthy.

We even have the stronger property of *solidity*:

$$W \equiv V \Rightarrow W \equiv_{\mathsf{f}} V.$$

Examples:

theory	solid?
PA	*no*
$I\Sigma_1 + \mathsf{incon}(I\Sigma_1)$	*yes*
G^c	*no*

Characterization Theorem
W is trustworthy iff $W \rhd_{\mathsf{f}} \mathsf{FOL}_2$.

Application

$\widetilde{V}$ is obtained by expanding the signature with unary P and binary R. Relativize the quantifiers in V to P. Then

$$V \equiv \widetilde{V} \rhd_{\mathsf{f}} \mathsf{FOL}_2.$$

Alternative: no P, but replace $=$ in T by binary E.

If V has infinite model then we need neither P nor E.

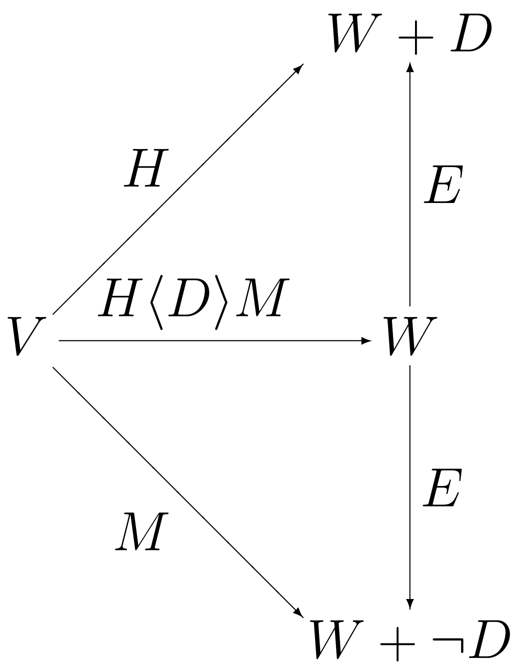
We indicate some crucial ideas of the proof the characterization theorem.

Proof (of "$\Leftarrow$"):
Suppose $K : W \rhd_{\mathsf{f}} \mathsf{FOL}_2$. There is an interpretation ω such that $\mathsf{FOL}_2 + \mathsf{Q}^{\omega}, \omega$ is Σ-sound.

Hence, $W + \mathsf{Q}^{\omega K}$ is Σ-sound, w.r.t. $K' := K \circ \omega$.

Suppose $M : W \rhd V$ and $V \nvdash C$. We show how to construct $M^* : W \rhd V$ s.t. $W \nvdash C^{M^*}$.

Let $D := (\mathsf{Q}^{K'} \wedge \mathsf{con}^{K'}(V + \neg C))$. Let $M^* := H\langle D \rangle M$.



Suppose $W \vdash C^{M^*}$. Then, $W + D \vdash C^H$. On the other hand $W + D \vdash \neg C^H$. Ergo, $W \vdash \neg D$. I.o.w.

$$W + \mathsf{Q}^{K'} \vdash \mathsf{prov}_V^{K'}(C).$$

By Σ-soundness: $V \vdash C$. Quod non. Ergo: $W \nvdash C^{M^*}$.

We generalize this construction for all C simulaneously. This employs the usual diagonal trickery. ❑

Corollary
Every RE theory W is finitely axiomatizable in FOL_2 modulo an interpretation, i.e., there are A and M such that:

$$W \vdash B \Leftrightarrow \mathsf{FOL}_2 + A^M \vdash B^M.$$

Inspection of the proof of the characterization theorem shows:
W is trustworthy iff there is an interpretation K such that $W + \mathsf{Q}^K$ is Σ-sound w.r.t. K.

A New Proof of a Theorem of Friedman

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History:

19xx Friedman shows that consistent, finitely axiomatized, sequential theories are trustworthy.

1985 Smoryński reports Friedman's result —without proof.

1987 Krajíček proves that for every consistent, finitely axiomatized, sequential theory T, there is a T-cut I not containing the inconsistency-statement for T.

