Discrete Spectral Geometry

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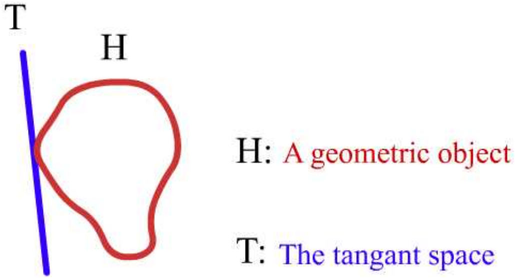
Hossein Hajiabolhassan is mathematically dense in this talk!!

May 31, 2006

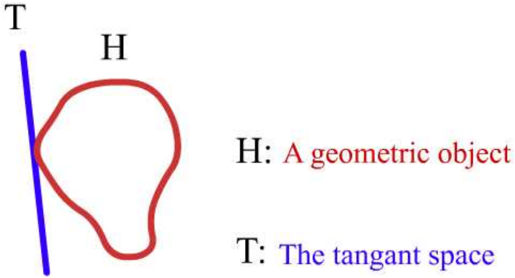
Outline

Basic Objects



- H is a geometric objects, we are going to analyze.

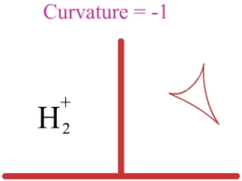
Basic Objects



- There are usually some generic (i.e. well-known, typical, close at hand, important, ...) types of these objects.

Basic Objects (continuous case)

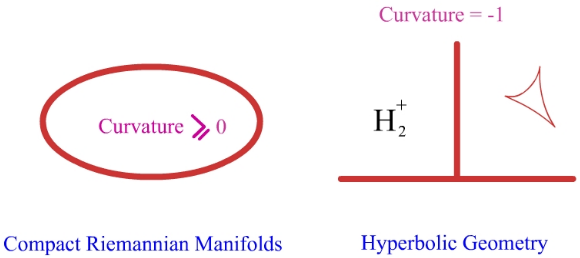
Compact Riemannian Manifolds



Hyperbolic Geometry

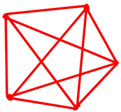
- These are models and objects of Euclidean and non-Euclidean geometry.

Basic Objects (continuous case)

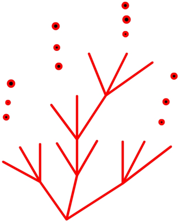


- Two important generic objects are the sphere and the upper half plane.

Basic Objects (discrete case)



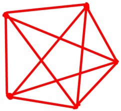
Complete graph



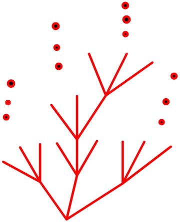
k-regular tree

- Some important generic objects in this case are complete graphs, infinite k-regular tree and fractal-meshes in $\mathbf{R}^n$.

Basic Objects (discrete case)



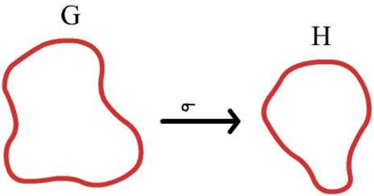
Complete graph



k-regular tree

- These are models and objects in network analysis and design, discrete state-spaces of algorithms and discrete geometry (e.g. in geometric group theory).

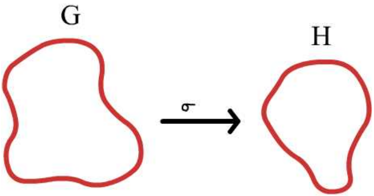
Comparison method



How different is G?

- The basic idea here is to try to understand (or classify if we are lucky!) an object by comparing it with the generic ones.

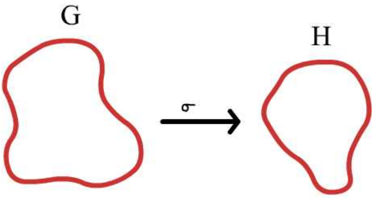
Comparison method



How different is G?

- The space we consider is the space of natural (structure preserving) maps as $\sigma \in \mathrm{Hom}(G, H)$.

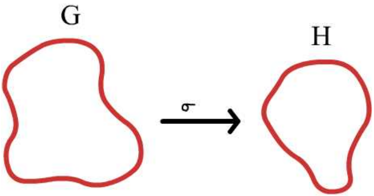
Comparison method



How different is G?

- But it is not usually easy to extract enough information from the space of natural maps !

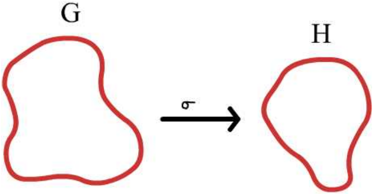
Comparison method



How different is G?

