Ortho-radial drawings

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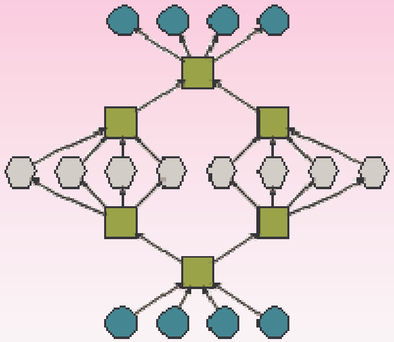
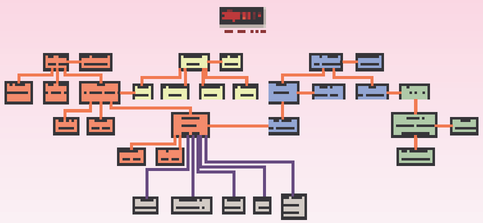
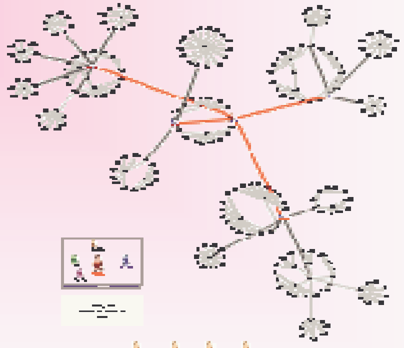
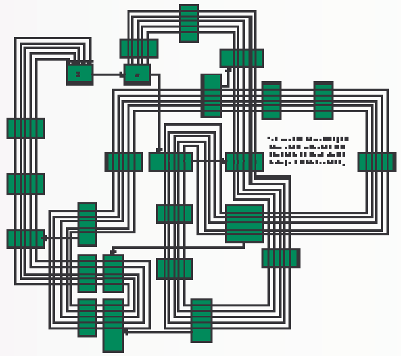
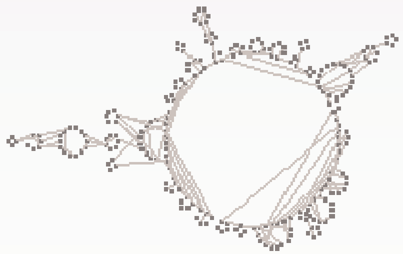
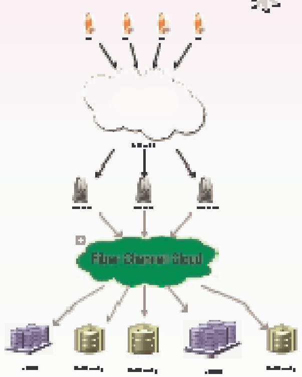
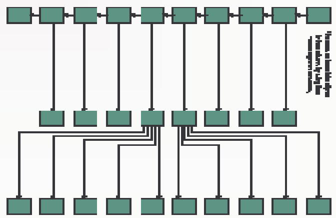
Graph Drawing

- A drawing of a graph G is a mapping from vertex set and edge set of *G* into R^n , n=2,3, in which vertices have distinct coordinates and each edge is a simple Jordan curve.

