Cluster Algebra Structures and Semicanonical Bases for Unipotent Groups

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Outline

Notation & Overview
- Notation
- Overview

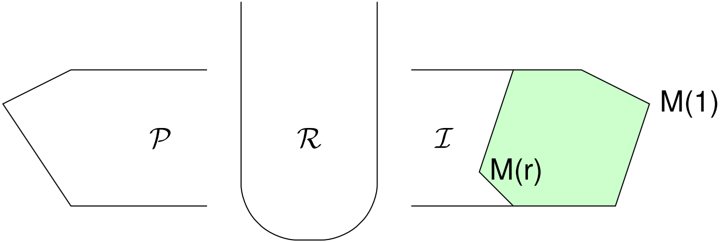
Cluster Tilting in 2-CY Subcategories
- The Category $\mathcal{C}_M$
- Connecting the canonical cluster tilting objects

Semicanonical- and PBW bases
- Basic Construction
- Comultiplication and Applications

Coordinate Rings
- Unipotent- and Kac-Moody Groups
- Unipotent Cells

Preinjective modules for quivers

- Q Quiver without oriented cycles, vertices $Q_0 = \{1, 2, \dots, n\}$
- $M = \oplus_{i=1}^{r} M(i)$ "preinjective" module.



Preprojective Algebras

Definition

- $\overline{Q}$ *double* of Q: extra arrow $i \xleftarrow{\overline{a}} j$ for each arrow $i \xrightarrow{a} j$ of Q.
- $\Lambda := \mathbb{C}\overline{Q}/(\sum_{a \in Q_1} [\overline{a}, a])$ *preprojective algebra* of Q.

Note

- $\mathbb{C}Q \hookrightarrow \Lambda$, so have restriction functor $?|_Q \colon \Lambda\text{-mod} \to \mathbb{C}Q\text{-mod}$.
- Λ depends not on orientation of Q
- $\mathrm{Hom}_{\mathbb{C}}(\Lambda, \mathbb{C})|_Q \cong \prod_{J \text{ indec preinj}} J$
- $M \in \mathbb{C}Q\text{-mod}$ can be interpreted as Λ-module via $M(\overline{a}) = 0$ for all $a \in Q_1$.

Kac-Moody Lie Algebras

- Q defines a symmetric (generalized) Cartan matrix $C = C_{|Q|}$.
- $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ triangular decomposition of KM-algebra defined by C.
- $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{n}_\alpha$ decomposition of $\mathfrak{n}$ into root spaces.
- $U(\mathfrak{n})$ universal enveloping algebra, a graded cocommutative Hopf algebra. Chevalley generators $e_1, \dots, e_n$ with $e_i \in U(\mathfrak{n})_{\alpha_i}$.
- The graded dual $U(\mathfrak{n})^*_{\mathrm{gr}}$ is commutative Hopf algebra. Think of it as coordinate ring of a (pro-) unipotent group N.

References

This is a report on joint work with Bernard Leclerc (Caen) & Jan Schröer (Bonn).
Preprint: `arXiv:math/0703039v3` 121p.

There is related work by Aslak Buan, Osamu Iyama, Idun Reiten & Jeanne Scott:
Cluster structures for 2-Calabi-Yau categories and unipotent groups.
`arXiv:math/0701557v3`.

Overview I

1. The stably 2-Calabi-Yau category

$$\mathcal{C}_M = \{X \in \Lambda\text{-mod} \mid X|_Q \in \mathrm{Add}(M)\}$$

with canonical maximal rigid object T_M “built” from M.
Combinatorics of mutations in $\mathcal{C}_M$.

2. Cluster Character $\delta_? \colon \Lambda\text{-mod}_0 \to U(\mathfrak{n})^*_{\mathrm{gr}}$ restricts to

$$\mathcal{C}_M \to \mathcal{A}_M := \mathrm{span}_{\mathbb{C}}\{\delta_X \mid X \in \mathcal{C}_M\} \subset U(\mathfrak{n})^*_{\mathrm{gr}}$$

provides $\mathcal{A}_M$ with a cluster algebra structure and initial seed given by T_M.
$\mathcal{A}_M$ is a polynomial ring, comes with a semicanonical basis containing the cluster monomials.