- Therefore we consider invariant or isotone parameters (as ζ_H) and prove no-map theorems.

Comparison method



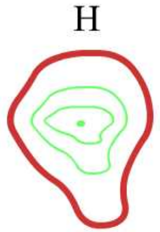
How different is G?

- A typical no-map theorem (in this sense) is:

$$\exists\ \sigma \in \mathrm{Hom}^*[G, H] \quad \Rightarrow \quad Condition(\zeta_G, \zeta_H);$$

where $Condition(\zeta_G, \zeta_H)$ is a condition or relation on ζ_G and ζ_H (e.g. $\zeta_G = \zeta_H$ or $\zeta_G \leq \zeta_H$).

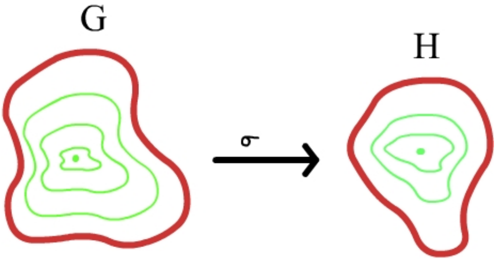
Spectral Geometry (main setup)



The diffusion reflects some geometric properties of H.

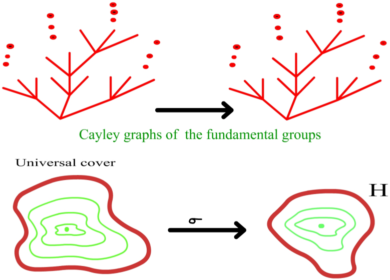
- Here usually ζ_G is related to the spectrum of a nice linear operator that is related to the geometry of G through the behaviour of a diffusion process on G.
(e.g. Laplacian and the heat equation!)

Spectral Geometry (main setup)



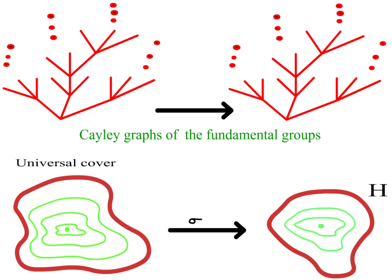
- Hence we will talk about spectral parameters as ζ_G and the corresponding no-map theorems coming from comparison of diffusion processes linked by a natural map.

A GENERAL SETUP



- It was observed by J. Milnor(1968) (continued by Gromov et.al.) that there is a close relationship between the covering space theory of H and the fundamental group of H.
  (e.g. if H is a k-regular graph then the universal cover is the infinite k-regular tree.)

A GENERAL SETUP



- It seems that the spectral radius of the universal cover has a very close relationship to the spectral gap of the symmetric spaces constructed over it and some versions of Riemann Hypothesis for the corresponding zeta (or L) functions seems to be true.

A continuous example (Selberg's Conjecture)

- Let H_2^+ be the the upper half plane and

$$\Gamma_n \stackrel{\text{def}}{=} \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbf{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod n\}.$$

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- Then $\Gamma_n \setminus H_2^+$ is a finite volume Riemann surface. Selberg used Weil's theorem on the correctness of Riemann Hypothesis for curves over finite fields and proved $\lambda_1(\Gamma_n \setminus H_2^+) \geq \frac{3}{16}$.

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- He also conjectured that $\lambda_1(\Gamma_n \setminus H_2^+) \geq \frac{1}{4} = \lambda_0(H_2^+)$.

A DISCRETE EXAMPLE (RAMANUJAN GRAPHS)

- Let K be the (L^2)-Markov kernel of the natural random walk on the k-regular tree. Then the spectral radius of K is equal to $2\sqrt{k-1}$.

A discrete example (Ramanujan graphs)

- Let K be the (L^2)-Markov kernel of the natural random walk on the k-regular tree. Then the spectral radius of K is equal to $2\sqrt{k-1}$.
- (Alon-Boppana 1986) Let $G_{n,k}$ be a family of k-regular connected graphs $(|V(G_{n,k}|) = n)$. Then,

$$\lim \sup_{n \longrightarrow \infty} \lambda_1(G_{n,k}) \leq 1 - \frac{2\sqrt{k-1}}{k}.$$

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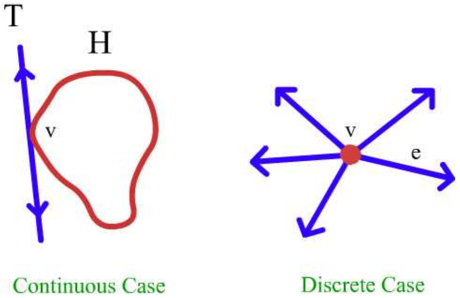
- Graphs satisfying the extremal case are called Ramanujan graphs and satisfy the Riemann Hypothesis for the Ihara zeta-function.

