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Set Theory with a Class of Indiscernibles

A. Enayat
American University
Washington, DC, USA

We focus on an extension of Zermelo-Fraenkel set theory, $ZFCI$, which strongly negates Leibniz's dictum on the identity of indiscernibles by asserting "there are a proper class of indiscernibles". $ZFCI$ is a theory formulated in the language $\{\in, I(x)\}$, where $I(x)$ is a unary predicate to distinguish the indiscernibles. As we shall see, $ZFCI$ goes well beyond ZFC since it proves the existence of n-Mahlo cardinals for each concrete natural number n. One can precisely describe the first order consequences of $ZFCI$ in the usual language of set theory $\{\in\}$. In order to do so, let Φ be the set of sentences of the following form (for each concrete natural number n) :

"there is an n-Mahlo cardinal κ such that V_κ is a Σ_n elementary submodel of the universe".

Here is the first main result:

Theorem A. *For any sentence σ in the usual language of set theory $\{\in\}$, the following two conditions are equivalent:*

(a) $ZFCI \vdash \sigma$;

(b) $ZFC + \Phi \vdash \sigma$.

One can also *iterate* the idea of adding indiscernibles by introducing countably many new unary predicates $\{I_n : n \in \omega\}$ in order to formulate a theory $ZFCI^\omega$ extending $ZFCI$ by adding axioms asserting, for each n, that I_{n+1} is a proper class of indiscernibles for formulae in the language $\{\in, I_1, \cdots, I_n\}$. As it turns out, this new system will not produce any new theorems of set theory beyond those of $ZFCI$, i.e., Theorem A can be improved to the following result:.

Theorem B. *For any sentence S in the usual language of set theory $\{\in\}$,*

the following five conditions are equivalent:

(a) $ZFCI^\omega \vdash \sigma$;

(b) $ZFCI \vdash \sigma$;

(c) $ZFC + \Phi \vdash \sigma$.