Overview II

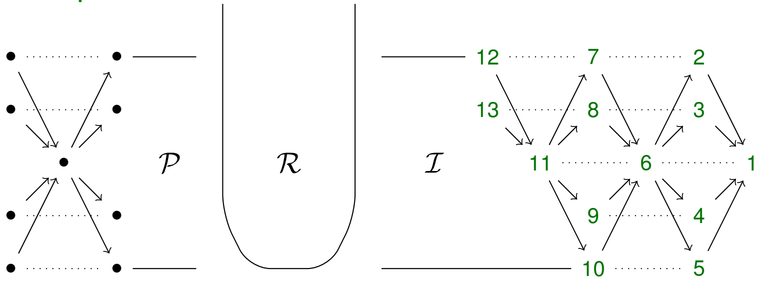
3. Preinjective Module $M = \oplus_{i=1}^{r} M(i)$ defines *adaptable* element $w = w_M$ of length r in the Coxeter group associated to $C_{|Q|}$. This is also the Weyl group of $\mathfrak{g}$ and of the Kac-Moody group $G_{\min}$.
   $N = N(w) \circ N'(w)$ and $N(w) = N/N'(w)$ as homogeneous spaces

$$\mathcal{A}_M = \mathbb{C}[N]^{N'(w)} = \mathbb{C}[N(w)]$$
$$(\mathcal{A}_M)_{\delta(\text{proj.-inj})} = \mathbb{C}[N^w]$$
$$N^w := N \cap B_- w B_- \text{ unipotent cell}$$

The "preinjective" module M

$M = \oplus_{i=1}^{r} M(i)$ direct sum of a family of indecomposable pairwise non-isomorphic preinjective $\mathbb{C}Q$-modules. Family is closed under successors and we assume $\mathrm{Hom}_Q(M(i), M(j)) = 0$ if $i < j$.

Example



Properties of the Category $\mathcal{C}_M$

Theorem

- *$\mathcal{C}_M$ is a stably 2-CY category, closed under factor modules, with $T_M := \oplus_{i=1}^r T_M(i)$ and $T_M^\vee := \oplus_{i=1}^r T_M^\vee(i)$ canonical cluster tilting objects.*
- *The quiver of $\mathrm{End}_\Lambda(T_M)$ and of $\mathrm{End}_\Lambda(T_M^\vee)$ is obtained easily from the quiver of $\mathrm{End}_Q(M)$.*
- *Have a cluster structure on the cluster tilting objects, i.e. exchange of summands corresponds to two exact sequences, quiver of $\mathrm{End}_\Lambda(T)$ changes according to FZ-quiver mutation.*

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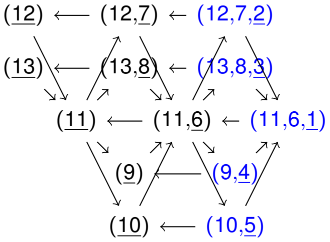
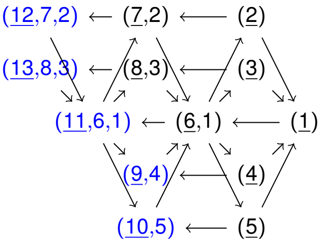
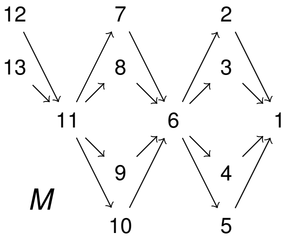
T_M and $T_M^{\vee}$ - an Example

T_M



$T_M^{\vee} = \Omega_{\mathcal{C}_M} T_N$





$(11,\underline{6})$ means

$T_M(6)|_Q = M(11) \oplus M(6)$

Remarks

- $X \in \mathcal{C}_M$ rigid $\Rightarrow$ X uniquely determined by $X|_Q \in \mathrm{Add}(M)$
- Can work out how $T^*(k)|_Q$ is obtained from quiver of $\mathrm{End}_\Lambda(T)$ and $(T(k)|_Q)_{i=1,2,\dots,r}$ similar to "denominator mutation".
- the previous two remarks can be nicely interpreted using that $B = \mathrm{End}_\Lambda(T_M)$ is quasi-hereditary and

$$\mathrm{Hom}_\Lambda(-, T_M)\colon \mathcal{C}_M \xrightarrow{\sim} \mathcal{F}(\Delta) \subset B\text{-mod}$$

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Connecting T_M and $T_M^\vee$

Theorem

There exists and explicit sequence of mutations connecting T_M to $T_M^\vee$.
On this "path" each $M(i)$ $(i = 1, 2, \ldots, r)$ appears as a direct summand of a cluster tilting object.

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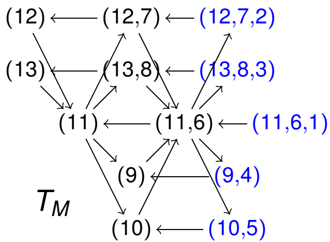

