Cluster Algebra Structures and Semicanonical Bases for Unipotent Groups

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Outline

Notation & Overview
  Notation
  Overview

Cluster Tilting in 2-CY Subcategories
  The Category $C_M$
  Connecting the canonical cluster tilting objects

Semicanonical- and PBW bases
  Basic Construction
  Comultiplication and Applications

Coordinate Rings
  Unipotent- and Kac-Moody Groups
  Unipotent Cells
Preinjctive modules for quivers

- **Q** Quiver without oriented cycles, vertices
  \[Q_0 = \{1, 2, \ldots, n\}\]
- **M** = \(\bigoplus_{i=1}^{r} M(i)\) “preinjective” module.
Preprojective Algebras

Definition

- $\overline{Q}$ double of $Q$: extra arrow $i \xleftarrow{\bar{a}} j$ for each arrow $i \rightarrow j$ of $Q$.
- $\Lambda := \mathbb{C}\overline{Q}/(\sum_{a \in Q_1} [\bar{a}, a])$ preprojective algebra of $Q$.

Note

- $\mathbb{C}Q \hookrightarrow \Lambda$, so have restriction functor $\mathbb{C}(\Lambda, \mathbb{C})|_Q \cong \prod_{\text{indec preinj } J} J$
- $\Lambda$ depends not on orientation of $Q$
- $M \in \mathbb{C}Q$-mod can be interpreted as $\Lambda$-module via $M(\bar{a}) = 0$ for all $a \in Q_1$. 
Kac-Moody Lie Algebras

- $Q$ defines a symmetric (generalized) Cartan matrix $C = C|_Q$.
- $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ triangular decomposition of KM-algebra defined by $C$.
- $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} n_\alpha$ decomposition of $\mathfrak{n}$ into root spaces.
- $U(\mathfrak{n})$ universal enveloping algebra, a graded cocommutative Hopf algebra. Chevalley generators $e_1, \ldots, e_n$ with $e_i \in U(\mathfrak{n})_{\alpha_i}$.
- The graded dual $U(\mathfrak{n})^*_{\text{gr}}$ is commutative Hopf algebra. Think of it as coordinate ring of a (pro-) unipotent group $N$. 
This is a report on joint work with Bernard Leclerc (Caen) & Jan Schröer (Bonn).

There is related work by Aslak Buan, Osamu Iyama, Idun Reiten & Jeanne Scott:
*Cluster structures for 2-Calabi-Yau categories and unipotent groups.*
Overview I

1. The stably 2-Calabi-Yau category

\[ C_M = \{ X \in \Lambda \text{-mod} \mid X|_Q \in \text{Add}(M) \} \]

with canonical maximal rigid object \( T_M \) “built” from \( M \). Combinatorics of mutations in \( C_M \).

2. Cluster Character \( \delta ? : \Lambda \text{-mod}_0 \rightarrow U(n)^*_{\text{gr}} \) restricts to

\[ C_M \rightarrow A_M := \text{span}_\mathbb{C}\{ \delta_X \mid X \in C_M \} \subset U(n)^*_{\text{gr}} \]

provides \( A_M \) with a cluster algebra structure and initial seed given by \( T_M \).

\( A_M \) is a polynomial ring, comes with a semicanonical basis containing the cluster monomials.
Overview II

3. Preinjective Module \( M = \bigoplus_{i=1}^{r} M(i) \) defines *adaptable* element \( w = w_M \) of length \( r \) in the Coxeter group associated to \( C_{|Q|} \). This is also the Weyl group of \( g \) and of the Kac-Moody group \( G_{\text{min}} \).

\( N = N(w) \circ N'(w) \) and \( N(w) = N/N'(w) \) as homogeneous spaces

\[
\mathcal{A}_M = \mathbb{C}[N]^{N'(w)} = \mathbb{C}[N(w)]
\]

\[
(\mathcal{A}_M)_{\delta(\text{proj.-inj})} = \mathbb{C}[N^w]
\]

\( N^w := N \cap B^- wB^- \) unipotent cell
The “preinjective” module $M$

$M = \bigoplus_{i=1}^{r} M(i)$ direct sum of a family of indecomposable pairwise non-isomorphic preinjective $\mathbb{C}Q$-modules. Family is closed under successors and we assume $\text{Hom}_Q(M(i), M(j)) = 0$ if $i < j$.