Other continuous-discrete connections

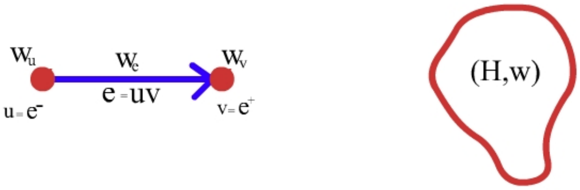
- Discrete approximation of Markov processes (Started by Varopoulos).
- Manifolds from graph constructions. (e.g. Riemann surfaces through 3-regular graphs (Mangoubi's thesis))
- Graph on surface embeddings (Robertson-Seymour well-ordering theorem).
- Discretization of manifolds (This workshop).
- Amalgam constructions (Combinatorial group theory, 3TQFT).
- Geometric group theory and its consequences in spectral geometry (Amenable groups, groups of automata, ...).
- Honeycombs and tensor products.
- ...

The tangent space



- The tangent space at vertex v is the set of out-going vectors from v!

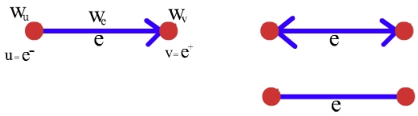
The measure



A weighted graph A weighted manifold

- Discrete case: Hereafter, we assume that $\forall\ u \in V(G) \quad w_u \stackrel{\text{def}}{=} \sum_{u \to v} w_{uv}$.
- Continuous case: Hereafter, we assume that w is a Riemannian measure on H.

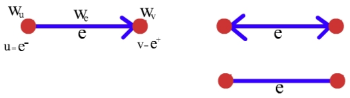
Chain maps



A simple edge = A bidirected edge

- Discrete case: Given a graph $H = (V(H), E(H))$,
- $C^0(H) \stackrel{\text{def}}{=} \{f \mid f : V(H) \longrightarrow \mathbf{R} \text{ with compact support}\}$.
- $C^1(H) \stackrel{\text{def}}{=} \{f \mid f : E(H) \longrightarrow \mathbf{R} \text{ with compact support}\}$.

The Gradient



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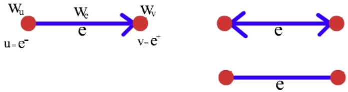
- Discrete symmetric case: Given a graph $H = (V(H), E(H))$,
- $\partial^* = \nabla_w : C^0(H) \longrightarrow C^1(H)$,
- $\nabla_w f(e) \stackrel{\text{def}}{=} (f(e^+) - f(e^-))w_e$.

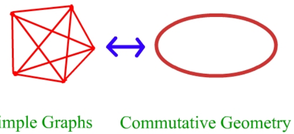
THE DIVERGENCE



A simple edge = A bidirected edge

- Discrete symmetric case: Given a graph $H = (V(H), E(H))$,
- $\partial = \mathrm{div}_w : C^1(H) \longrightarrow C^0(H)$,
- $\mathrm{div}_w f(u) \stackrel{\text{def}}{=} \frac{1}{w_u}\big(\sum_{u=e^+} f(e) - \sum_{u=e^-} f(e)\big)$.

THE LAPLACE OPERATOR (SYMMETRIC CASE)



- Discrete symmetric case: Given a graph $H=(V(H),E(H))$,
- $\Delta : C^0(H) \longrightarrow C^0(H)$,
- $\Delta_w \stackrel{\text{def}}{=} \operatorname{div}_w \nabla_w = Id - K$,
- $K(u,v) \stackrel{\text{def}}{=} \begin{cases} p(u,v) \stackrel{\text{def}}{=} \frac{w_{uv}}{w_u} & u \leftrightarrow v \\ 0 & u \not\leftrightarrow v, \end{cases}$, K is Markov!!!

Symmetric case (summary)

Discrete	*Continuous*
$\Delta_w = \mathrm{div}_w \nabla_w$	$\Delta_w = w^{-1}\mathrm{div}(w\nabla)$
$\mathcal{E}_w(f) =< \Delta f, f >_w = \|\|\nabla f\|\|_w^2$	$\mathcal{E}_w(f) = \int f(\Delta f) dw = \int \|\nabla f\|^2 dw$

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- Conservation of energy $- - -- >$ Symmetry ?!
- It is not easy to generalize this approach to the non-symmetric (e.g. directed, non-commutative, ...) case !!