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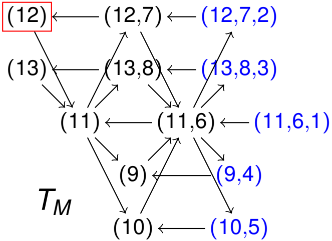
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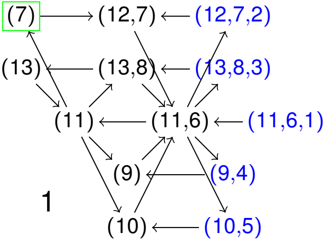
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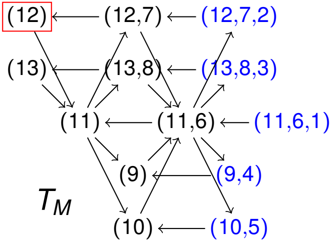
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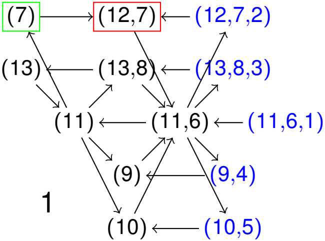
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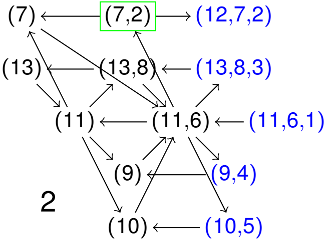
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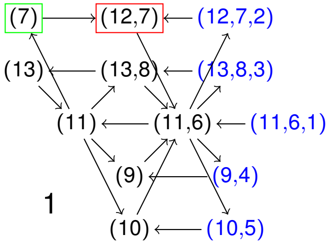
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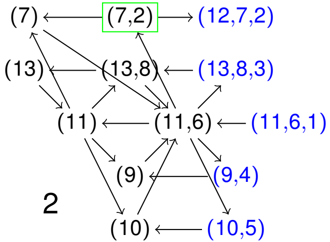
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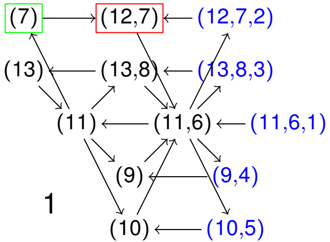
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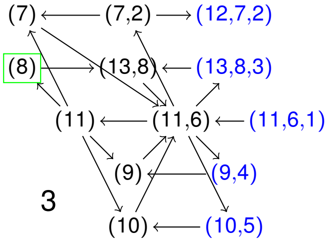
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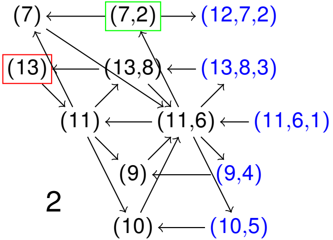
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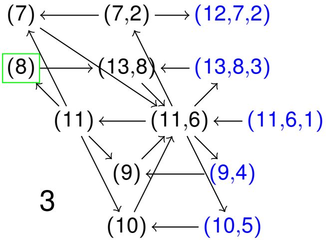
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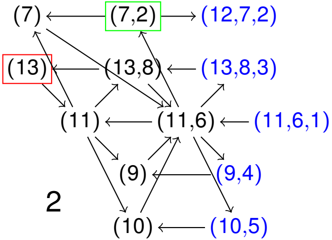
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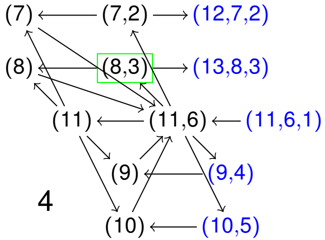
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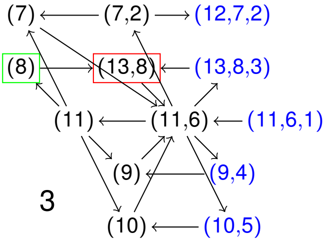
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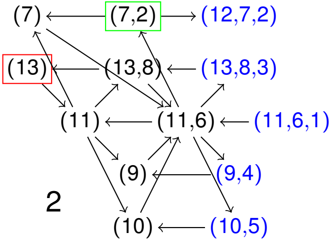
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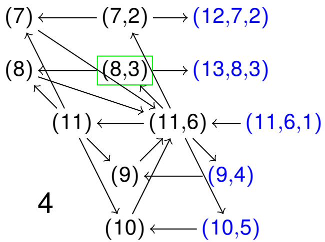
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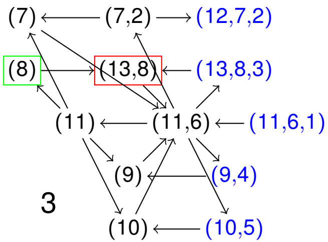
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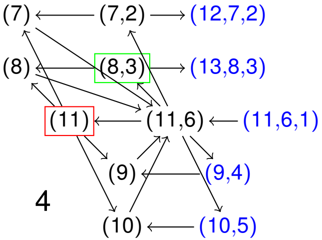
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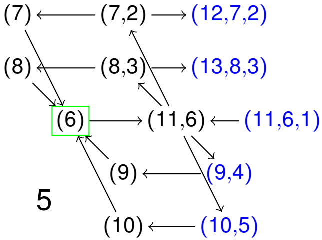
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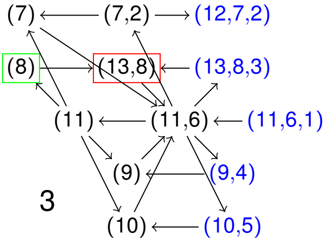
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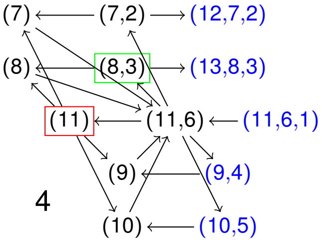
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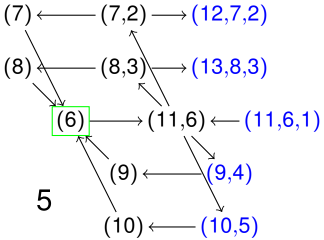
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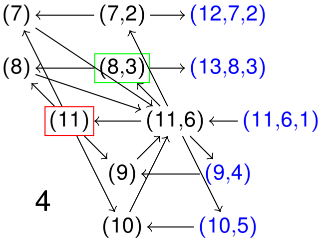