Example
Properties of the Category $\mathcal{C}_M$

**Theorem**

- $\mathcal{C}_M$ is a stably 2-CY category, closed under factor modules, with $T_M := \bigoplus_{i=1}^r T_M(i)$ and $T_M^\vee := \bigoplus_{i=1}^r T_M^\vee(i)$ canonical cluster tilting objects.

- The quiver of $\text{End}_\Lambda(T_M)$ and of $\text{End}_\Lambda(T_M^\vee)$ is obtained easily from the quiver of $\text{End}_Q(M)$.

- Have a cluster structure on the cluster tilting objects, i.e. exchange of summands corresponds to two exact sequences, quiver of $\text{End}_\Lambda(T)$ changes according to FZ-quiver mutation.
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$T_M$ and $T_M^\lor$ - an Example

$T_M$

$T_M = \Omega_{C_M} T_N$

$(12,7,2) \mapsto (7,2) \mapsto (2)$

$(13,8,3) \mapsto (8,3) \mapsto (3)$

$(11,6,1) \mapsto (6,1) \mapsto (1)$

$(9,4) \mapsto (4)$

$(10,5) \mapsto (5)$

$(11,6) \text{ means } T_M(6) |_Q = M(11) \oplus M(6)$
Remarks

• $X \in C_M$ rigid $\Rightarrow$ $X$ uniquely determined by $X|_Q \in \text{Add}(M)$

• Can work out how $T^*(k)|_Q$ is obtained from quiver of $\text{End}_\Lambda(T)$ and $(T(k)|_Q)_{i=1,2,...,r}$ similar to “denominator mutation”.

• the previous two remarks can be nicely interpreted using that $B = \text{End}_\Lambda(T_M)$ is quasi-hereditary and

$$\text{Hom}_\Lambda(-, T_M): C_M \xrightarrow{\sim} \mathcal{F}(\Delta) \subset B\text{-mod}$$
Remarks

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$$\text{Hom}_\Lambda(-, T_M): C_M \xrightarrow{\sim} \mathcal{F}(\Delta) \subset B\text{-mod}$$
Connecting $T_M$ and $T^\vee_M$

**Theorem**

There exists and explicit sequence of mutations connecting $T_M$ to $T^\vee_M$.

On this “path” each $M(i)$ ($i = 1, 2, \ldots, r$) appears as a direct summand of a cluster tilting object.
“Proof”

\[ T_M \]

\[
\begin{array}{c}
(12) \leftarrow (12,7) \leftarrow (12,7,2) \\
(13) \leftarrow (13,8) \leftarrow (13,8,3) \\
(11) \leftarrow (11,6) \leftarrow (11,6,1) \\
(9) \leftarrow (9,4) \\
(10) \leftarrow (10,5)
\end{array}
\]
Notation & Overview

Cluster Tilting in 2-CY Subcategories

Semicanonical- and PBW bases

Coordinate Rings

“Proof”
\[
T_M \quad \text{“Proof”}
\]
“Proof”

```
(7)  (12,7)  (12,7,2)
  ↖      ↕      ↗
(13)  (13,8)  (13,8,3)
    ↖      ↕      ↗
  (11)  (11,6)  (11,6,1)
        ↖      ↕      ↗
      (9)  (9,4)  (9,4)
```

```
1  (10)  (10,5)
```

```
(7)  (7,2)  (12,7,2)
  ↖      ↕      ↗
(13)  (13,8)  (13,8,3)
    ↖      ↕      ↗
  (11)  (11,6)  (11,6,1)
        ↖      ↕      ↗
      (9)  (9,4)  (9,4)
```

```
2  (10)  (10,5)
```
"Proof"
"Proof"
“Proof”