Markov kernels (Discrete case)

- Let $K(u,v)$ be an ergodic Markov kernel on $C^0(H)$ with a nowherezero stationary distribution π, i.e. $\pi K = \pi$ (e.g. natural random walk on a connected simple graph). Note that in this case we have,

$$\forall\ u \in V(H) \quad \sum_{v \in V(H)} K(u,v) = 1 \text{ and } \sum_{u \in V(H)} \pi(u) = 1.$$

- We consider the following inner-product on $C^0(H)$,

$$< f, g >_\pi \stackrel{\text{def}}{=} \sum_{u \in V(H)} f(u) g(u) \pi(u).$$

Generalized discrete energy spaces I

Concept	*Data*
The Laplace Operator	$\Delta = Id - K$
Dirichlet (Energy) Form	$\mathcal{E}(f,g) =< \Delta f, g >_\pi$
Continuous Heat semigroup	$P_t = e^{-t\Delta}$
Discrete Heat semigroup	$K^n = (Id - \Delta)^n$

Generalized discrete energy spaces II

- Note that $\mathcal{E}(f,f) =< (Id-K)f, f >_\pi =< (Id - \frac{1}{2}(K+K^*))f, f >_\pi$. Hence, one can define the generalized Laplace operator as $\Delta_K \stackrel{\text{def}}{=} Id - \frac{1}{2}(K+K^*)$, which is not only self-adjoint, but also,
$$\mathcal{E}(f,f) = \frac{1}{2}\sum_{u,v} |f(u)-f(v)|^2 K(u,v)\pi(u).$$
This shows that we can interpret $K(u,v)\pi(u)$ as the conductance if f is assumed to be a potential.
- Also, we have
$$\frac{\partial}{\partial t}||P_t f||^2 = -2\mathcal{E}(P_t f, P_t f).$$

Generalized discrete energy spaces (Summary)

Symmetric	*Nonsymmetric*
$K = K^*$	$\overline{K} = \frac{1}{2}(K + K^*)$
$\nabla f(uv) = \lvert f(u) - f(v) \rvert$	$\overrightarrow{\nabla} f(uv) = (f(u) - f(v))^+$
$\overline{\phi}(u,v) = \frac{1}{2}(\phi(u,v) + \phi(v,u))$	$\phi(u,v) = K(u,v)\pi(u)$
$\Delta = Id - \overline{K}$	$\overrightarrow{\Delta} = Id - K$

Magic formula (Summary) !

- ϕ is a nowherezero flow i.e.

$$\sum_v \phi(u,v) = \sum_v \phi(v,u).$$

Compare to the case of for a Riemannian manifold i.e.

$$\partial(Q) = Vol(Boundary(Q)).$$

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-

$$||\nabla f||^2_{2,\overline{\phi}} =< \Delta f, f >_\pi =< \vec{\Delta} f, f >_\pi .$$

Min-Max principle (finite case)

Let

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1},$$

be the eigenvalues of Δ_K. Then, for any $0 \leq k < n$,

$$\lambda_k = \min_{W \in \mathcal{W}_{k+1}} \max_{0 \neq f \in W} \left\{ \frac{\mathcal{E}(f,f)}{\|f\|_\pi^2} \right\} = \max_{W \in \mathcal{W}_k^\perp} \min_{0 \neq f \in W} \left\{ \frac{\mathcal{E}(f,f)}{\|f\|_\pi^2} \right\},$$

in which

$$\mathcal{W}_k \stackrel{\text{def}}{=} \{W \leq L^2(\pi_G) \mid \dim(W) \geq k\},$$

$$\mathcal{W}_k^\perp \stackrel{\text{def}}{=} \{W \leq L^2(\pi) \mid \dim(W^\perp) \leq k\}.$$

- This variational description of λ_k is NOT suitable for perturbation analysis!

Spectral decomposition theorem (finite case)

Let

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1},$$

be the eigenvalues of Δ_K, and also, let $\{\psi_i\}$ be a corresponding orthonormal basis consisting of eigenvectors of Δ_K. Then, by Spectral Decomposition Theorem for self-adjoint operators we have,

$$(\frac{1}{2}(K + K^*))^m(u, v) = \sum_{i=0}^{n-1}(1 - \lambda_i)^m \psi_i(u)\psi_i(v)\pi(v).$$

- How can we use eigenfunctions to get more information?

Summing up

- Define a well-defined and nice self-adjoint operator that defines a diffusion process on the base-space (e.g. Δ_w or Δ_K).
- Obtain information (e.g. estimates) about the eigenvalues and the eigenfunctions of this operator (or functions of these).
- Use a variational principle along with estimates of the Dirichlet (energy) form to compare these quantities and obtain no-map theorems.