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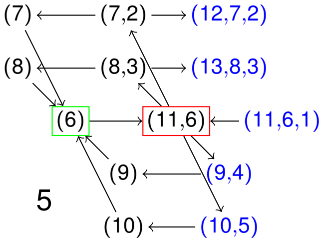
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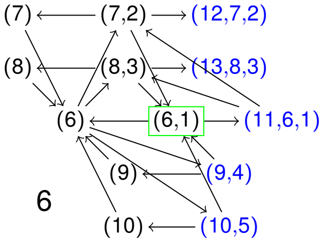
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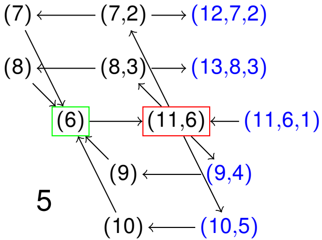
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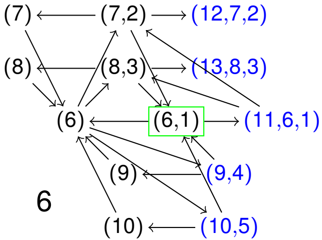
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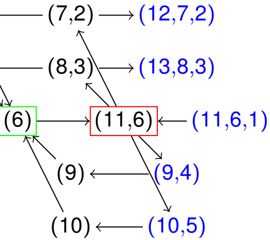
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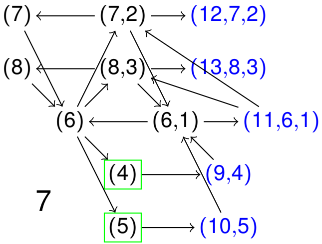
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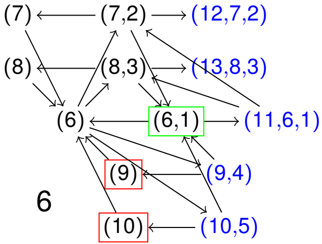
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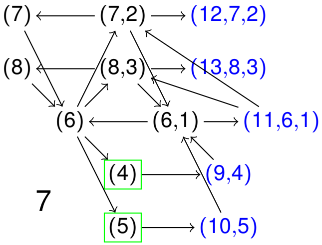
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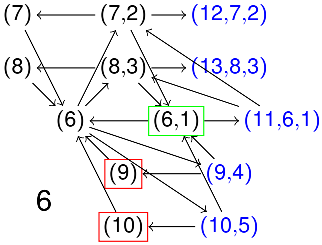
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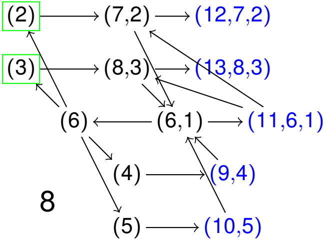
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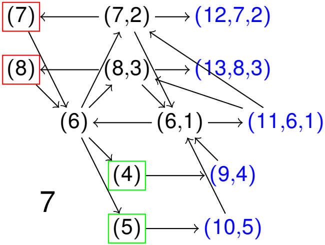
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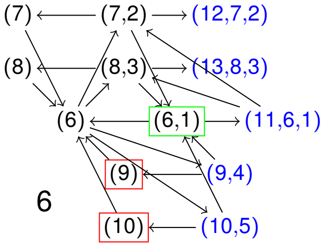
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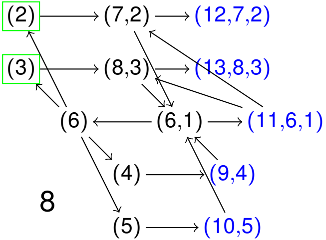
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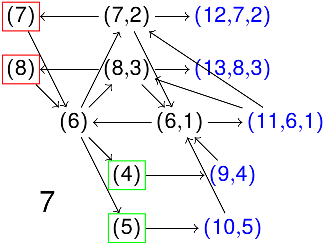
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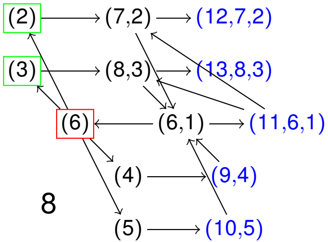
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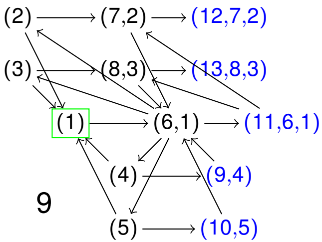
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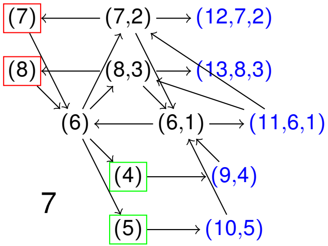
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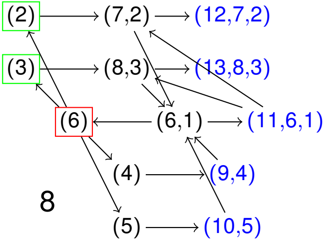
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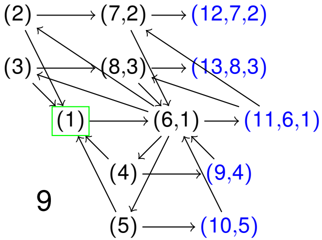
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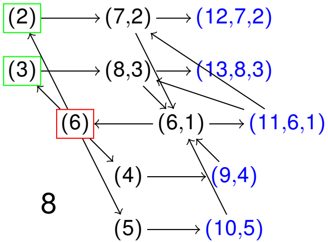
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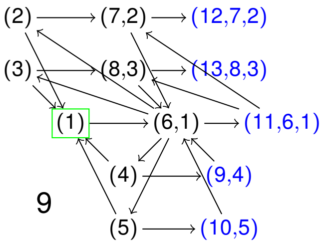
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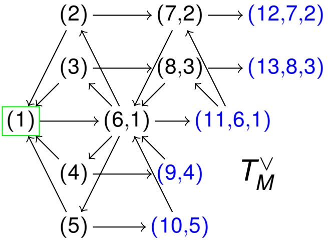
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Basic Notations