(7) ← (7,2) → (12,7,2)

(8) ← (8,3) → (13,8,3)

(11) ← (11,6) ← (11,6,1)

(9) ← (9,4)

(10) ← (10,5)

(7) ← (7,2) → (12,7,2)

(8) ← (13,8) → (13,8,3)

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(10) ← (10,5)
“Proof”

\[ (7) \rightarrow (7,2) \rightarrow (12,7,2) \]

\[ \rightarrow (8) \rightarrow (8,3) \rightarrow (13,8,3) \]

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(5) → (10,5)

(4) → (9,4)

(9) ← (9,4)

(10) ← (10,5)

(6) → (11,6,1) → (11,6,1)

(10) ← (10,5)
“Proof”
“Proof”
"Proof"
“Proof”
“Proof”
“Proof”

(2) \rightarrow (7,2) \rightarrow (12,7,2)

(3) \rightarrow (8,3) \rightarrow (13,8,3)

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\text{T}^\vee_M
Basic Notations

- $\alpha$ dimension vector
- $\Lambda(\alpha)$ (affine) Variety of nilpotent representations of $\Lambda$ with dimension vector $\alpha$.
- $\text{Rep}_Q(\alpha)$ affine space of representations of $Q$ with dimension vector $\alpha$.
- In both cases $GL(\alpha)$ orbits are isoclasses of the corresponding representations.
- Note that $\text{Rep}_Q(\alpha)$ is naturally an irreducible component of $\Lambda(\alpha)$. 
Algebra of Constructible Functions

- $\widehat{\mathcal{M}}(\alpha)$ Space of $\text{GL}(\alpha)$-invariant constructible functions $\Lambda(\alpha) \to \mathbb{C}$.
- $\bigoplus_{\alpha} \widehat{\mathcal{M}}(\alpha)$ becomes a graded assoc. algebra via
  $$ (f * g)(x) := \int_{u \leq x} f(u)g(x/u) \quad \text{top. Euler Characteristic} $$

- $\mathcal{M}$ subalgebra generated by $\mathbb{1}_i \in \mathcal{M}(\alpha_i)$. Here $\alpha_i$ simple root, so $\Lambda(\alpha_i) = \{\text{pt}\}$.
- $\Psi: U(n) \to \mathcal{M}$ defined by $e_i \mapsto \mathbb{1}_i$ (where $e_i$ Chevalley generator) is surjective algebra homomorphism. In fact an isomorphism (uses semicanonical bases).
**Algebra of Constructible Functions**

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Two Bases

**Theorem (Lusztig)**

*Each* $\mathcal{M}(\alpha)$ *has a basis* $S(\alpha) := (s_Z)_{Z \in \text{Irr}(\Lambda(\alpha))}$ *such that*

\[ s_Z|_{Z'} = \delta_{Z,Z'} \mathbb{1}_{Z'} \]  

*generically.*

The union $S = \cup_\alpha S(\alpha)$ is the *Semicanonical Basis* of $U(n)$.

**Definition**

Let $(p_i)_{i \in \mathbb{N}}$ be a basis of $\mathfrak{n}$ consisting of *root vectors* and

\[ p_i \in \mathfrak{n}_{\text{dim} M(i)} \text{ for } i = 1, 2, \ldots, r \text{ (real roots)} \]

then the $p^{(m)} := \frac{1}{m_s! \ldots m_1!} p_{1}^{m_1} \ast \cdots \ast p_{s}^{m_s}$ *for* $m \in \mathbb{N}_{0}^{(\mathbb{N})}$

form an appropriate (scaled) PBW-basis of $U(n)$. 

