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Graph Algebras

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We consider graphs with k nodes labeled 1,...,k and any number of unlabeled nodes. We form the product of two such graphs by gluing together their labeled nodes. (In other words, we consider decomposing along a cutset of nodes as factorization.) By extending this to quantum graphs (formal linear combinations of such k-labeled graphs), we get a commutative algebra. Every graph parameter defines a bilinear form on this algebra and through this, a factor algebra.

This factor algebra is often finite dimensional and is very useful in studying the graph parameter. For example, it was proved by Freedman, Lovász and Schrijver that the parameter can be represented as the number of homomorphisms into a fixed weighted graph (e.g., the number of 4-colorings) if and only if this factor algebra is finite dimensional for every k, its dimension grows only exponentially with k, and the inner product is positive definite. Further applications of this algebraic setup include a characterization of generalized quasirandom graphs by Vera Sós and the author.

The construction has an analogue using edge-cuts in place of node-cuts, which leads to non-commutative algebras and where many questions are still unsettled. There is a corresponding class of graph parameters, called edge-coloring models, which were characterized by Balazs Szegedy using these algebras, in a way analogous to (but much more involved than) the characterization of homomorphism functions mentioned above. Through more recent work of Alexander Schrijver, this research has very close ties with the representation theory of tensor algebras.