Summing up

- Define a well-defined and nice self-adjoint operator that defines a diffusion process on the base-space (e.g. Δ_w or Δ_K).
- Obtain information (e.g. estimates) about the eigenvalues and the eigenfunctions of this operator (or functions of these).
- Use a variational principle along with estimates of the Dirichlet (energy) form to compare these quantities and obtain no-map theorems.

This approach shows that estimating the Dirichlet (energy) form is a basic problem!!

Comments

- Differences between the discrete and the continuous cases. eigenvalues can be computed in polynomial time while min-cut is NP-complete.

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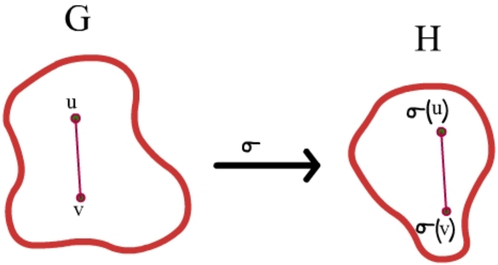
- The space of eigenfunctions is richer and more complex (e.g. nodal domains and star-partitions in this talk).

COMMENTS

- Differences between the discrete and the continuous cases. eigenvalues can be computed in polynomial time while min-cut is NP-complete.
- The space of eigenfunctions is richer and more complex (e.g. nodal domains and star-partitions in this talk).
- There is a direct relationship between the continuous and discrete heat-kernels in the symmetric (reversible) case, but this is not necessarily true in the nonsymmetric (directed) case.

A graph homomorphism



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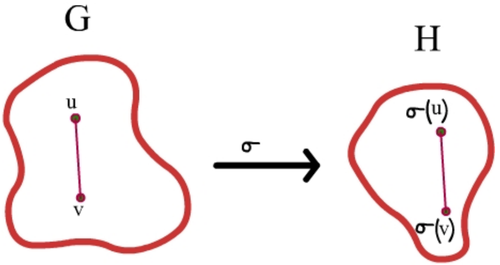
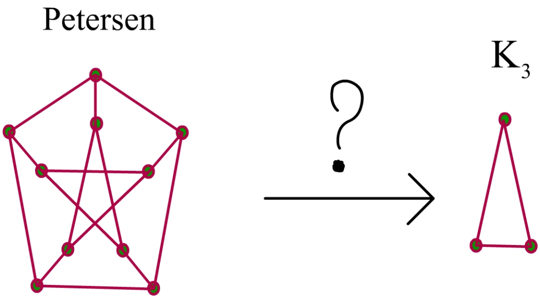


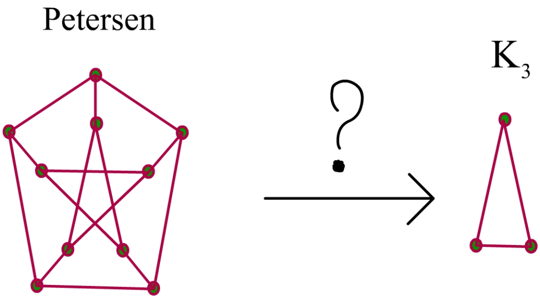
A graph homomorphism

- A graph homomorphism σ from a graph G to a graph H is a map $\sigma : V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ implies $\sigma(u)\sigma(v) \in E(H)$.

A question

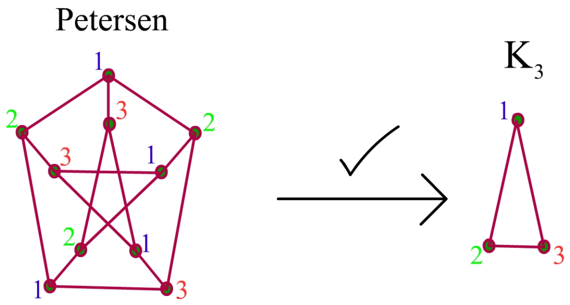


A QUESTION



- Does there exist a homomorphism from the Petersen graph to the triangle K_3?