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$$p_i \in n_{\dim M(i)} \text{ for } i = 1, 2, \ldots, r \text{ (real roots)}$$

then the $p^{(m)} := \frac{1}{m_s! \cdots m_1!} p_s^{m_s} \ast \cdots \ast p_1^{m_1}$ for $m \in \mathbb{N}_0^{(\mathbb{N})}$

form an appropriate (scaled) PBW-basis of $U(n)$. 
Comultiplication

**Theorem**

*Can define a comultiplication* \( \Delta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{M} \) *such that for* 

\[ f \in \mathcal{M}(\alpha) \text{ with} \]

\[ \Delta(f) = \sum_{\alpha' + \alpha'' = \alpha} f' \otimes f'' \text{ get} \quad (f' \otimes f'')(x', x'') = f(x' \oplus x''). \]

*Coincides with usual comultiplication of* \( U(\mathfrak{n}) \). 

Despite the tautological appearance this is quite non-trivial, proof uses several fibre bundle constructions.
Consequences of Comultiplication formula

Corollary

(a) If $f \in n_\alpha \subset M(\alpha) \Rightarrow \text{supp}(f) \subset \Lambda(\alpha)^{\text{indec}}$.

(b) If $\alpha$ is a real root then $f|_{\text{Rep}_Q(\alpha)} = c \mathbb{1}_{O_N} \neq 0$

N unique indec. rep. of $Q$ with $\dim N = \alpha$.

(c) $p^{(m_r)}_r \ast \cdots \ast p^{(m_1)}_1|_{\text{Rep}_Q(\beta)} = \mathbb{1}_{O(M')}$

where $M' = \bigoplus_{i=1}^r M(i)^{m_r}$, $\dim M' = \beta$

Proof.

(a) Use $f$ primitive: $\Delta(f) = 1 \otimes f + f \otimes 1$

(b) Kac’s theorem: For real roots there exists a unique indec. representation of $Q$.

(c) Use $\text{Ext}^1_Q(M(i), M(j)) = 0$ for $i \geq j$. 
Consequences of Comultiplication formula

Corollary

(a) If \( f \in n_{\alpha} \subset \mathcal{M}(\alpha) \Rightarrow \text{supp}(f) \subset \Lambda(\alpha)^{\text{indec}}. \)

(b) If \( \alpha \) is a real root then \( f \mid_{\text{Rep}_Q(\alpha)} = c \mathbb{1}_{\mathcal{O}_N} \neq 0 \). N unique indec. rep. of \( Q \) with \( \dim N = \alpha. \)

(c) \( p_{r}^{(m_r)} \ast \cdots \ast p_{1}^{(m_1)} \mid_{\text{Rep}_Q(\beta)} = \mathbb{1}_{\mathcal{O}(M')} \)
   where \( M' = \oplus_{i=1}^{r} M(i)^{m_r}, \dim M' = \beta. \)

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Graded Dual and Evaluation Forms

The *Graded Dual* \( \mathcal{M}^* = U(\mathfrak{n})_{\text{gr}}^* \) is a *commutative* Hopf-algebra with multiplication \( \Delta^* \). Consider the evaluation form

\[
\delta?: \Lambda - \text{mod} \to \mathcal{M}^*, \ x \mapsto (f \mapsto (f(x))).
\]

**Theorem**

\( \delta \) is a cluster character i.e.

(a) \( \delta_X \cdot \delta_Y = \delta_{X \oplus Y} \).

(b) If \( \dim \text{Ext}_\Lambda^1(x, y) = 1 \) with corresponding non split s.e.s

\[
0 \to x \to E' \to y \to 0 \quad \text{and} \quad 0 \to y \to E'' \to x \to 0
\]

then \( \delta_X \cdot \delta_Y = \delta_{E'} + \delta_{E''} \).

**Proof.**

(a) follows from descr. of comult. (b) after [CK].
Graded Dual and Evaluation Forms

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$$\delta? : \Lambda\text{-mod} \to \mathcal{M}^*, \ x \mapsto (f \mapsto (f(x))).$$

**Theorem**

δ is a cluster character i.e.

(a) $\delta_X \cdot \delta_Y = \delta_{X \oplus Y}.$

(b) If $\dim \text{Ext}^1_{\Lambda}(x, y) = 1$ with corresponding non split s.e.s

$$0 \to x \to E' \to y \to 0 \quad \text{and} \quad 0 \to y \to E'' \to x \to 0$$

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**Theorem**

$\delta$ *is a cluster character* i.e.

