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Proofs, Programs and Abstract Complexity

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Axiom systems are ubiquitous in mathematical logic, one famous and well studied example being first order Peano arithmetic. Foundational questions asked about axiom systems comprise analysing their provable consequences, describing their class of provable recursive functions (i.e. for which programs can termination be proven from the axioms), and characterising their consistency strength. One branch of *proof theory*, called *ordinal analysis*, has been quite successful in giving answers to such questions, often providing a unifying approach to them. The main aim of ordinal analysis is to reduce such questions to the computation of so called *proof theoretic ordinals*, which can be viewed as abstract measures of the complexity inherent in axiom systems. Gentzen's famous consistency proof of arithmetic using transfinite induction up to (a notation of) Cantor's ordinal ϵ_0, can be viewed as the first computation of the proof theoretic ordinal of Peano arithmetic.

Bounded arithmetic, as we will consider it, goes back to Buss [Bus86]. Bounded arithmetic theories can be viewed as subsystems of Peano arithmetic which have strong connections to complexity classes like the polynomial time hierarchy of functions. Ever since their introduction, research on bounded arithmetic has aimed at obtaining a good understanding of the three questions mentioned above for the bounded arithmetic setting, with varying success. While a lot of progress has been obtained in relating definable functions to complexity classes, very little can be said about how the provable consequences are related (this problem is called the separation problem for bounded arithmetic), or how the consistency strength of bounded arithmetic theories can be characterised.

A natural question to ask is whether proof theoretic ordinals can give answers for bounded arithmetic. However, results by Sommer [Som93] have shown that this is not the case, proof theoretic ordinals are useless in the setting of bounded arithmetic. But there are adaptations of proof theoretic ordinals denoted *dynamic ordinals* which can be viewed as suitable abstract measures of the complexity of bounded arithmetic theories [Bec03, Bec06]. In my presentations I will try to draw pictures of this situation, starting from ordinal analysis for Peano arithmetic, via their adaptation to dynamic ordinals, leading to dynamic ordinal analysis for bounded arithmetic theories. I will try to convince you that dynamic ordinals can equally be viewed as suitable measures of the proof and computation strength of bounded arithmetic theories, which can be used to give answers to (some of the) above questions.

References

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