- α dimension vector
- $\Lambda(\alpha)$ (affine) Variety of nilpotent representations of Λ with dimension vector α.
- $\mathrm{Rep}_Q(\alpha)$ affine *space* of representations of Q with dimension vector α.
- In both cases $\mathrm{GL}(\alpha)$ orbits are isoclasses of the corresponding representations.
- Note that $\mathrm{Rep}_Q(\alpha)$ is naturally an irreducible component of $\Lambda(\alpha)$.

Algebra of Constructible Functions

- $\widetilde{\mathcal{M}}(\alpha)$ Space of $\mathrm{GL}(\alpha)$-invariant constructible functions $\Lambda(\alpha) \to \mathbb{C}$.
- $\oplus_\alpha \widetilde{\mathcal{M}}(\alpha)$ becomes a graded assoc. algebra via

$$(f * g)(x) := \int_{u \leq x} f(u)g(x/u) \quad \text{top. Euler Characteristic}$$

- $\mathcal{M}$ subalgebra generated by $1\!\!1_i \in \mathcal{M}(\alpha_i)$. Here α_i simple root, so $\Lambda(\alpha_i) = \{\mathrm{pt}\}$.
- $\Psi \colon U(\mathfrak{n}) \to \mathcal{M}$ defined by $e_i \mapsto 1\!\!1_i$ (where e_i Chevalley generator) is surjective algebra homomorphism. In fact an isomorphism (uses semicanonical bases).

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Two Bases

Theorem (Lusztig)

Each $\mathcal{M}(\alpha)$ *has a basis* $\mathcal{S}(\alpha) := (s_Z)_{Z \in \mathrm{Irr}(\Lambda(\alpha))}$ *such that*

$$s_Z|_{Z'} = \delta_{Z,Z'} \mathbb{1}_{Z'} \quad \textit{generically}.$$

The union $\mathcal{S} = \cup_\alpha \mathcal{S}(\alpha)$ is the *Semicanonical Basis* of $U(\mathfrak{n})$.

Definition

Let $(p_i)_{i\in\mathbb{N}}$ be a basis of $\mathfrak{n}$ consisting of *root vectors* and

$$p_i \in \mathfrak{n}_{\underline{\dim} M(i)} \text{ for } i = 1, 2, \ldots, r \text{ (real roots)}$$

then the $\mathbf{p}^{(\mathbf{m})} := \frac{1}{m_s! \cdots m_1!} p_s^{m_s} * \cdots * p_1^{m_1}$ for $\mathbf{m} \in \mathbb{N}_0^{(\mathbb{N})}$

form an appropriate (scaled) PBW-basis of $U(\mathfrak{n})$.