Graph colouring



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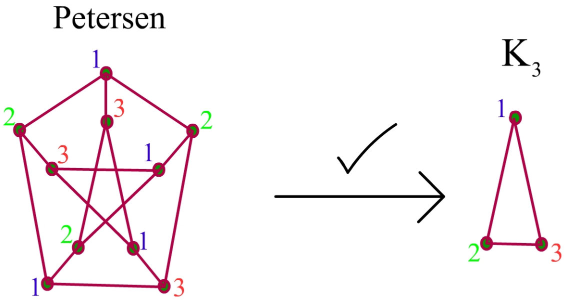
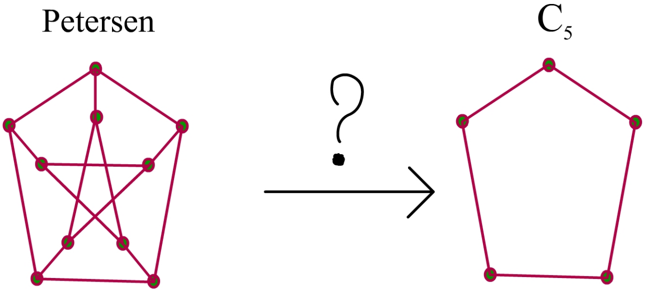


- Homomorphisms to K_n is equivalent to colouring the vertices of the graph by n colours such that the terminal ends of each edge have different colours.

Another question!



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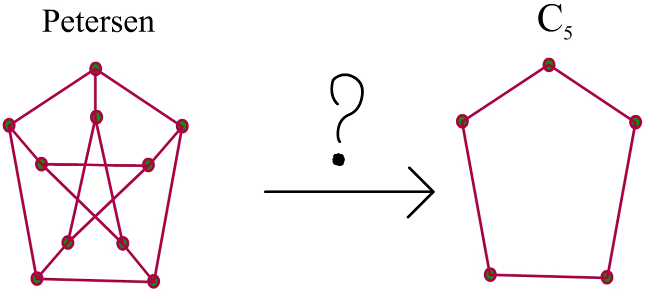

- Does there exist a homomorphism from the Petersen graph to the 5-cycle C_5?

Graph homomorphisms and combinatorics

- Graph homomorphisms are natural maps in the category of graphs.
- Many different concepts in combinatorics are related to the homomorphism problem, *e.g.*
  - The ordinary colouring problem.
  - The circular colouring problem.
  - The fractional colouring problem.
  - The graph partitioning problem, specially, existence results in design theory.
  - The Hamiltonicity problem.

Algorithmic considerations

- The following problem is NP-complete (P. Hell & J. Nesetril 1990).

Problem:	**HCOL**.
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Given:	A graph G.
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- There are close connections to $P \neq NP$ problem !
- Eigenvalues are polynomially computable for the case of finite graphs !
- The case of directed graphs is completely different!

Nodal domains

- Given a graph G with the vertex set $V(G)$, let ψ be an eigenfunction of Δ. Then a strong positive (resp. negative) sign graph P of ψ, is a maximal connected subgraph of G, on vertices $v_i \in V(G)$ such that $\psi(v_i) > 0$ (resp. $\psi(v_i) < 0$). Also, we define $\kappa(\psi)$ to be the whole number of both positive and negative strong sign graphs of ψ.

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- Estimating $\kappa(\psi)$ is an important problem in Geometry, Computer Science and Analysis of Algorithms.
- Note that in the discrete case an eigenfunction has no continuity property and the problem is much harder than when we are dealing with Riemannian manifolds!!

Hilbert–Courant theorem (a generalization)

- (A. Daneshgar & H. Hajiabolhassan 2003)

  For any pair of graphs G and H with $|V(G)| = n$ and $|V(H)| = m$, and for any $1 \leq k \leq m$, If $\sigma \in \mathrm{Hom}^{\mathrm{v}}(G, H)$ and ψ_k is an eigenfunction for the eigenvalue λ_k^H, then $\max(\lambda_k^G, \lambda_{\kappa(\psi_k)}^G) \leq \frac{\mathcal{M}^\sigma}{\mathcal{S}_\sigma} \lambda_k^H$.

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- Considering the identity automorphism will give rise to the Hilbert–Courant theorem in the discrete case (P. Stadler 2000).
- The theorem can be generalized to other kernels (possibly non-positive-definite) with applications in combinatorics. (e.g. generalize Fisher's inequality for G-designs).

Isolated and separated eigenfunctions

- An eigenfunction f of a matrix A is a separated eigenfunction if for any edge uv, we have $f(u)f(v) \geq 0$. Also, we define an eigenfunction f to be an isolated eigenfunction if for any edge uv, we have $f(u)f(v) \leq 0$.

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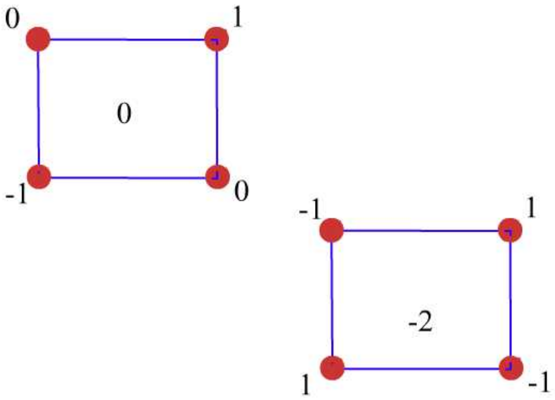
- Note that the eigenvalue corresponding to a separated (resp. isolated) eigenfunction is always non-negative (resp. non-positive). Also, it is an easy observation that the subgraph induced on the non-zero vertices of an isolated eigenfunction is always a bipartite graph.