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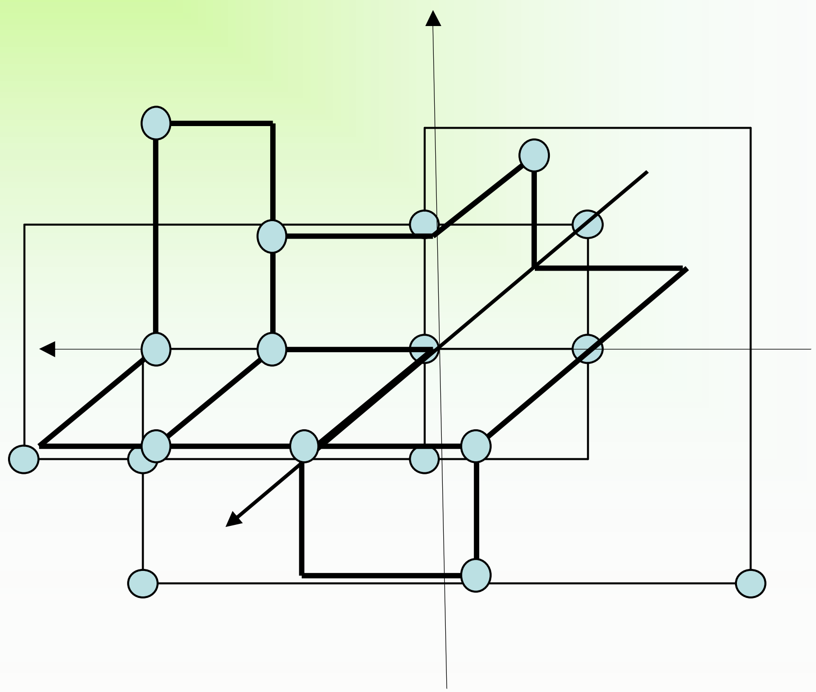
Embedding

- An embedding is an assignment of a cyclic order of neighbours of each vertex.
- An embedding of a graph G is a planar embedding if there is a non-crossing drawing of G such that the neighbors of each vertex appear around the vertex with the same order as in the embedding.

Orthogonal Drawing

- An orthogonal drawing of a graph *G* is a planar drawing of *G* where each vertex has integer coordinates, and each edge is represented by a chain of segments that are parallel to one of the axes.

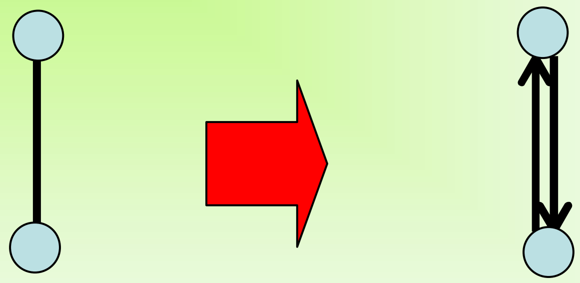
Three Dimensions



Orthogonal Drawings with given shape

Dart:

The term dart is used for each of the two possible orientations *(u,v)* and *(v,u)* of an undirected edge *uv*.

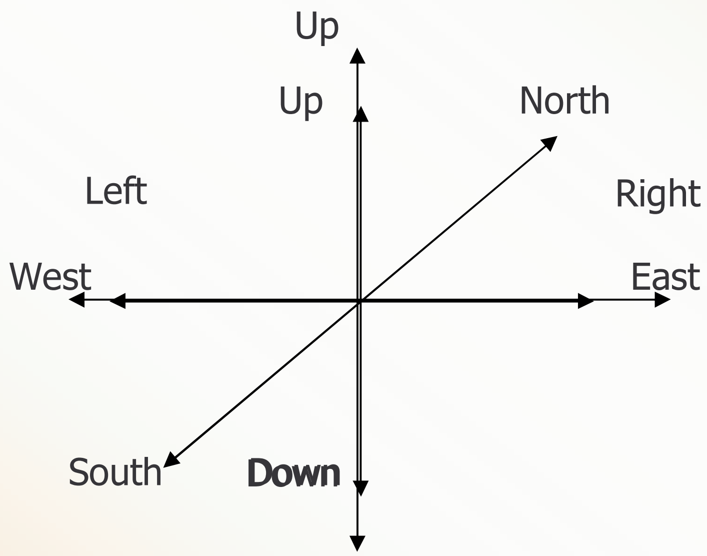


Shape:

A shape of a graph *G* is a labeling of darts of *G* with directions.

