Infinite dimensional tilting theory

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Notation. Let R be a ring (associative, with 1),

Mod-R the category of all right R-modules,

mod-R the subcategory of all modules M admitting a projective resolution

$$\cdots \to P_{k+1} \to P_k \to \cdots \to P_1 \to P_0 \to M \to 0$$

where all P_i are finitely generated.

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(T1) $\operatorname{pdim} T < \infty$;

(T2) $\operatorname{Ext}^i_R(T, T^{(I)}) = 0$ for each $i > 0$ and all sets I;

(T3) There exists a long exact sequence

$$0 \to R_R \to T_0 \to \cdots \to T_r \to 0$$

with $T_i \in \mathrm{Add}\, T$ for each $0 \le i \le r$.

Tilting classes

If T is a tilting module, then

$$T^{\perp} = \{M \in \mathrm{Mod}{-}R \mid \mathrm{Ext}^i_R(T, M) = 0 \text{ for all } i \geq 1\}$$

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Two tilting modules T and T' are **equivalent** if $T^{\perp} = T'^{\perp}$.

Tilting classes

Given a tilting module T, we set

$$\mathcal{B} = T^{\perp} \qquad \mathcal{A} = {}^{\perp}(T^{\perp})$$

and consider the *cotorsion pair*

$$(\mathcal{A}, \mathcal{B}).$$

Tilting classes

Properties:

- $(\mathcal{A}, \mathcal{B})$ is complete:
  For every $M \in \mathrm{Mod}\text{-}R$ there are short exact sequences

$$0 \to M \xrightarrow{f} B \to A \to 0$$

$$0 \to B' \to A' \xrightarrow{g} M \to 0$$

  where $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$.

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- $\mathrm{pdim}\mathcal{A} = \sup\{\mathrm{pdim} A \mid A \in \mathcal{A}\} \leq \mathrm{pdim}\, T$ is finite.
- $\mathcal{A} \cap \mathcal{B} = \mathrm{Add}\, T$ is closed under coproducts.

Tilting classes

Theorem (A–Coelho 2001).
Let $\mathcal{B} \subseteq \mathrm{Mod}\text{-}R$, and $\mathcal{A} = {}^{\perp}\mathcal{B}$. Then

$\mathcal{B}$ is a tilting class if and only if

1. $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair,
2. pdim$\mathcal{A}$ is finite,
3. $\mathcal{A} \cap \mathcal{B}$ is closed under coproducts.

Tilting classes

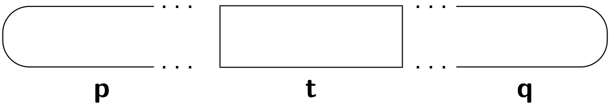
Theorem (Bazzoni–Eklof–Herbera–Sťovíček–Trlifaj 2005).
Every tilting class is of the form

$$\mathcal{B} = \{B \mid \mathrm{Ext}^1_R(\mathcal{S}, B) = 0\}$$

where $\mathcal{S} \subset \mathrm{mod}\text{-}R$ with $\mathrm{pdim}\mathcal{S} < \infty$.

Example 1: Tilting modules and representation type.

Let R be a (connected) hereditary finite dimensional algebra.
The Auslander-Reiten-quiver of R is of the form



p is the preprojective component
q is the preinjective component
t is a family of regular components.

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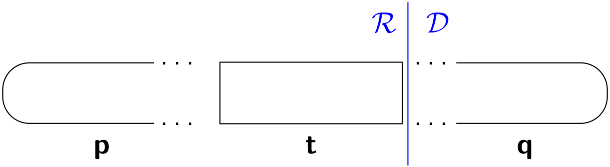
There is a torsion theory $(\mathcal{R}, \mathcal{D})$ maximal w.r.t. $\mathbf{q} \subset \mathcal{D}$ and $\mathbf{t} \subset \mathcal{R}$



with a large tilting module $W \in \mathrm{Mod}\text{-}R$ such that $W^{\perp} = \mathcal{D}$
(Ringel 1979, Reiten - Ringel 2006).

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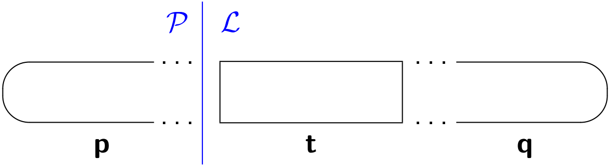
There is a torsion theory $(\mathcal{P}, \mathcal{L})$ maximal w.r.t. $\mathbf{p} \subset \mathcal{P}$ e $\mathbf{t} \subset \mathcal{L}$



with a large tilting module $L \in \mathrm{Mod}\text{-}R$ such that $L^{\perp} = \mathcal{L}$
(Lukas 1991, Kerner–Trlifaj 2005).