1993 Visser provides a purely syntactical proof of Krajíček's theorem.

2002 Visser realizes that Friedman's result and a version of Krajíček's result are 'equivalent'.

To prove:
Suppose U is consistent, finitely axiomatized and sequential. To show: U is trustworthy.

Let $N : T \rhd \mathsf{F}$, where F is a suitable fragment of arithmetic.

Sufficient:
There is N-cut I s.t., for all Σ^0_1-sentences S:

$$T \vdash S^I \Rightarrow S \text{ is true.}$$

S^I is pro-version of S or ‘boosted’ S: more difficult to get true, since a smaller witness is demanded.

Idea 1:
Can we use that other boosting trick: the FGH-theorem?

FGH Theorem
Suppose $N : T \rhd \mathsf{F}$. Let S be Σ^0_1. Take R be such that:

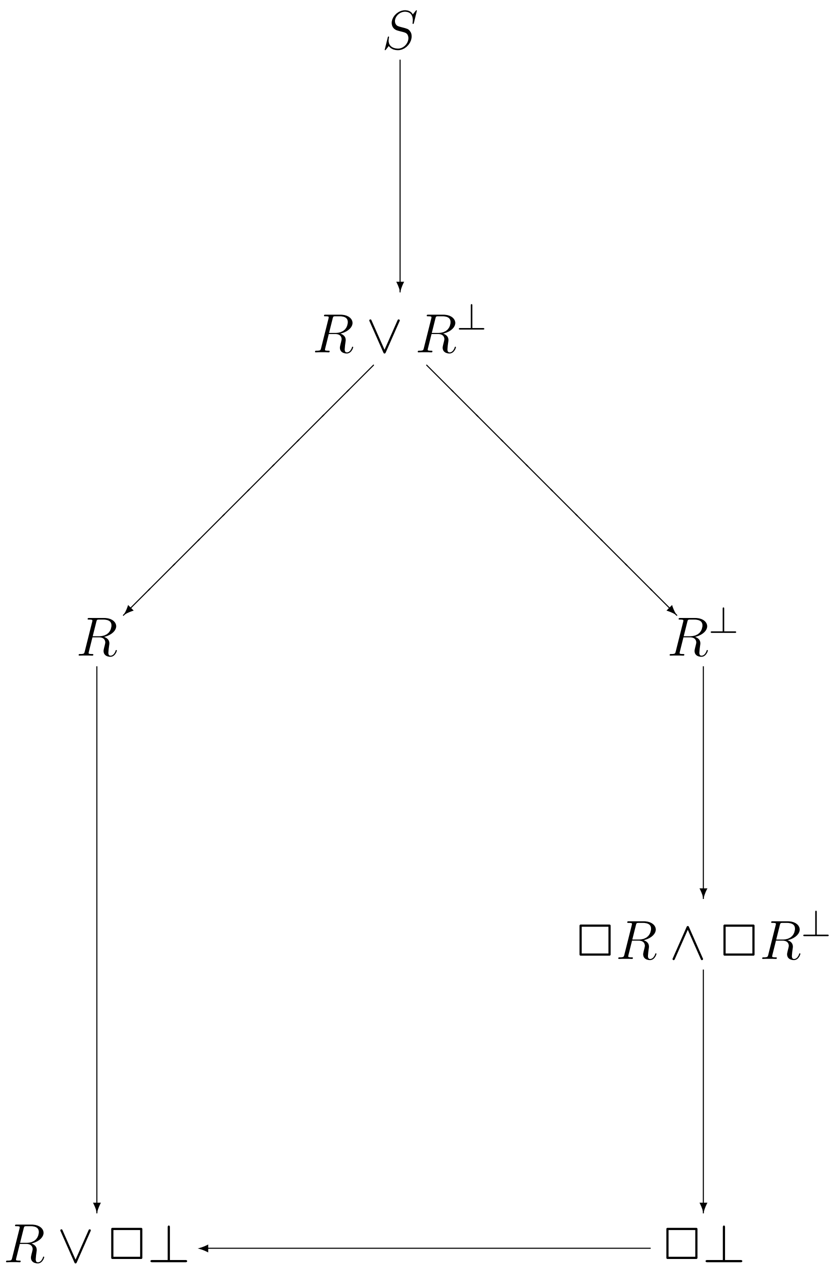
$$\mathsf{Q} \vdash R \leftrightarrow S \leq \Box_T R^N.$$