A toy example!



A result for the kernel

- (A. Daneshgar & H. Hajiabolhassan 2003)

  Let G and H be two graphs with $|V(G)| = n$ and $|V(H)| = m$, and α_k's be the eigenvalues of the adjacency matrix in non-increasing order. Then for any separated eigenfunction f_k of the eigenvalue α^H_k and for any isolated eigenfunction f_l of the eigenvalue α^H_l,

  a) if $\sigma \in \mathrm{Hom}^{\mathrm{v}}(G,H)$ we have

$$\alpha^G_{n-\kappa(f_k)+1} \leq \frac{\mathcal{M}^{\sigma}}{\mathcal{S}_{\sigma}}\, \alpha^H_k \quad \& \quad \alpha^G_{\kappa(f_l)} \geq \frac{\mathcal{M}^{\sigma}}{\mathcal{S}_{\sigma}}\, \alpha^H_l .$$

  b) if $\sigma \in \mathrm{Hom}^{\mathrm{e}}(G,H)$ we have $\alpha^G_{\kappa(f_k)} \geq \frac{\mathcal{M}_{\sigma}}{\mathcal{S}^{\sigma}}\, \alpha^H_k .$

A comparison theorem

(A. Daneshgar & H. Hajiabolhassan 2002)

- Let G and H be two graphs with $|V(G)| = n$ and $|V(H)| = m$.
  a) If $\sigma \in \mathrm{Hom}^{\mathrm{v}}(G, H)$, then for all $1 \leq k \leq m$,
  $$\lambda_k^G \leq \frac{\mathcal{M}^\sigma}{\mathcal{S}_\sigma} \lambda_k^H.$$
  b) If $\sigma \in \mathrm{Hom}^{\mathrm{e}}(G, H)$, then for all $1 \leq k \leq m$,
  $$\lambda_{n-m+k}^G \geq \frac{\mathcal{M}_\sigma}{\mathcal{S}^\sigma} \lambda_k^H.$$
  c) If $\sigma \in \mathrm{Hom}(G, H)$ and H is both vertex and edge transitive then,
  $$\lambda_n^G \geq \frac{2|E(G)|}{n\Delta_H} \lambda_m^H.$$

Spectral gap and connectivity

- Let K be a Markov kernel. Then the smallest non-zero eigenvalue λ of $\Delta_K = Id - \frac{1}{2}(K + K^*)$ is called the spectral gap, and by the Min-Max principle we have,

$$\lambda = \min_{0 \neq f} \left\{ \frac{\mathcal{E}(f,f)}{\|f\|_\pi^2} \right\}.$$

- The spectral gap controls the rate of convergence of the diffusion and hence is a measure of connectedness.
- What happens if we consider the L^1 version of the quotient

$$\frac{\mathcal{E}(f,f)}{\|f\|_\pi^2} = \frac{\int |\nabla f|^2 d\pi}{\int |f|^2 d\pi}?$$

Cheeger's constant

- The L^1 version of the spectral gap, called the Cheeger's constant h, reduces to the concept of minimum weighted cut in the discrete case as,

Discrete	*Continuous*
$\min\limits_{\lvert A\rvert_\pi \leq 1/2} \left\{ \dfrac{\lvert\partial A\rvert_\pi}{\lvert A\rvert_\pi} \right\}$	$\min\limits_{Vol_n(A)\leq 1/2} \left\{ \dfrac{Vol_{n-1}(\partial A)}{Vol_n(A)} \right\}$

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- Cheeger's constant is also a measure of connectivity that guarantees fast diffusion!!
- Examples: $h(H_2^+) = 1$. also, $h = 0$ is related to amenability!

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  - Relation to the first kth eigenvalues.
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  - Have a nice functional description.
- A straight-forward generalization:

$$\tilde{\imath}_n(G) \stackrel{\text{def}}{=} \min_{\{Q_i\}_1^n \in \mathcal{P}_n(G)} \frac{1}{n} \left(\sum_{i=1}^{n} \frac{\vec{\partial}(Q_i)}{\pi(Q_i)} \right),$$

where $\mathcal{P}_n(G)$ is the class of all n-partitions of G.

