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Double Negation of Intermediate Value Theorem

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In the context of intuitionistic analysis, we assume the set $\mathcal{F}$ consisting of all continuous functions ϕ from $[0,1]$ to $\mathbb{R}$ such that $\phi(0)=0$ and $\phi(1)=1$, and $\mathcal{I}_0$ the set of ϕ's in $\mathcal{F}$ that there exists $x\in[0,1]$ such that $\phi(x)=\frac{1}{2}$. It is well-known that there are *weak counterexamples* to the intermediate value theorem, and with the help of *Brouwer's continuity principle* $\mathcal{I}_0\neq\mathcal{F}$. However, there exists no satisfying answer to $\mathcal{I}_0^{\neg\neg} =^? \mathcal{F}$. We try to answer to this question by reducing it to some properties about intuitionistic decidable subsets of $\mathbb{N}$. Using this reduction, it is shown that $\mathcal{F}_{mon}\neq(\mathcal{I}_0)_{mon}^{\neg\neg}$ cannot be derived form some well known intuitionistic axioms, *Weak Continuity principle*, *Kripke Schema*, and *Fan principle*. Also, assuming an equivalent form of *Markove's principle*, i.e., $\forall x\in\mathbb{R}(x\not=0\rightarrow x\neq 0)$, it is derived $\mathcal{F}_{mon}=(\mathcal{I}_0)_{mon}^{\neg\neg}$. It is proved that the converse does not hold, and the assumption $\mathcal{F}_{mon}=(\mathcal{I}_0)_{mon}^{\neg\neg}$ does not imply $\forall x\in\mathbb{R}(x\not=0\rightarrow x\neq 0)$. We also introduce the notion of *strong Specker double sequence*, and prove that existence of a *strong Specker double sequence* implies existence of $\phi\in\mathcal{F}$ such that $\neg\exists x\in[0,1]\phi(x)=\frac{1}{2}$.