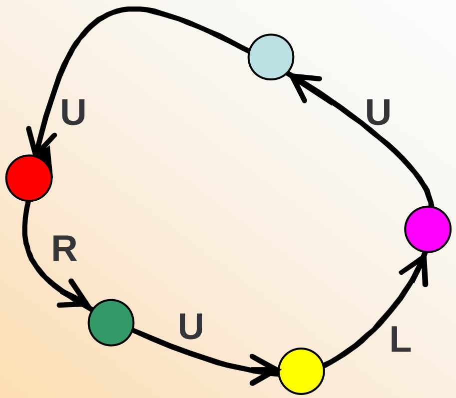
Directions

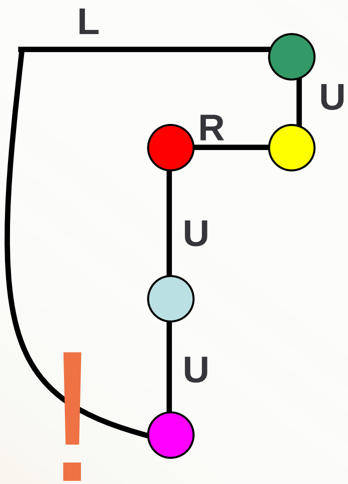
Two Dimensions
Three Dimensions



Question

For which shapes is there an orthogonal drawing in which the darts of G are drawn in the same direction as their labels in the shape?





Which shapes have a drawing?

- Vijayan and Wigderson (1985):

  A necessary and sufficient condition to determine which 2D-shapes have orthogonal drawing.

- Di Battista, Liotta and Lubiw (2001,2002):

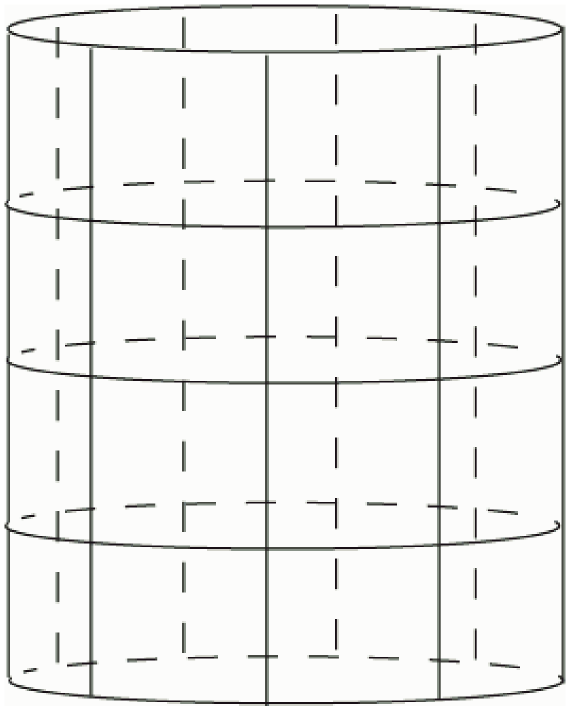
  solved this problem for 3D-shapes of cycles and paths.

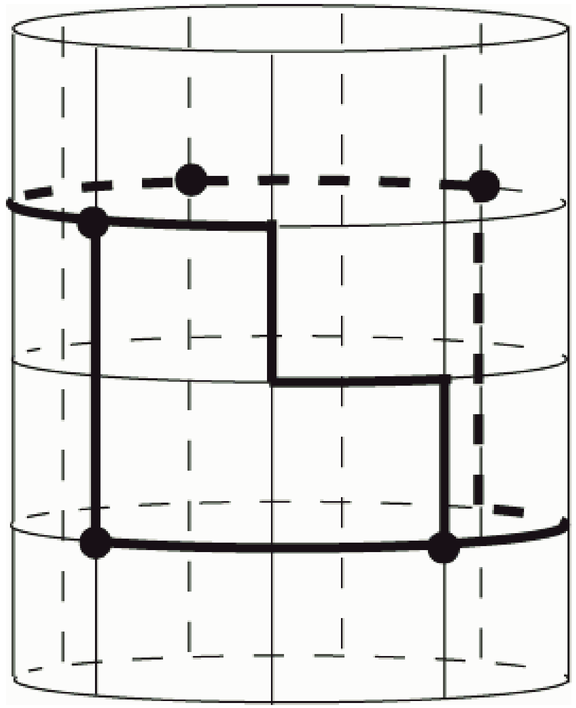
- Di Giacomo, Liotta and Patrignani (2002):

  A sufficient condition for 3D- shapes of theta graphs.

