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Existence and a Priori Estimate for Elliptic Problems with Subquadratic Gradient Dependent Terms

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In this lecture I will consider the nonlinear elliptic model problem

$$u \in H_0^1(\Omega), \qquad -div\, A(x)Du + \alpha_0 u = \gamma |Du|^q + f(x) \ \text{ in } \mathcal{D}'(\Omega),$$

with A a coercive matrix with bounded coefficients, $\alpha_0 \geq 0$, $0 \leq q \leq 2$ and $f \in L^m(\Omega)$ for some suitable m. This is a model problem, and there are many possible variants of it.

In the case where $0 \leq q < 1$, existence is classical for $f \in H^{-1}(\Omega)$. When γ is large, the case where $q = 1$ and $f \in H^{-1}(\Omega)$ is difficult but has been solved by G. Bottaro and M.E. Marina in 1973. On the other hand, the case $q = 2$ has been treated by many authors, including in particular in a series of papers by L. Boccardo, J.-P. Puel and myself. In a more recent paper, V. Ferone and myself proved the existence of a solution u wich further satisfies $e^{\gamma u} - 1 \in H_0^1(\Omega)$, and an a priori estimate for such solutions, when $f \in L^{\frac{N}{2}}(\Omega)$.

In this lecture I will mainly report about recent joint work with Nathalie Grenon and Alessio Porretta, the announcement of which has been published in C. R. Acad. Sci. Paris, Série I, 342, (2006), pp. 23-28. When $1 + \frac{2}{N} \leq q < 2$ and $f \in L^m(\Omega)$ with $m = \frac{N(q-1)}{q}$ (we also solved the case where $1 \leq q < 1 + \frac{2}{N}$, but I will not discuss it since it uses the notion of renormalized solution), and when either $\alpha_0 > 0$ or f is sufficiently small in $L^m(\Omega)$, we prove the existence of a solution u which enjoys the further regularity $|u|^\sigma \in H_0^1(\Omega)$ with $\sigma = \frac{(N-2)(q-1)}{2(2-q)}$, as well as an a priori estimate for any solution which enjoys this further regularity. One of the main interests of our result lies in the priori estimate, the proof of which is non standard.