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Comultiplication

Theorem

Can define a comultiplication $\Delta\colon \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ *such that for* $f \in \mathcal{M}(\alpha)$ *with*

$$\Delta(f) = \sum_{\alpha' + \alpha'' = \alpha} f' \otimes f'' \quad \textit{get} \quad (f' \otimes f'')(x', x'') = f(x' \oplus x'').$$

Coincides with usual comultiplication of $U(\mathfrak{n})$*.*

Despite the tautological appearance this is quite non-trivial, proof uses several fibre bundle constructions.

Consequences of Comultiplication formula

Corollary

(a) *If* $f \in \mathfrak{n}_\alpha \subset \mathcal{M}(\alpha) \Rightarrow \mathrm{supp}(f) \subset \Lambda(\alpha)^{indec}$.

(b) *If* α *is a real root then* $f|_{\mathrm{Rep}_Q(\alpha)} = c\mathbb{1}_{\mathcal{O}_N} \neq 0$
N unique indec. rep. of Q with $\underline{\dim} N = \alpha$.

(c) $p_r^{(m_r)} * \cdots * p_1^{(m_1)}|_{\mathrm{Rep}_Q(\beta)} = \mathbb{1}_{\mathcal{O}(M')}$

where $M' = \oplus_{i=1}^r M(i)^{m_r}$, $\underline{\dim} M' = \beta$

Proof.

(a) Use f primitive: $\Delta(f) = 1 \otimes f + f \otimes 1$

(b) Kac's theorem: For real roots there exists a unique indec. representation of Q.

(c) Use $\mathrm{Ext}^1_Q(M(i), M(j)) = 0$ for $i \geq j$.

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Graded Dual and Evaluation Forms

The *Graded Dual* $\mathcal{M}^* = U(\mathfrak{n})^*_{\mathrm{gr}}$ is a *commutative* Hopf-algebra with multiplication Δ^*. Consider the evaluation form

$$\delta_? \colon \Lambda\text{-mod} \to \mathcal{M}^*,\ x \mapsto (f \mapsto (f(x)).$$

Theorem

δ is a cluster character i.e.

(a) $\delta_X \cdot \delta_Y = \delta_{X \oplus Y}$.

(b) *If* $\dim \operatorname{Ext}^1_\Lambda(x, y) = 1$ *with corresponding non split s.e.s*

$$0 \to x \to E' \to y \to 0 \quad \textit{and} \quad 0 \to y \to E'' \to x \to 0$$

then $\quad \delta_x \cdot \delta_y = \delta_{E'} + \delta_{E''}$.

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$$\mathcal{S}^*_M := (\delta_{g_Z})_{Z \in \mathrm{Irr}(\Lambda_M(?))} \subset \mathcal{S}^*.$$

Proof.

(a) follows from Corollary. Key: $p^{\mathbf{m}}(x) = 0$ if $x \in \mathcal{C}_M$ and $m_s > 0$ for some $s > r$.

(b) Clearly, $\mathcal{S}^*_M \subset \mathcal{S}^*$. Moreover $\mathcal{S}^*_M$ spans $\mathcal{A}(\mathcal{C}_M)$ by a "count" for each homogeneous component using (a). □

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Two bases for for $\mathcal{A}(\mathcal{C}_M)$

Theorem

(a) $\mathcal{A}(\mathcal{C}_M) := \mathrm{span}_{\mathbb{C}}\{\delta_X \mid X \in \mathcal{C}_M\} \overset{!}{=} \mathbb{C}[\delta_{M(1)}, \dots, \delta_{M(r)}] \subset \mathcal{M}^*$ *is a polynomial ring. Thus* $(\delta_{M'})_{M' \in \mathrm{Add}(M)} \subset \mathcal{P}^*$ *is dual PBW-basis for* $\mathcal{A}(\mathcal{C}_M)$.

(b) *With* $\Lambda_M(\alpha) := \{x \in \Lambda(\alpha) \mid x|_Q \in \mathrm{Add}(M)\} \overset{open}{\subset} \Lambda(\alpha)$ *can find for each* $Z \in \mathrm{Irr}(\Lambda_M(\alpha))$ *an element* $g_z \in Z$ *s.t.*

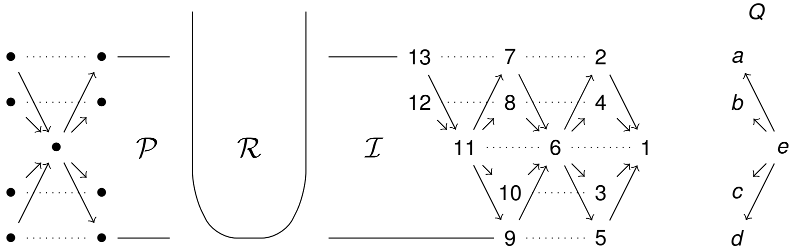
$$\mathcal{S}_M^* := (\delta_{g_Z})_{Z \in \mathrm{Irr}(\Lambda_M(?))} \subset \mathcal{S}^*.$$

Proof.