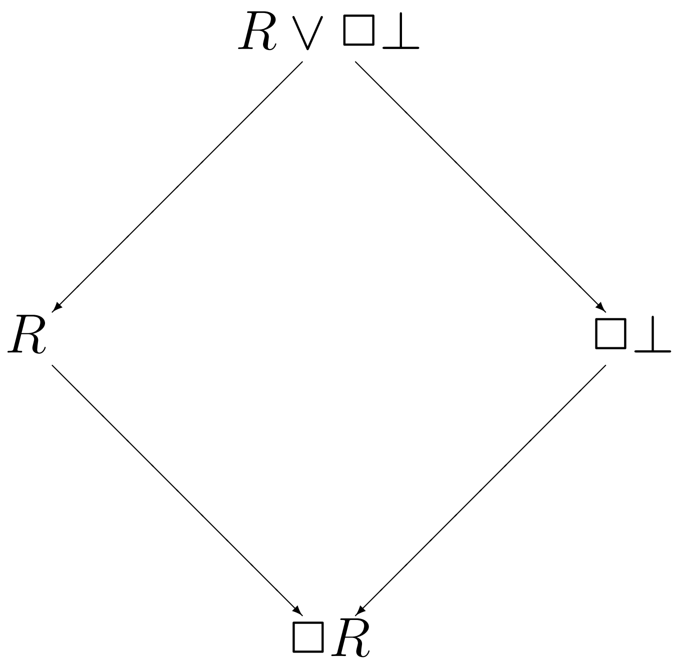
We have:

$$\begin{aligned} \mathsf{EA} \vdash (S \vee \Box_T \bot) &\leftrightarrow (R \vee \Box_T \bot) \\ &\leftrightarrow \Box_T R^N \end{aligned}$$

or, equivalently,

$$\mathsf{EA} + \mathsf{con}(T) \vdash$$
$$(S \leftrightarrow R) \wedge (S \leftrightarrow \Box_T R^N).$$


S
$R \vee R^{\perp}$
R
$R^{\perp}$
$\Box R \wedge \Box R^{\perp}$
$R \vee \Box\bot$
$\Box\bot$