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Theorem (A–Herbera–Kerner–Trlifaj 2007).

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1. R is *tame* if and only if L is endonoetherian.
2. R is of *finite representation type* if and only if both L and W are endonoetherian.

Example 2: Tilting modules and finitistic dimensions.

Let R be right noetherian. Set

$$\mathcal{P} = \{M \in \mathrm{Mod}\text{-}R \mid \mathsf{pdim} M < \infty\}$$

$$\mathcal{P}^{<\infty} = \{M \in \mathrm{mod}\text{-}R \mid \mathsf{pdim} M < \infty\}$$

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The big and the little *finitistic dimension* of R are defined as

$$\mathrm{Findim}\, R = \mathrm{pdim}\, \mathcal{P}$$

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Open Problem: Is $\mathrm{findim}\, R < \infty$ for any artin algebra R ?

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Theorem 1 (A–Trlifaj 2002, A–Mendoza 2008).
Let R be right noetherian. Then

1. $\mathrm{findim} R < \infty \Leftrightarrow \mathcal{B} = T^{\perp}$ for some tilting module T.
   In this case $\mathrm{pdim}\, T = \mathrm{findim} R$.

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1. $\mathrm{findim} R < \infty \;\Leftrightarrow\; \mathcal{B} = T^{\perp}$ for some tilting module T.
   In this case $\mathrm{pdim}\, T = \mathrm{findim} R$.
2. $\mathrm{Findim} R = \mathrm{findim} R \;\Leftrightarrow\;$ $\mathrm{pdim}\, T$ coincides with $\mathrm{pdim}\{$modules with finite $\mathrm{Add}\, T$-resolution$\}$.

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So $\mathrm{Findim} R = \mathrm{findim} R < \infty$ (Auslander–Reiten 1991, Huisgen-Zimmermann–Smalø 1998).

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and $T = I_0 \oplus \ldots \oplus I_n$ where

$$0 \to R \to I_0 \to \ldots \to I_n \to 0$$

is a minimal injective coresolution.

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So $\mathrm{Findim} R = \mathrm{findim} R < \infty$ (A–Herbera–Trlifaj 2006).

Example 2: Tilting modules and finitistic dimensions.

Theorem (A–Mendoza 2008).
If R is right noetherian, then for every tilting module T we have

$$\operatorname{Findim} R \leq \operatorname{pdim} T + \operatorname{idim} T$$

Example 3: Tilting modules and localization.

$\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is a tilting $\mathbb{Z}$-module.
Its tilting class is the class of *divisible* groups.

This pattern occurs in many situations !

Example 3: Tilting modules and localization.

Theorem (Schofield) Let Σ be a set of morphisms between finitely generated projective right R-modules. Then there is a ring homomorphism $\lambda\colon R \to R_\Sigma$ such that

1. λ is Σ-*inverting:* if $\alpha\colon P \to Q$ belongs to Σ, then $\alpha \otimes_R 1_{R_\Sigma}\colon P \otimes_R R_\Sigma \to Q \otimes_R R_\Sigma$ is an isomorphism
2. λ is *universal* with respect to *1.*

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$\lambda\colon R \to R_\Sigma$ is a ring epimorphism with $\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$, the universal localization of R at Σ.

Example 3: Tilting modules and localization.

Let now $\mathcal{U} \subset \mathrm{mod}\text{-}R$ be a set of R-modules of pdim 1.
For each $U \in \mathcal{U}$, fix a projective resolution in $\mathrm{mod}\text{-}R$

$$0 \to P \overset{\alpha_U}{\to} Q \to U \to 0$$

and set $\Sigma = \{\alpha_U \mid U \in \mathcal{U}\}$.

$R_{\mathcal{U}}$ denotes the universal localization of R at Σ.

Example 3: Tilting modules and localization.

Theorem (A–Sánchez 2007).
Let $\mathcal{U} \subset \mathrm{mod}\text{-}R$ be a set of R-modules of pdim 1.
If R embeds in $R_{\mathcal{U}}$ and pdim$R_{\mathcal{U}} \leq 1$, then

$$R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$$

is a tilting module.

Example 3: Tilting modules and localization.

