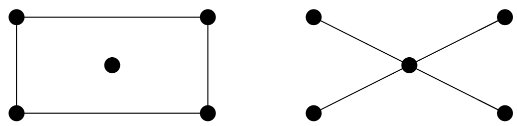
IPM Combinatorics II: Design Theory, Graph Theory, and Computational Methods
April 22-27, 2006, IPM, Tehran

Spectral Characterizations of Graphs

W. H. Haemers
Tilburg University
Tilburg, The Netherlands

The spectrum of a (finite) graph is the multiset of the eigenvalues of its adjacency matrix. Clearly the spectrum is determined by the graph, but the converse is in general not true. The two graphs below, for example, both have spectrum $\{-2\ ,\ 0\ (3\times),\ 2\}$.



A key question is, whether this phenomenon is an exception or not. This question is addressed in [2]. Here we report on old and new developments. There are several construction methods for nonisomorphic cospectral graphs, and it is known that for almost all trees there exists another tree with the same spectrum (see [5]). However, the fractions of graphs on n vertices for which such a cospectral mate is known tends to 0 if $n \to \infty$. On the other hand, it is difficult to show for a given graph G that there exists no graph nonisomorphic but cospectral with G. Such a graph is called DS (determined by its spectrum). Several special graphs are proven to be DS. This includes the complete graphs K_n, the regular complete bipartite graphs $K_{k,k}$ (note that the above example gives a counter example for a nonregular complete bipartite graph), the cycles and the line graphs of $K_{k,k}$ and K_n provided $k \neq 4,\ n \neq 8$. A typical proof of such a spectral characterization goes in two steps. First one obtains structural properties from the spectrum and then it is shown that these properties determine the graph. For distance-regular graphs (DRG's for short) this approach is very suitable. First one needs to prove that a graph with the spectrum of a DRG is a DRG, and in the second step one needs to show that the DRG is uniquely determined by its parameters (intersection array). For many DRG's the second step is given in [1]. Here the game is to take a known DRG, and try to find a nonisomorphic cospectral mate, or show that it does not exist. Part of the talk will report on recent progress in this game [3].

The paradox is, that the above approach for proving that a given graph is DS, needs some nice structures (like being a DRG). However, most of the time these graphs turn out to be not DS. On the other hand computer investigations on graphs up to 11 vertices indicate that it is conceivable that almost all graphs are DS. Therefore one would like a tool to check whether an arbitrary graph is DS. Such a tool has recently been developed by Wang and Xu [6], and we will also pay attention to this development.

References

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