R ∨ □⊥
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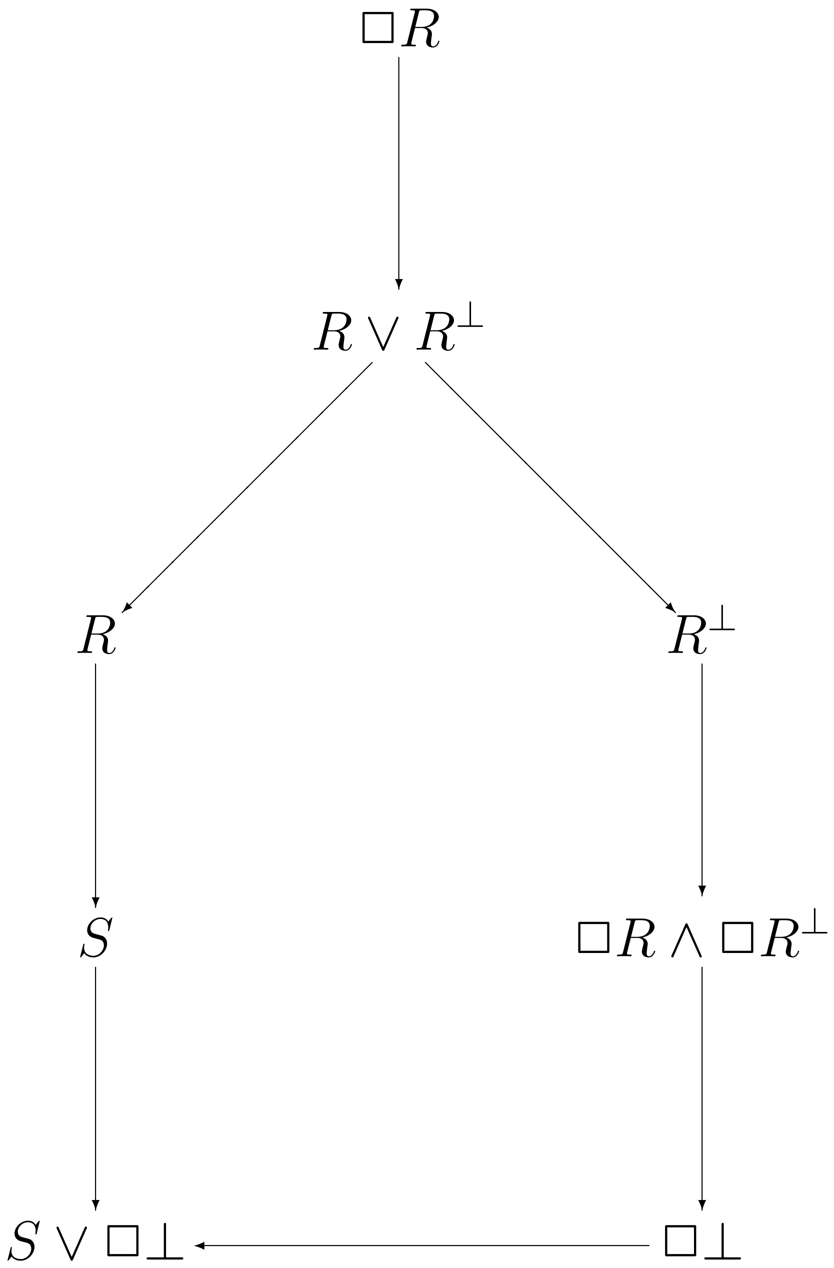

□R
R ∨ R^⊥
R
R^⊥
S
□R ∧ □R^⊥
S ∨ □⊥
□⊥

idea 2

Use restricted provability.

Suppose T is finitely axiomatized. FGH-theorem also works for $\Box_{T,n}$ with:

$$\mathsf{Q} \vdash R_n \leftrightarrow S \leq \Box_{T,n} R_n^N.$$

$\mathsf{EA} + \mathsf{con}^N(T) \vdash$
$\Box_{T,n} R_n^N \rightarrow S.$

$\mathsf{EA} + \mathsf{supexp} + \mathsf{con}^N(T) \vdash$
$\Box_T R_n^N \rightarrow S.$

n should be large enough: this only depends on the complexities of T, N, S.