Application 1: Classification of tilting modules.
Over a *Dedekind domain*, every tilting module is equivalent to a module of the form

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where $\mathcal{U} = \{R/\mathfrak{m} \mid \mathfrak{m} \in \mathfrak{P}\}$ and $\mathfrak{P}$ is a set of maximal ideals of R

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Similar results also for
Prüfer domains, commutative Gorenstein rings, HNP–rings ...

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Over the *Kronecker-algebra*

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1. a finite dimensional tilting module
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In particular, $W \sim R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ where $\mathcal{U}$ is the set of all simple regular modules.

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Application 2 (A–Herbera–Trlifaj 2005).
Let R be commutative, and let S be a multiplicative subset consisting of non-zero-divisors. Set $Q = S^{-1}R$.

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The following are equivalent.

1. $\mathrm{pdim} Q_R \leq 1$.
2. $\mathrm{Gen} Q_R$ is the class of S-divisible modules.
3. Q/R is a direct sum of countably presented submodules.

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(For domains: Hamsher 1971, Matlis 1973, Lee 1989).

Example 4.

Let R be a commutative domain, and Q its quotient field.

The *Fuchs' divisible module* δ is a tilting module of pdim$\delta = 1$.
Its tilting class is the class of all *divisible modules*.
(Facchini 1987)

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- If $\mathrm{pdim} Q_R \leq 1$, then δ is equivalent to $Q \oplus Q/R$.
- If $\mathrm{pdim} Q_R > 1$:

  Question: Is δ related to the localization $\lambda : R \to Q$?

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Consider the perpendicular category

$$\hat{T}_1 = \{M \in \text{Mod-}R \mid \operatorname{Hom}_R(T_1, M) = \operatorname{Ext}^1_R(T_1, M) = 0\}$$

Ring epimorphisms

There is a ring epimorphism $\lambda : R \to S$ which induces an equivalence

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Note: If $\mathrm{Hom}_R(T_1, T_0) = 0$, then λ is

- injective
- a homological epimorphism

Ring epimorphisms

Theorem (Geigle-Lenzing 1991). The following statements are equivalent for a ring homomorphism $\lambda : R \to S$.

1. λ is a ring epimorphismus, and $\mathrm{Tor}_i^R(S, S) = 0$ for all $i \geq 1$.
2. $\mathrm{Ext}^i_R(M, N) \cong \mathrm{Ext}^i_S(M, N)$ for all $M, N \in \mathrm{Mod}{-}S$, $i \geq 1$.

Then λ is said to be a homological ring epimorphism.

Ring epimorphisms

Theorem (A–Sánchez 2007). The following are equivalent.

1. There is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathrm{Add}\, T$ and $\mathrm{Hom}_R(T_1, T_0) = 0$.

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1. There is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathrm{Add}\, T$ and $\mathrm{Hom}_R(T_1, T_0) = 0$.
2. There is an injective ring epimorphism $\lambda : R \to S$ such that $\mathrm{Tor}_1^R(S, S) = 0$ and
$$S \oplus S/R$$
is a tilting module equivalent to T.

Example 4.

Let R be a commutative domain, and Q its quotient field.

- If $\mathrm{pdim}\, Q_R > 1$,
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Recollements

Let T and T_1 be as above. Consider

$$\mathcal{X} = \mathrm{Tria}\, T_1$$

the smallest full triangulated subcategory of $\mathcal{D}(R)$ which contains T_1 and is closed under small coproducts,

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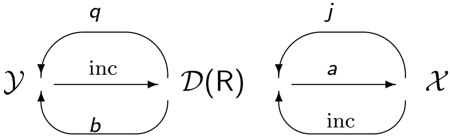
$$\mathcal{Y} = \mathsf{Ker}\,\mathsf{Hom}_{\mathcal{D}(R)}(\mathcal{X}, -)$$

Note: $\mathcal{Y}$ is closed under small coproducts,
so $\mathcal{X}$ is a smashing subcategory of $D(R)$.