(a) $\delta_x \cdot \delta_y = \delta_{x \oplus y}$.

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Two bases for for \( \mathcal{A}(\mathcal{C}_M) \)

**Theorem**

1. \( \mathcal{A}(\mathcal{C}_M) := \text{span}_\mathbb{C}\{\delta_X \mid X \in \mathcal{C}_M\} \cong \mathbb{C}[\delta_{M(1)}, \ldots, \delta_{M(r)}] \subset \mathcal{M}^* \) is a polynomial ring. Thus \((\delta_{M'})_{M' \in \text{Add}(M)} \subset \mathcal{P}^* \) is dual PBW-basis for \( \mathcal{A}(\mathcal{C}_M) \).

2. With \( \Lambda_M(\alpha) := \{x \in \Lambda(\alpha) \mid x|_Q \in \text{Add}(M)\} \subset \Lambda(\alpha) \) can find for each \( Z \in \text{Irr}(\Lambda_M(\alpha)) \) an element \( g_Z \in Z \) s.t.

\[
S^*_M := (\delta_{g_Z})_{Z \in \text{Irr}(\Lambda_M(?))} \subset S^*.
\]

**Proof.**

1. (a) follows from Corollary. Key: \( p^m(x) = 0 \) if \( x \in \mathcal{C}_M \) and \( m_s > 0 \) for some \( s > r \).

2. (b) Clearly, \( S^*_M \subset S^* \). Moreover \( S^*_M \) spans \( \mathcal{A}(\mathcal{C}_M) \) by a “count” for each homogeneous component using (a).
Two bases for for $\mathcal{A}(\mathcal{C}_M)$

**Theorem**

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(b) With \( \Lambda_M(\alpha) := \{x \in \Lambda(\alpha) \mid x \mid_Q \in \text{Add}(M)\} \subset \Lambda(\alpha) \) can find for each \( Z \in \text{Irr}(\Lambda_M(\alpha)) \) an element \( g_Z \in Z \) s.t.

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(b) With $\Lambda_M(\alpha) := \{x \in \Lambda(\alpha) \mid x \mid Q \in \text{Add}(M)\}^{\text{open}} \subset \Lambda(\alpha)$ can find for each $Z \in \text{Irr}(\Lambda_M(\alpha))$ an element $g_Z \in Z$ s.t.

$$S_M^* := (\delta_{g_Z})_{Z \in \text{Irr}(\Lambda_M(?))} \subset S^*.$$

**Proof.**

(a) follows from Corollary. Key: $p^m(x) = 0$ if $x \in \mathcal{C}_M$ and $m_s > 0$ for some $s > r$.

(b) Clearly, $S_M^* \subset S^*$. Moreover $S_M^*$ spans $\mathcal{A}(\mathcal{C}_M)$ by a “count” for each homogeneous component using (a).
Two bases for for $\mathcal{A}(\mathcal{C}_M)$

Theorem

(a) $\mathcal{A}(\mathcal{C}_M) := \text{span}_\mathbb{C}\{\delta_X | X \in \mathcal{C}_M\} \overset{!}{=} \mathbb{C}[\delta_{M(1)}, \ldots, \delta_{M(r)}] \subset \mathcal{M}^*$ is a polynomial ring. Thus $(\delta_{M'})_{M' \in \text{Add}(M)} \subset \mathcal{P}^*$ is dual PBW-basis for $\mathcal{A}(\mathcal{C}_M)$.

(b) With $\Lambda_M(\alpha) := \{x \in \Lambda(\alpha) | x|_Q \in \text{Add}(M)\}^{\text{open}} \subset \Lambda(\alpha)$ can find for each $Z \in \text{Irr}(\Lambda_M(\alpha))$ an element $g_Z \in Z$ s.t.

$$S^*_M := (\delta_{g_Z})_{Z \in \text{Irr}(\Lambda_M(?))} \subset S^*.$$ 

Proof.

