A NEW DIMENSION FUNCTION ON RINGS

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Rosenberg and Zelinsky [10] studied rings over which every module of finite length has an injective hull of finite length. As special case Michler and Villamayor [7] considered rings over which every simple module is injective.

Definition. The ring R is a (right) V-ring if it satisfies the equivalent conditions:
(1)- Every simple right R-module is injective.
(2)- Mod-R has a semisimple cogenerator.
(3)- For every module M_R, $\text{Rad}(M) = 0$.(Jacobson radical)
(4) Any right ideal is an intersection of maximal right ideals.

As a consequence of (4):

$$\forall I \trianglelefteq_r R,\ I^2 = I.$$

Thus if R is a commutative V-ring, it has to be Von-Neumann regular . In fact

If R is commutative, then R is a V-ring iff R is regular.

But we cannot drop commutativity: The endomorphism ring of an infinite dimensional left vector space is regular but not a right V-ring.

Examples.

Cozzens [1]: There exists a Noetherian V domain with a unique simple module. This is an Ore extension example.

Osofsky [9]: There exists a Noetherian V domain with infinitely many nonisomorphic simple modules.

McConnell and Robson [6]: The first Weyl-algebra A_1 over a field of characteristic zero has infinitely many non isomorphic simple modules, and no simple A_1-module is injective.

Furthermore in this connection:

Matlis [5]: If R is commutative Noetherian $E(M_R)$ is Artinian whenever M_R is simple.

Snider [11]: If G is a nilpotent by finite group, then for the integral group ring $\mathbb{Z}[G] = R$, $E(M_R)$ is Artinian if M_R is simple.

All these results motivated Hirano's study of n-V-rings, and π-V-rings [4]:

Definition. A ring R is called a π-V-ring if
$\forall$ simple M_R, $\text{length}_R(E(M)) < \infty$, and
R is called a k-V-ring if $\forall$ simple M_R, $\text{Len}_R(E(M)) \leq k$.

Thus 1-V-ring $\equiv$ V-ring.

Rosenberg and Zelinsky : If R is left and right Artinian PI-ring, then R is (left and right) π-V-ring.

In particular a module finite algebra over a commutative Artinian ring is a π-V-ring.

Notation. π-$\underline{\text{V}}$ is the class of all π-V-rings.
n-$\underline{\text{V}}$ is the class of all n-V-rings.

So

$$1 - \underline{V} \subseteq 2 - \underline{V} \subseteq ... \subseteq n - \underline{V} \subseteq ... \subseteq \pi - \underline{V}$$

Hirano characterized n-V-rings and π-V-rings and proved:

1- $R \in \pi$-$\underline{\text{V}}$ iff $\forall$ M_R with $\text{Len}_R(M) < \infty$, $\text{Len}_R E(M) < \infty$ iff $\forall$ M_R, $\alpha_\pi(M) = 0$ where $\alpha_\pi(M) = \cap N$, $N \leq M$ with $\text{Len}_R(M/N) < \infty$.
2- $R \in n$-$\underline{\text{V}}$ iff $\forall$ M_R, $\alpha_n(M) = 0$ where $\alpha_n(M) = \cap N$, $N \leq M$ with $\text{Len}_R(M/N) \leq n$.
3- Let S be a finite normalizing extension of R with a set of generators consisting of k-elements. Then

$$R \in \pi - \underline{V} \Rightarrow S \in \pi - \underline{V}$$

$$R \in n - \underline{V} \Rightarrow S \in kn - \underline{V}$$

4- If R is commutative, then
$R \in \pi - \underline{V}$ iff all localizations of R at maximal ideals are Artinian.

$R \in n - \underline{V}$ iff for all maximal ideals T of R, $\text{Len}(R_T) \leq n$ as an R_T-module.

Question. Given $n \in \mathbb{N}$, does there exist a ring R such that
$R \in n - \underline{V}$ but $R \notin (n-1)\underline{V}$? I.e, $(n-1) - \underline{V} \subsetneqq n - \underline{V}$?

