Abstract of Courses

Quivers and Representations

Let k be a field. Consider Q a quiver (=finite oriented graph). We consider the category of representations of Q and show it is equivalent to the module category of the path algebra kQ. We study special representations of Q and prove Gabriel's Theorem: Q accepts only finitely many indecomposable representations up to isomorphism if and only if the underlying graph $|Q|$ of Q is of Dynkin type ($\mathbb{A}_n, \mathbb{D}_n, \text{or } \mathbb{E}_{\mathrm{p}}, \mathrm{p} = 6, 7, 8$). For a general quiver Q we consider those dimension vectors v for which there are indecomposable representations X with $\underline{\dim} X = v$. We consider Kac's Theorem characterizing those dimension vectors. For an algebra given as a quotient $A = kQ/I$ we introduce the module variety of $\mathrm{mod}_A(v)$ of A-modules with vector dimension v and study elementary geometric properties.

Derived and Triangulated Categories

Derived categories were invented at the beginning of the sixties by Grothendieck as the language in which to formulate his duality theory for schemes. The theory was worked out in the thesis of his pupil J.-L. Verdier. Grothendieck-Verdier introduced the notion of a triangulated category in order to axiomatize the properties of derived categories (independently, the notion was introduced in topology by Puppe as an axiomatization of the properties of the stable homotopy category of finite CW-complexes). In the last forty years, the use of derived categories has spread from algebraic geometry through algebraic analysis to representation theory of Lie algebras, associative algebras, finite groups, ... In this series of lectures, we will present the basic constructions of derived categories and derived functors and show their axiomatization via triangulated categories and triangle functors. The abstract notions will be illustrated with numerous examples coming from commutative algebra and representation theory of finite-dimensional algebras.

Cluster Algebras and Cluster Categories

Cluster algebras were invented by S. Fomin and A. Zelevinsky in joint work in 2000. Their aim was to find a combinatorial approach to two theories due to G. Lusztig: the theory of canonical bases in quantum groups and the theory of total positivity in algebraic groups. Despite great progress in the last few years, we are still far from these initial aims. Nevertheless, cluster algebras have themselves become a thriving subject notably thanks to their surprising links with many other subjects: These include Poisson Geometry, Teichmüller spaces, integrable systems, combinatorics and combinatorial polyhedra, and, last not least, the representation theory of quivers and finite-dimensional algebras. Here, the idea is that of categorification: One tries to interpret cluster algebras as combinatorial (perhaps K-theoretic) invariants associated with categories of representations. The lift from combinatorial objects to categories provides rich additional structure, which sometimes allows one to prove statements which seem out of reach of the purely combinatorial methods. In this series of lectures, we will present the basic notions of cluster algebras and relate them to representation theory following the recent work of Buan, Caldero, Chapoton, Geiss, Keller, Leclerc, Marsh, Reineke, Reiten, Schröer, Todorov, ...

Auslander-Reiten Theory

By clever homological analysis AR-theory reveals a hidden combinatorial structure of the category of finite dimensional modules over a finite dimensional algebra (and other categories of a similar nature). For instance, AR-theory allows to arrange indecomposable objects systematically in Auslander-Reiten components, each having the structure of a translation quiver. The theory has a homological incarnation in Auslander-Reiten duality $D\mathrm{Ext}^1(\mathrm{X},\mathrm{Y}) \cong \overline{\mathrm{Hom}}(\mathrm{Y},\tau\mathrm{X})$, being close to Serre duality; it extends to suitable derived and triangulated categories. AR-theory therefore is fundamental for the representation theory of finite dimensional algebras, in particular, playing a decisive rôle for classification.

Tilting Theory

For a tilting module (more generally a tilting complex) T over a finite dimensional algebra A the categories of modules mod-A and mod-B, where B is the endomorphism ring of T, offer a lot of similarities despite the fact that mod-A and mod-B usually are far from being equivalent. The similarity between A and B is best expressed by the fact that the (bounded) derived categories of mod-A and mod-B are triangle-equivalent, implying that A and B share important homological invariants. The concept of tilting is by now established as a central tool in representation theory, linking the topic with many interesting subjects: Lie theory, Derived and triangulated categories, Algebraic Geometry and all types of classification results.