(a) follows from Corollary. Key: $p^m(x) = 0$ if $x \in \mathcal{C}_M$ and $m_s > 0$ for some $s > r$.

(b) Clearly, $S^*_M \subset S^*$. Moreover $S^*_M$ spans $\mathcal{A}(\mathcal{C}_M)$ by a “count” for each homogeneous component using (a).
Preinjective modules and adaptable elements of $W$

The preinjective module $M = \bigoplus_{i=1}^{13} M(i)$ induces a reduced expression for the adaptable element $w = w_M = s_a s_b s_e s_c s_d s_b s_a s_e s_d s_b s_c s_a s_e$ in the (affine) Coxeter Group $W$ of type $\tilde{D}_4$.

$$\Phi_w := \{ \alpha \in \Phi^+ \mid \langle -w(\alpha) \rangle \in \Phi^+ \}$$

$$= \{ \dim M(1), \dim M(2), \ldots, \dim M(r) \}$$
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(pro-) Unipotent groups

Recall: $U(n)^*_{gr}$ the graded dual of $U(n)$ is a commutative Hopf algebra, with comultiplication given by the dual $\mu^*$ of the multiplication map for $U(n)$. We have

$$U(n)^*_{gr} = \mathbb{C}[p_1^*, p_2^*, \cdots] = S(n^*_{gr}) \quad \text{(as a ring)}$$

Get a pro-unipotent group:

$$N := (\text{Spec } U(n)^*_{gr})(\mathbb{C}) \quad \text{as a set } \hat{n} \quad \text{(completion of } n = \bigoplus_\alpha n_\alpha)$$

carries a group structure via $\mu^*$, such that $\text{Lie}(N) = \hat{n}$.

This is the positive part of the maximal Kac-Moody group $G_{\text{max}}$ associated to $\mathfrak{g}$. 
Coordinate Ring of $\mathbb{N}(w)$

\[
\hat{n} = (\bigoplus_{\alpha \in \Phi_w} n_\alpha) \oplus (\prod_{\beta \in \Phi^+ \setminus \Phi_w} n_\beta)
\]

$\mathbb{N}(w) \oplus \hat{n}'(w)$ (as subalgebras)

Yields a decomposition

\[
N = \mathbb{N}(w) \odot \mathbb{N}'(w)
\]

\[
N \to N/N'(w) = \mathbb{N}(w) \quad \text{(right lateral classes)}
\]

From this it is elementary to derive for $w = w_M$

\[
\mathbb{C}[\mathbb{N}(w)] = \mathbb{C}[\mathbb{N}]^{N'(w)} = A(C_M) \subseteq U(n)^*_{\text{gr}}
\]
Description of Unipotent Cells

In the small Kac-Moody Group $G_{\text{min}}$ of Kac and Peterson we can define

$$N^w := B_-wN(w)^t \cap N_{\text{min}}$$

Note that $B_-wN(w)^t = B_-wB_-$ is a “Bruhat Cell”. Analyzing this we get the following picture:

Thus $\mathbb{C}[N^w] = A(C_M)_{\delta_P}$ (make coefficients invertible) carries cluster algebra structure categorified by $(C_M, T_M^\vee)$. 

$$D = \{ n \in N \mid \delta_P(n) = 0 \}$$

$P$ proj. generator of $C_M$. 

$N(w)$

$N'(w)$

$N^w$

$D$
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Concluding Remarks

• $N_{\text{min}} = \bigcup_{w \in W} N_w$. An element $n \in N$ is totally non-negative if $n \in N_w$ (for an appropriate $w$) is totally positive with respect to the corresponding cluster structure.

• If $Q$ Dynkin quiver we recover for $w$ adaptable result from [BFZ Cluster III]: Get even a “honest” (rather than an “upper”) cluster algebra structure for the coordinate ring of the reduced double Bruhat Cell $G^{e,w} = N_w$.

• In the general case this proves the conjecture from [BIRS] on the cluster algebra structure of $\mathbb{C}[N^w]$ for $w$ adaptable.
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