IPM Combinatorics II: Design Theory, Graph Theory, and Computational Methods
April 22-27, 2006, IPM, Tehran

The Frankl-Wilson Inequalities and P-ary t-designs

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Let $L \subseteq \{0, 1, \ldots, k-1\}$. A family $\mathcal{F}$ of k-subsets of a v-set X is said to be *L-intersecting* when $|A \cap B| \in L$ for all distinct $A, B \in \mathcal{F}$. For $S \subseteq X$, we use $\lambda(S)$ to denote the number of members of $\mathcal{F}$ that contain S, counting multiplicities if $\mathcal{F}$ is a multiset. We define a *p-ary t-design* with parameters (v, k, λ_0) to consist of a v-set X and a family $\mathcal{F}$ of k-subsets of X so that $\lambda(T) \equiv \lambda_0 (\text{mod } p)$ for every t-subset S of X. We prove

Theorem. *Let p be a prime and suppose $f(x)$ is a rational polynomial of degree d, $d \leq k$, so that $f(\ell)$ is an integer $\equiv 0 \pmod p$ for $\ell \in L$, but $f(k)$ is an integer $\not\equiv 0 \pmod p$. Then for any L-intersecting family $\mathcal{F}$ of k-subsets of a v-set,*

$$|\mathcal{F}| \leq \binom{v}{d}.$$

If equality holds, then $\mathcal{F}$ is the set of blocks of a p-ary t-design for $t = d, d+1, \ldots, 2d$.

The first part of the theorem is from a 1984 paper of P. Frankl and the author, and it implies that $|\mathcal{F}| \leq \binom{v}{|L|}$. The second part motivates us to investigate p-ary t-designs.

It should be noted that while an (ordinary) t-design is also an s-design for $s < t$, a p-ary t-design is not necessarily also a p-ary s-design for $s < t$. Given v, k, and p, we characterize the sets of integers $J \subseteq \{1, 2, \ldots, k-1\}$ so that there exist families $\mathcal{F}$ which are p-ary s-designs for $s \in J$ and *not* p-ary s-designs for $s \in \{1, 2, \ldots, k-1\} \setminus J$. We also consider Fisher-type inequalities on the number of blocks of p-ary t-designs.

Our primary tool is the algebra of higher inclusion matrices.