If yes, we say that R has V-dimension n, and write V-dim$(R) = n$.
If $R \in \pi - \underline{V}$ and $\forall n \in \mathbb{N}$, $R \notin n - \underline{V}$, we write V-dim$(R) = \infty$.

Note. V-dim(R) is not defined if $R \notin \pi - \underline{V}$.

We approach the above question through formal triangular matrix rings, and prove:

Theorem. Given $n \in \mathbb{N}$, there exist formal triangular matrix rings T_1, T_2 and T_3 with V-dim$(T_1) = n$, V-dim$(T_2) = \infty$, and $T_3 \notin \pi - \underline{V}$.

Rudiments of Formal Triangular Matrix Rings

Let A and B be rings, ${}_BM_A$ a bimodule, and

$$T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}.$$

The following is recalled from [3].
The category Mod$-T$ is equivalent to a category Ω of triples $(X, Y)_f$ where $X \in$Mod$_A$, $Y \in$ Mod$-B$ and $f : Y \bigotimes_B M \longrightarrow X$ is a map in Mod$-A$.
Morphisms from $(X, Y)_f$ to $(U, V)_g$ in Ω are pairs (φ_1, φ_2)
$\varphi_1 : X \longrightarrow U$ is a map in Mod$-A$, $\varphi_2 : Y \longrightarrow V$ is a map in Mod$-B$ such that

$$\begin{array}{ccc} Y \bigotimes M & \longrightarrow & X \\ \downarrow \varphi_2 & & \downarrow \varphi_1 \\ V \bigotimes M & \longrightarrow & U \end{array}$$

is commutative. In fact the right T-module corresponding to $(X,Y)_f$ is the additive group $X \oplus Y$ with the right $T-$action:

$$(x,y)\begin{pmatrix} a & 0 \\ m & b \end{pmatrix} = (xa + f(y\otimes m), yb)$$

If $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1e_2 = e_2e_1 = 0$ and $e_1 + e_2 = 1$. So $A \cong e_1Te_1$ and $B \cong e_2Te_2$.

For any Z_T we have $Z_T = Ze_1 \oplus Ze_2$, $(Ze_1)_{e_1Te_1}$ and $(Ze_2)_{e_2Te_2}$ are modules. Put $X = Ze_1$, $Y = Ze_2$, so we get X_A and Y_B.

Define: $f : Y \bigotimes_B M \longrightarrow X$ by $f(y,m) = y\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} = y\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix}e_1 \in X$
f is A-linear, and $Z_T = (X \otimes Y)_T$ corresponds to $(X,Y)_f \in \Omega$.

If $Z' \le Z_T$ then $Z' = Z'e_1 \oplus Z'e_2 \equiv (X' \oplus Y')_T$ with $X' = Z'e_1$, $Y' = Z'e_2$, $X' \le X_A$, $Y' \le Y_B$ and Z' corresponds to $(X',Y')_{f'}$ where
$f' = fo(j_2 \otimes Id_M) : Y' \otimes M \longrightarrow X'$ and $j_2 : Y' \longrightarrow Y$ is the inclusion map.
Conversely if $X' \le X_A$, $Y' \le Y_B$, $f' = fo(j_2 \otimes Id_M)$ satisfy $Imf' \le X'$ then $(X',Y')_{f'} \in \Omega$, and $(X' \oplus Y')_T \le (X \oplus Y)_T$ which in Ω means that
$(j_1,j_2) : (X',Y')_{f'} \longrightarrow (X,Y)_f$ is a monomorphism.

Lemma. If Z_T is simple then Z corresponds to either $(X,0)_0$ or $(0,Y)_0$.