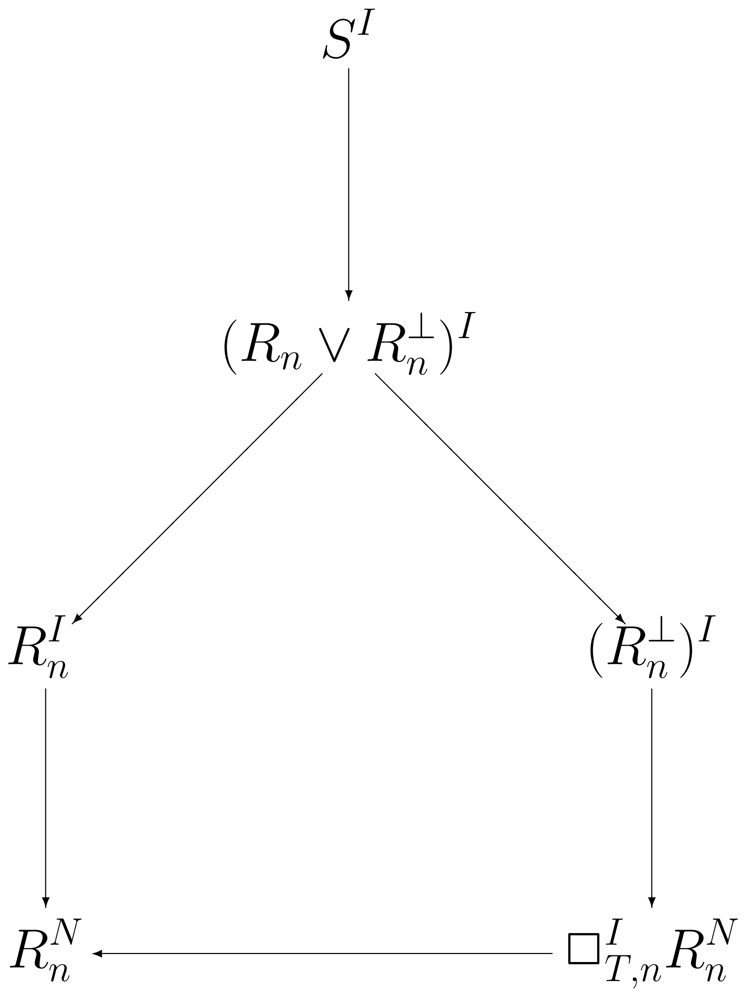
Idea 3

Suppose T is finitely axiomatized and sequential. Then, there is N-cut I s.t.
$T \vdash \Box^{I}_{T,n} A \to A$.

We can make it work for fixed S: Pick n large enough. Find n-reflecting I. We show:

$$U \vdash S^{I} \to R^{N}_{n}.$$

Reason in U.


S^I
$(R_n \vee R_n^\perp)^I$
R_n^I
$(R_n^\perp)^I$
R_n^N
$\Box_{T,n}^I R_n^N$

So for every S, we can find such an I. How to eliminate dependence on S?

Idea 4

Use Σ^0_1-truth predicate true_Σ.

$\mathsf{EA} \vdash S \leftrightarrow \mathsf{true}_\Sigma(\underline{\#S})$.

For any N-cut I we can find subcut J s.t.

$$U \vdash S^J \rightarrow \mathsf{true}^I_\Sigma(\underline{\#S}).$$

Complexity of $\mathsf{true}_\Sigma(\underline{\#S})$ is independent of S.

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Propositional Provability Logic

Löb's Logic GL.

L1 $\vdash \phi \Rightarrow \vdash \Box\phi$

L2 $\vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

L3 $\vdash \Box\phi \to \Box\Box\phi$

L4 $\vdash \Box(\Box\phi \to \phi) \to \Box\phi$

Let T be an arithmetical theory. Let $(\cdot)^*$ from the modal language to the arithmetical language satisfy:

- $(\cdot)^*$ commutes with the propositional connectives,
- $(\Box\phi)^* := \mathsf{prov}_T(\underline{\sharp\phi^*})$

$\mathsf{Prl}_T := \{\phi \mid \forall(\cdot)^* \;\; T \vdash \phi^*\}$.

Theorem
[Löb, Henkin][Wilkie & Paris, Buss] $\mathsf{GL} \subseteq \mathsf{Prl}_T$, for all theories T with p-time decidable axiom set extending Buss' theory S^1_2.

Theorem [Solovay]
$\mathsf{Prl}_T = \mathsf{GL}$
where T is Σ-sound, p-time axiomatized and extends EA.

Open Problem
What is the provability logic of S^1_2, $I\Delta_0 + \Omega_1$?

Can we Escape this Awesome Stability?

Extend the modal language:
E.g. with $\phi \rhd \psi$ for:
$(T + f^*\phi) \rhd (T + f^*\psi)$.
Or: to predicate logic ...

Consider another underlying logic:
E.g. study the provability logic of HA.

Leivant's Principle
$\vdash \Box(\phi \vee \psi) \rightarrow \Box(\phi \vee \Box\psi)$.

The case of Predicate Logic
Let's call the corresponding logic $\mathsf{Prl}_{\mathsf{pred}}(T)$.

Theorem (Vardanyan, McGee)
$\mathsf{Prl}_{\mathsf{pred}}(\mathsf{PA})$ is complete Π^0_2.

Inspection shows that Vardanyan's theorem generalizes to all extensions of $I\Sigma_1$ that do not prove their own inconsistency.

Can we extend this to more theories?

