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Heat Kernels and the Riesz Transform

(3 Lectures)

T. Coulhon
University of Cergy-Pontoise
France

Let M be a complete non-compact Riemannian manifold, μ the Riemannian measure, r the Riemannian gradient, and ∇ the (positive) Laplace-Beltrami operator on M. Denote by $|.|$ the length in the tangent space, and by $||.||_p$ the norm in $L^p(M;\mu)$, $1 \leq p \leq \infty$. It was asked in by Strichartz in 1983 for which complete non-compact Riemannian manifolds M and which $p \in (1,\infty)$ one has

$$C_p^{-1}||\Delta^{\frac{1}{2}} f||_p \leq ||\nabla f||_p \leq C_p||\Delta^{\frac{1}{2}} f||_p. \tag{1}$$

For $p = 2$, on any complete Riemannian manifold, one has the equality

$$||\nabla f||_2 = ||\Delta^{\frac{1}{2}} f||_2$$

and this may even be used to define the Laplace-Beltrami operator. For $p \neq 2$, already in the Euclidean space, the above equivalence of semi-norms is by no means a trivial matter, and a lot of work has been devoted to its proof for several classes of manifolds. In the last few years, it has appeared that, at least in the class of manifolds with the doubling property, the validity of (1) is strongly related with estimates of the heat kernel and its gradient. The aim of this course will be to explain these results.