Let now $(X,Y)_f \in \Omega$. Define $\tilde{f} : Y \longrightarrow Hom_A(M,X)$, $\tilde{f}(y)(m) = f(y\otimes m)$

Proposition.(Müller [8]) The injective envelope of $Z(X \oplus Y)_T$ corresponds to
$(E(X)$,Hom$_A(M,E(X) \oplus E(\ker\tilde{f}))_\delta$ where
$\delta : \{\text{Hom}_A(M,E(X)) \oplus E(\ker\tilde{f})\} \otimes_B M \longrightarrow E(X)$
$\delta((\eta,\mu)\otimes m) = \eta(m)$, $\forall\eta \in$Hom$_A(M,E(X))$, $\mu \in E(\ker\tilde{\beta})$, $m \in M$.
Furthermore if $\varphi_1 : X \longrightarrow E(X)$ is the natural embedding,
$\varphi_2 : Y \longrightarrow$Hom$_A(M,E(X)) \oplus E(\ker\tilde{f})$ is given by $\varphi_2(y) = (\iota o\tilde{f}(y), \tau(y))$ where $\tau : Y \longrightarrow E(\ker\tilde{f})$ is an extension of the inclusion $\ker\tilde{f} \hookrightarrow E(\ker\tilde{f})$,
then (φ_1,φ_2) corresponds to an essential monomorphism: $Z \longrightarrow E(Z)$.

Conclusion: If $Z = (X \oplus Y)_T$ is simple, then $E(Z)$ corresponds to either
$(E(X)$,Hom$_A(M,E(X))_\delta$ or $(0,E(Y))_0$ where δ :Hom$_A(M,E(M)) \otimes_B M \longrightarrow E(X)$
with $\delta(\eta)\otimes m = \eta(m)$.

Theorem.(Haghany [2]) Let $Z = (X \oplus Y)_T$. Then K-dim(Z) exists if and only if K-dim(X_A) and K-dim(Y_B) exist, and in this case K-dim(Z_T) =max{K-dim(X_A), K-dim(Y_B)}. In particular Z_T is Artinian if and only if X_A and Y_B are Artinian. These are valid if we replace Krull-dimension by Noetherian dimension, hence in particular Z_T is Noetherian if and only if X_A and Y_B are Noetherian.

Lemma. Let $Z = (X \oplus Y)_T$. Then Z_T has finite length if and only if X_A and Y_B are of finite length. If Z_T is of finite length, then

$$Le_T(Z) = Le_A(X) + Le_B(Y)$$

Proof. $(X \oplus 0)_T$ and X_A have isomorphic lattices of submodules. Suppose Z_T is of finite length, As $(X \oplus 0) \leq Z_T$, we get $\mathrm{Le}_A(X) < \infty$. Now $\frac{Z}{(X \oplus 0)}$ corresponds to $(0, Y)_0$, hence $\mathrm{Le}_B(Y) = \mathrm{Le}_T(\frac{Z}{(X \oplus 0)}) < \infty$. Since composition length is additive we have $\mathrm{Le}_T(Z) = \mathrm{Le}_A(X) + \mathrm{Le}_B(Y)$. Conversely suppose X_A and Y_B are of finite length. Then X_A and Y_B are both Artinian and Noetherian, and it follows that Z_T is both Artinian and Noetherian, hence Z_T is of finite length.

Theorem. Let $T = \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}$. Then V-dim$(T) = n$ if and only if the following hold.

(1)- $B \in n\underline{V}$ and $\forall$ simple X_A, $\mathrm{Le}_A(E(X)) + \mathrm{Le}_B(\mathrm{Hom}_A(M, E(X))) \leq n$ (*).

(2)- Either V-dim$(B) = n$ or $\exists$ simple X_A for which the equality in (*) holds.

Proof. Suppose $Z = (X \oplus Y)_T$ is simple. Then $E(Z)$ corresponds to $(0, E(Y))$ or to $(E(X), \mathrm{Hom}_A(M, E(X)))_\delta$. Now by previous Lemma, (1) $\Rightarrow T \in n-\underline{V}$;
(2)$\Rightarrow T \notin (n-1)-\underline{V}$. Consequently (1) and (2) imply V-dim$(T) = n$.
Conversely suppose V-dim$(T) = n$. Then (1) holds. Since $T \notin (n-1)-\underline{V}$, either $\exists$ simple $T-$module corresponding to $(X, 0)_0$ with X_A (necessarily simple) satisfying $\mathrm{Le}_T(E((X, 0)_0) \nleq n-1$ or $\exists$ simple $T-$module of the form $(0, Y)_0$ with Y_B simple satisfying $\mathrm{Le}_T(E((0, Y)_))) = \mathrm{Le}_B(E(Y)) \nleq n-1$, giving $B \notin (n-1)-\underline{V}$, consequently by (1) V-dim$(B) = n$. Hence (2) holds.