(a) follows from Corollary. Key: $p^{\mathbf{m}}(x) = 0$ if $x \in \mathcal{C}_M$ and $m_s > 0$ for some $s > r$.

(b) Clearly, $\mathcal{S}_M^* \subset \mathcal{S}^*$. Moreover $\mathcal{S}_M^*$ spans $\mathcal{A}(\mathcal{C}_M)$ by a “count” for each homogeneous component using (a). □

Preinjective modules and adaptable elements of W



The preinjective module $M = \oplus_{i=1}^{13} M(i)$ induces a reduced expression for the *adaptable* element

$$w = w_M = s_a s_b s_e s_c s_d s_b s_a s_e s_d s_b s_c s_a s_e$$

in the (affine) Coxeter Group W of type $\widetilde{\mathrm{D}}_4$.

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$$= \{\underline{\dim} M(1), \underline{\dim} M(2), \ldots, \underline{\dim} M(r)\}$$

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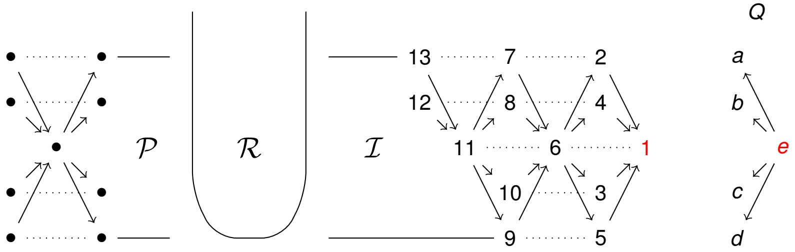

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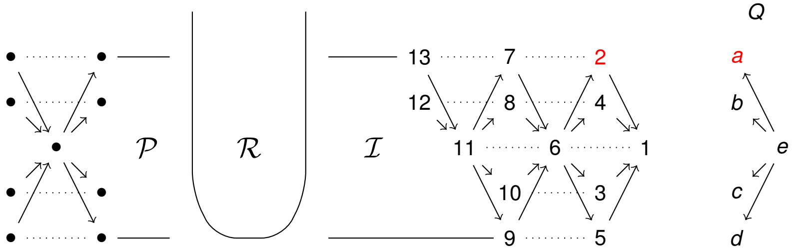

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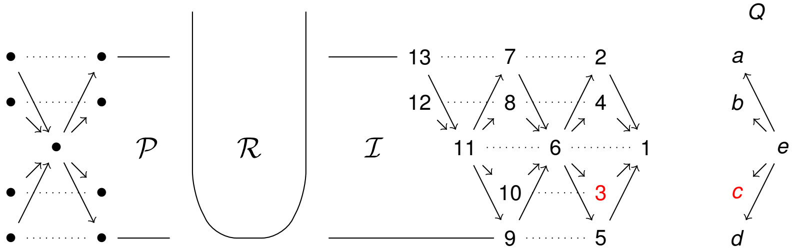

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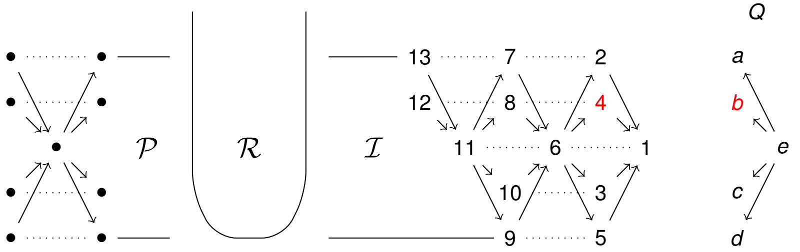

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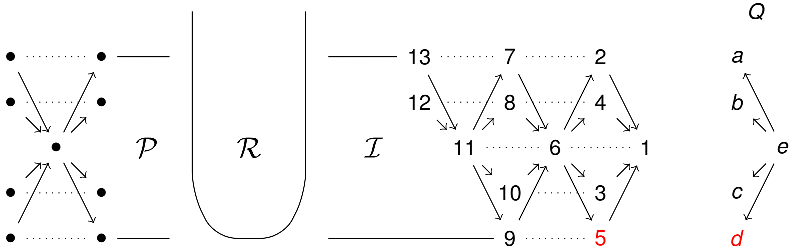