Recollements

Then there is a recollement



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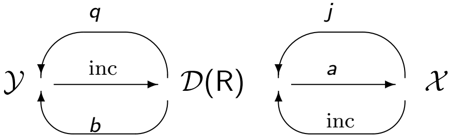


that is,

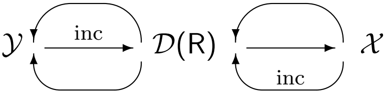
- $(q, \mathrm{inc}_{\mathcal{Y}})$, $(\mathrm{inc}_{\mathcal{Y}}, b)$, $(\mathrm{inc}_{\mathcal{X}}, a)$, and (a, j) are adjoint pairs
- $b \circ j = 0$
- j is a full embedding
- For each $C \in D(R)$ there are triangles

$$\mathrm{inc}_{\mathcal{Y}} b(C) \to C \to ja(C) \rightsquigarrow$$

$$\mathrm{inc}_{\mathcal{X}} a(C) \to C \to \mathrm{inc}_{\mathcal{Y}} q(C) \rightsquigarrow$$

Recollements

Theorem (A–König–Liu 2008). Every tilting module T of projective dimension one induces a recollement

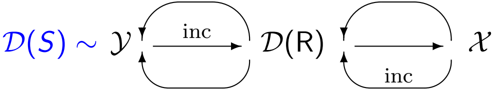


with the following properties:

- $\mathcal{X} = \mathrm{Tria}\, T_1$ where T_1 is an exceptional object of $\mathcal{D}(R)$.
- $\mathcal{Y} = \mathrm{Tria}\, T_2$ where T_2 is a self-compact object of $\mathcal{D}(R)$.

Recollements

Theorem (A–König–Liu 2008). Every tilting module T of projective dimension one induces a recollement



with the following properties:

- $\mathcal{X} = \mathrm{Tria}\, T_1$ where T_1 is an exceptional object of $\mathcal{D}(R)$.
- $\mathcal{Y} = \mathrm{Tria}\, T_2$ where T_2 is a self-compact object of $\mathcal{D}(R)$.
- T_2 is exceptional $\Leftrightarrow$ λ is a homological epimorphism.
  In this case λ_* induces an equivalence $\mathcal{D}(S) \sim \mathcal{Y}$.

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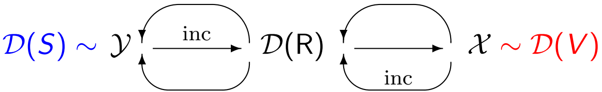

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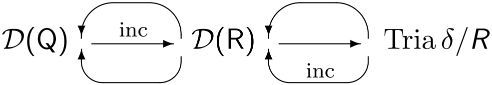
- T_2 is exceptional $\Leftrightarrow$ λ is a homological epimorphism.
  In this case λ_* induces an equivalence $\mathcal{D}(S) \sim \mathcal{Y}$.
- T_1 is self-compact $\Leftrightarrow$ there are a ring V and an equivalence $\mathcal{X} \sim \mathcal{D}(V)$ taking $T_1 \mapsto V_V$.
  This occurs iff $T \in \mathrm{mod}\text{-}R$ up to equivalence.

Example 4.

Let R be a commutative domain, and Q its quotient field.
The tilting module δ always induces a recollement



Example 3.

Over the *Kronecker-algebra* $\bullet \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \bullet$

- the tilting module L induces the trivial recollement with $\mathcal{Y} = 0$, $\mathcal{X} = \mathcal{D}(R)$.

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- the tilting module L induces the trivial recollement with $\mathcal{Y} = 0$, $\mathcal{X} = \mathcal{D}(R)$.

- the tilting module $W \sim R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$, where $\mathcal{U}$ is the set of all simple regular modules, induces a recollement

$$\mathcal{D}(R_{\mathcal{U}}) \; \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\text{inc}}}} \; \mathcal{D}(\mathrm{R}) \; \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow[\text{inc}]{}}} \; \text{Tria}\, W_1$$

where $R_{\mathcal{U}} \cong F^{d \times d}$ is a simple artinian ring.

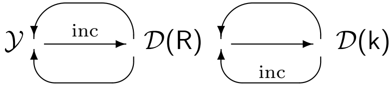
Example 5.

Over the quasi-hereditary algebra $R = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$

the characteristic tilting module

$$T = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \oplus 3$$

induces a recollement



where $\lambda : R \to R_{\mathcal{U}}$, the universal localization at $\mathcal{U} = \{ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \}$,

is *not* a homological epimorphism.

Example 5.

We choose the exact sequence

$$0 \to R \to T_0 \to T_1 \to 0$$

with

$$T_0 = \begin{array}{c}1\\2\\1\end{array} \oplus \begin{array}{c}2\\1\,3\\2\end{array} \oplus \begin{array}{c}2\\1\,3\\2\end{array} \quad \text{and} \quad T_1 = \begin{array}{c}2\\1\end{array}$$