Corollary. V-dim$(A \times B)$ =max{V-dim(A), V-dim(B)}.

Proof. In previous theorem put $M = 0$.

Corollary. Given $n \geqslant 2$, $\exists T_1$ with V-dim$(T_1) = n$.

Proof. Let F be a field, $M = F^{(n-1)}$ and $T_1 = \begin{pmatrix} F & 0 \\ M & F \end{pmatrix}$. Then V-dim$(F) = 1$ and F is the only simple F-module, so $\mathrm{Le}_F[\mathrm{Hom}_F(M, F)]$ =dim$(M) = n-1$. Consequently $\text{Le}_F(F)+\ \text{Le}_F[\text{Hom}_F(M,F)] = n$.

Theorem There exists a formal triangular matrix ring T_2 with V-dim$(T_2) = \infty$.

Proof. Let $A = A_1(k)$ be the first Weyl algebra over a field of characteristic zero. Choose a countable infinite set of non-isomorphic simple A-modules $X_1, X_2, \cdots$. Let $Z_i = X_i^{(i)}, M = \sum_{i\geqslant 1} \oplus Z_i$, and $T_2 = \begin{pmatrix} A & 0 \\ M & k \end{pmatrix}$. If X is an arbitrary simple A-module, we have:
$\text{Hom}_A(M, E(X)) = \text{Hom}_A(\sum_{i\geqslant 1} \oplus Z_i, E(X)) \cong \prod_{i\geqslant 1}\text{Hom}_A(Z_i, E(X))$.
Suppose $X \cong X_j$ for some j. Then
$\prod_{i\geqslant 1}\text{Hom}_A(Z_i, E(X)) \cong \text{Hom}_A(Z_j, E(X_j)) \cong (\text{Hom}_A(X_j, E(X_j)))^{(j)}$.
Since $\text{Hom}_A(X_j, E(X_j))$ embeds in $\text{Hom}_A(X_j, X_j)$, and the endomorphism rings of simple A-modules are finite dimensional k-vector spaces, we deduce that
$j \leqslant \dim_k \text{Hom}_A(M, E(X)) < \infty$. If on the other hand $X \ncong X_j$ for all $1 \leq j$, we conclude that $\text{Hom}_A(M, E(X)) = 0$. It follows that V-dim$(T) = \infty$.

We now construct a formal triangular matrix ring T_3 for which V-dimension is not defined.

Example: Let p be a fixed prime number, and set $T_3 = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_{p^\infty} & \mathbb{Z} \end{pmatrix}$.
We observe below that $T_3 \notin \pi-\underline{\text{V}}$. In fact simple $\mathbb{Z}$-modules are of the form $\bar{\mathbb{Z}}_q$ with $\mathbb{Z}_{q^\infty}$ as its $\mathbb{Z}$-injective envelope, (q is a prime number). Now triple $(\bar{\mathbb{Z}}_p, 0)_0$ corresponds to a simple T_3-module whose injective envelope gives $(\mathbb{Z}_{p^\infty}, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty}))_\delta$.
Since the endomorphism ring of $\mathbb{Z}_{p^\infty}$ is not of finite length as a $\mathbb{Z}$-module we deduce that $T_3 \notin \pi$-$\underline{\text{V}}$.

FURTHER DEVELOPMENTS

Co-Noetherian ring R is one such that the injective hull of every simple R-module is Artinian. Of course all $\pi-\underline{\text{V}}$ rings are co-Noetherian and there are many other examples including the first Weyl algebra over finitely generated commutative $\mathbb{Z}$-algebras. In a forthcoming paper, we have defined a dimension function that measures how distant a ring is from being co-Noetherian . The crucial step towards this is the following fact: Given a positive integer n there exists a ring R with a simple module whose injective envelope has Krull dimension n.
One may analogously consider co-Artinian rings using Noetherian dimension.

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