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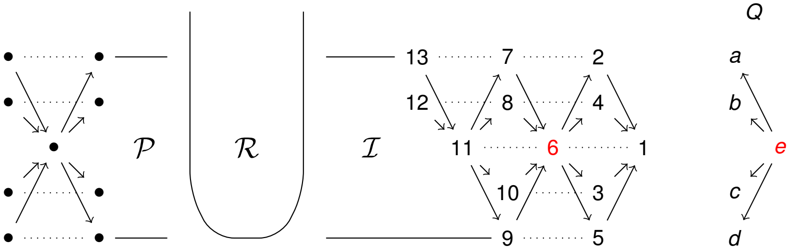

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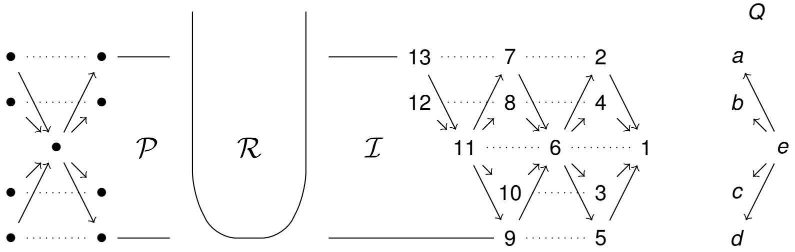

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(pro-) Unipotent groups

Recall: $U(\mathfrak{n})^*_{\mathrm{gr}}$ the graded dual of $U(\mathfrak{n})$ is a commutative Hopf algebra, with comultiplication given by the dual μ^* of the multiplication map for $U(\mathfrak{n})$. We have

$$U(\mathfrak{n})^*_{\mathrm{gr}} = \mathbb{C}[p_1^*, p_2^*, \cdots] = S(\mathfrak{n}^*_{\mathrm{gr}}) \quad \text{(as a ring)}$$

Get a pro-unipotent group:

$$N := (\mathrm{Spec}\, U(\mathfrak{n})^*_{\mathrm{gr}})(\mathbb{C}) \overset{\text{as a set}}{=} \widehat{\mathfrak{n}} \quad (\text{completion of } \mathfrak{n} = \oplus_\alpha \mathfrak{n}_\alpha)$$

carries a group structure via μ^*, such that $\mathrm{Lie}(N) = \widehat{\mathfrak{n}}$.

This is the positive part of the maximal Kac-Moody group $G_{\max}$ associated to $\mathfrak{g}$.

Coordinate Ring of N(w)

$$\widehat{\mathfrak{n}} = (\oplus_{\alpha\in\Phi_w}\mathfrak{n}_\alpha) \oplus (\Pi_{\beta\in\Phi^+\setminus\Phi_w}\mathfrak{n}_\beta)$$

$$\mathfrak{n}(w) \quad \oplus \quad \widehat{\mathfrak{n}}'(w) \qquad \text{(as subalgebras)}$$

Yields a decomposition

$$N = N(w) \circ N'(w)$$

$$N \to N/N'(w) = N(w) \quad \text{(right lateral classes)}$$

From this it is elementary to derive for $w = w_M$

$$\mathbb{C}[N(w)] = \mathbb{C}[N]^{N'(w)} = \mathcal{A}(\mathcal{C}_M) \subseteq U(\mathfrak{n})^*_{\mathrm{gr}}$$

Description of Unipotent Cells

In the small Kac-Moody Group $G_{\min}$ of Kac and Peterson we can define

$$N^w := B_- wN(w)^t \cap N_{\min}$$

Note that $B_- wN(w)^t = B_- wB_-$ is a "Bruhat Cell". Analyzing this we get the following picture:

Thus $\mathbb{C}[N^w] = \mathcal{A}(\mathcal{C}_M)_{\delta_P}$ (make coefficients invertible) carries cluster algebra structure categorified by $(\mathcal{C}_M, T_M^\vee)$.

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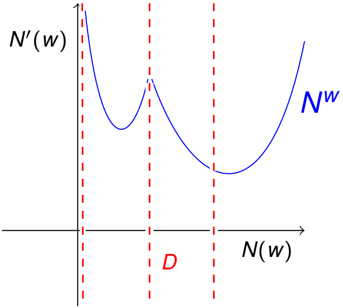

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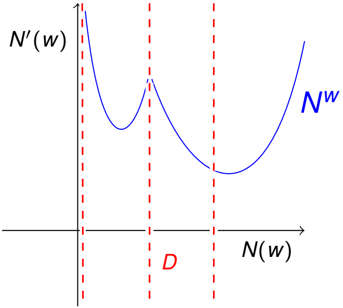

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Concluding Remarks

- $N_{\min} = \dot{\cup}_{w \in W} N^w$. An Element $n \in N$ is totally non-negative if $n \in N^w$ (for an appropriate w) is totally positive with respect to the corresponding cluster structure.
- If Q Dynkin quiver we recover for w adaptable result from [BFZ Cluster III]: Get even a"honest" (rather than an "upper") cluster algebra structure for the coordinate ring of the reduced double Bruhat Cell $G^{e,w} = N^w$.
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