Main obstacle: all proofs use formalization of Tennenbaum's Theorem. This uses too much induction.

So we look for a Tennenbaum-free proof.

Sketch of the proof

Consider theory T and Language for modal predicate logic with:
Z, S, A, M, E, X.

Define:

- $\nu_0(z) := \mathsf{Z}(z)$,
- $\nu_{n+1}(z) := \exists u\ (\nu_n(u) \wedge \mathsf{S}(u, z))$.

We write $C(\breve{n})$ for:
$\exists z\ (\nu_n(z) \wedge C(z))$.

We construct A and B in this language.

- A is X-free.
- $\vdash A \to \mathsf{Q}$.
- $T + A^{\mathsf{e}}$ is sufficiently sound.
- For every $(\cdot)^*$, there is an n such that:
  $T + A^* \vdash B(\breve{n}, X)^*$.
- There is class Γ s.t.
  for every m, there is $C \in \Gamma$, s.t. if $B(\breve{n}, X)^{[X:=C]}$ is true, then $n \geq m$.

Let $P := \forall x\, \exists y\ P_0(x, y)$,
$P_0 \in \Delta_0$. Take:

$$\tilde{P} := (A \to \exists z\ (B(z, X) \wedge \forall x{<}z\, \exists y\ P_0(x, y))).$$

To show: $P \Leftrightarrow \tilde{P} \in \mathsf{Prl}_{\mathsf{pred}}(T)$.

Proof from the conditions:

"$\Rightarrow$" supose P. Consider $(\cdot)^*$. There is n s.t.:

(a) $T + A^* \vdash B(\breve{n}, X)^*$.

We have:
$\forall x{<}n\, \exists y\ P_0(x, y)$.
Hence:
$\mathsf{Q} \vdash \forall x{<}\underline{n}\, \exists y\ P_0(x, y)$.
Ergo:
(b) $T{+}A^* \vdash (\forall x{<}\breve{n}\, \exists y\ P_0(x, y))^*$.

Combining (a) and (b):
$T \vdash (\tilde{P})^*$.

"$\Leftarrow$" Suppose $\tilde{P} \in \mathsf{Prl}_{\mathsf{pred}}(T)$.
Consider any m.
Pick C the formula in Γ for m. We find that:

$$T \vdash (A \to (\exists z\, B(z,X) \wedge \forall x{<}z\, \exists y\, P_0(x,y)))^{\mathsf{e}[X:=C]}.$$

Assuming that A is sufficiently sound, we find:

$$\exists n\, B(n,X)^{\mathsf{e}[X:=C]} \wedge \forall x{<}n\, \exists y\, P_0(x,y).$$

But $n \geq m$.

Our A, B, Γ:

- $A := \boxdot(\mathsf{EA} \wedge \forall y\,(\mathsf{Z}(y) \to \Box \mathsf{Z}(y)) \wedge \forall y \forall z\,(\mathsf{S}(y,z) \to \Box \mathsf{S}(y,z)))$,
- $B := \Box(\Box X \leftrightarrow \Box \mathsf{True}_{\Sigma^0_1}(x))$.
- $\Gamma = \{\Box^n_T \bot \mid n \in \omega\}$.

FGH-theorem

$\forall A\, \exists R \in \Sigma^0_1\ \mathsf{EA} \vdash \Box_U A \leftrightarrow \Box_U R^*$.

- A is X-free. *OK*
- $\vdash A \to \mathsf{Q}$. *OK*
- $T + A^{\mathsf{e}}$ is sufficiently sound.
  Assume T is Σ-sound and contains EA*.*
- For every $(\cdot)^*$, there is an n such that: $T + A^* \vdash B(\breve{n}, X)^*$.
  Use FGH-theorem.
- For every m, there is $C \in \Gamma$, s.t. if $B(\breve{n}, X)^{[X:=C]}$ is true, then $n \geq m$.
  Or else we get:
  $T \vdash \Box_T^i \bot \leftrightarrow \Box_T^j \bot$,
  for $j > i$. Contradicting Löb's theorem and Σ-soundness.