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- You can use Ky Fan's and Wielandt's variational principles (e.g. $\overline{\lambda}_n \leq \iota_n(G)$).
- You can use convex analysis.

Geometric graphs II

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- Program: What are the minimizers and maximizers?

A 0-UNIFORMIZER (I)

- Let G and H be two connected digraphs such that $|V(G)| = n$, $|V(H)| = m$ and also assume that the group of automorphisms of H, $\mathrm{Aut}(H)$, acts transitively on both $V(H)$ and $E(H)$. Let $\sigma \in \mathrm{Hom}(G, H)$.

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- Let $\mathrm{Aut}(H) = \{\zeta_i \mid i = 1, \ldots, t\}$ and define, $\tilde{G} \overset{\mathrm{def}}{=} \bigcup_{i=1}^{t} G_i$, where each connected component of $\tilde{G}$, such as G_i, is an isomorphic copy of G. Also, define the homomorphism $\tilde{\sigma}$ such that its restriction to G_i is $\zeta_i \circ \sigma$. It is easy to see that $\tilde{\sigma} \in \mathrm{Hom}^{\mathrm{e}}(G, H)$, $\mathcal{M}_{\sigma} = \frac{|E(G)|}{|E(H)|} \times t$ and $\mathcal{S}^{\sigma} = \frac{|V(G)|}{|V(H)|} \times t$.

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- Define a map $\tilde{\sigma} : V(G \square H) \longrightarrow V(H)$ as follows,

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- One can show that $\tilde{\sigma}$ is a homomorphism,

$$\mathcal{S}^{\tilde{\sigma}} = \mathcal{S}_{\tilde{\sigma}} = n \quad \text{and} \quad \mathcal{M}_{\tilde{\sigma}} \geq n + m\mathcal{M}_{\sigma}.$$

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- How can we find uniformizers?
- It seems that we need, surjectivity, symmetry on the range and nice amalgam constructions.

Cylinderical construction (Example)

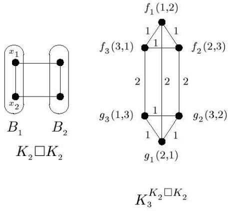

Figure 1: The cylinder $K_2 \square K_2$ and the exponential graph $K_3^{K_2 \square K_2}$.

- $Hom(G * H, K) \simeq Hom(G, K^H)$.

The general setup (I)

- Let H be a nice subgroup of a nice group G, and let $\chi : H \longrightarrow \mathcal{U}(\mathcal{H})$ be a representation of H. Let $\rho : G \longrightarrow \mathcal{U}(\mathcal{K})$ be the induced representation on G by χ and let $\sigma : L^1(G) \longrightarrow \mathcal{L}(\mathcal{K})$ be the algebra representation associated with ρ. Then for every $f \in L^1(G)$ the following kernel exists (at least in a weak sense)

$$K_f(x, y) \stackrel{\text{def}}{=} \int_H f(xhy^{-1})\chi(h)dh,$$

and moreover we have

$$\sigma(f)\theta(x) = \int_{G/H} K_f(x, y)\theta(y)d\eta.$$

The general setup (II)

- An important special case is when $H = \{0\}$ and then ρ will be the left regular representation of G and moreover $\sigma(f)$ is exactly convolution by f, i.e.

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- If f_S is the characteristic function of a generating set S then $\sigma(f_S)$ is exactly the adjacency matrix of the corresponding Cayley graph.
- The same construction works in general for coset graphs which gives rise to the most interesting symmetric examples! The adjacency operator is usually called the Hecke operator.

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- The algebra of bi-H-invariant functions on G is called the Hecke algebra.
  Also, a left-H-invariant function ω such that $f * \omega = \lambda_f \omega$ holds for any f in the Hecke algebra is called a spherical function.

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- Magic relationship!
  If f_S is in the Hecke algebra then any spherical function is also an eigenfunction of the Hecke operator and consequently an eigenfunction of the Laplacian.
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- What about more general cases?!

Coset graphs and character sums

- (Babai, Diaconis, M. Shahshahani)
  When S is a union of conjugacy classes (i.e. the case of quasi-Abelian Cayley graphs) then the eigenvalues can be computed from the character sums.
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  Generalizations of this to the case of general coset-graphs is being studied.
- Also, a crucial step is to construct ε-uniformizers for general coset-graphs.

Some general comments and questions

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- Try to introduce constructions that decode geometric properties into directed graphs.
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- Is it not a better strategy to consider the mean of the first k eigenvalues and characterize the extremal cases. By the way, can you say anything about this when $k = 2$?
- Can you use the symmetry of the generic geometric objects to construct ε-uniformizers in the continuous case?

The end.

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estimates of connectedness and density!!!

Thank You!