Patrignani (2005): This problem is NP-hard in 3D.

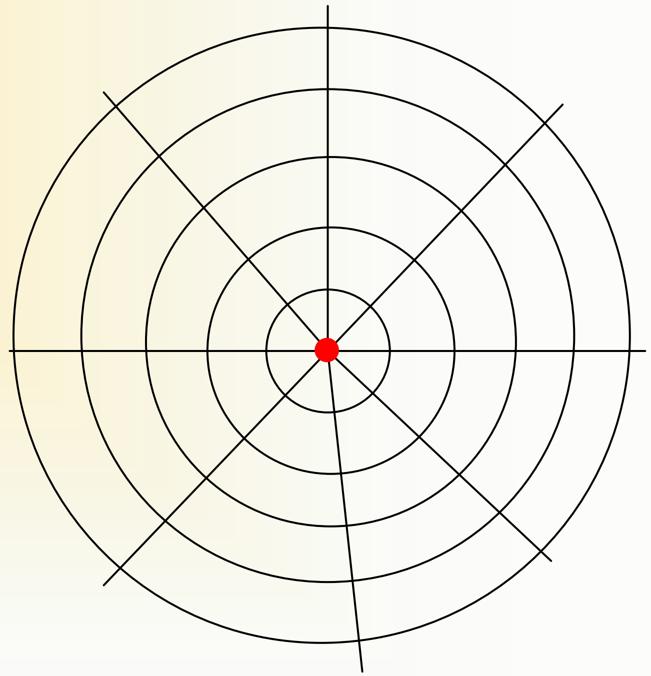
Extension of orthogonal drawings to drawings on a cylinder





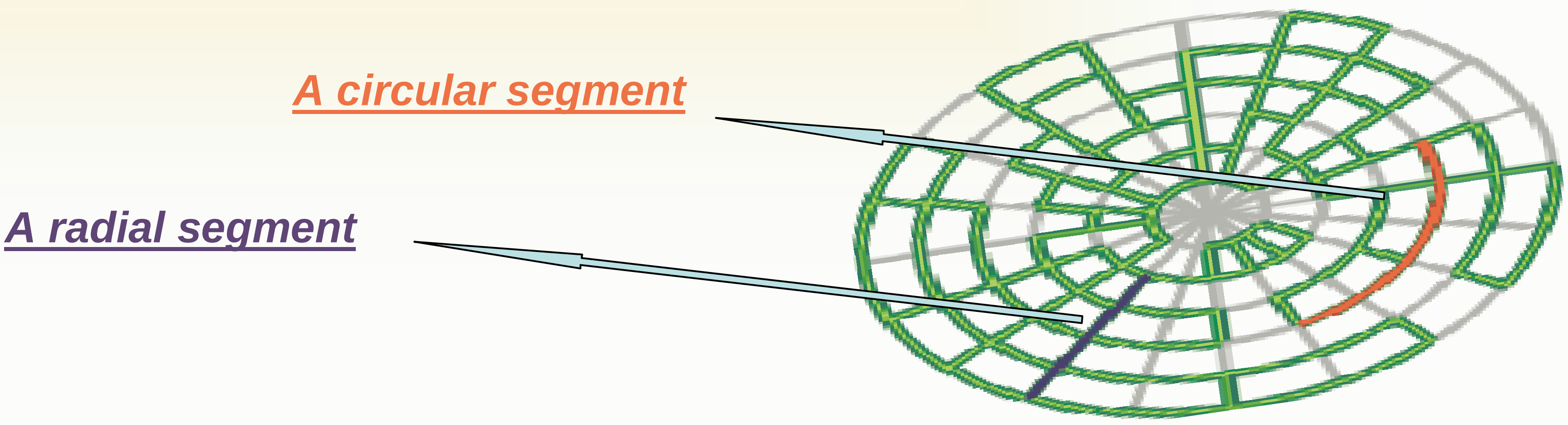
Ortho-radial grid

- Given a point *S*, the circles with center *S* and half lines starting at *S* define a grid called ortho-radial grid.



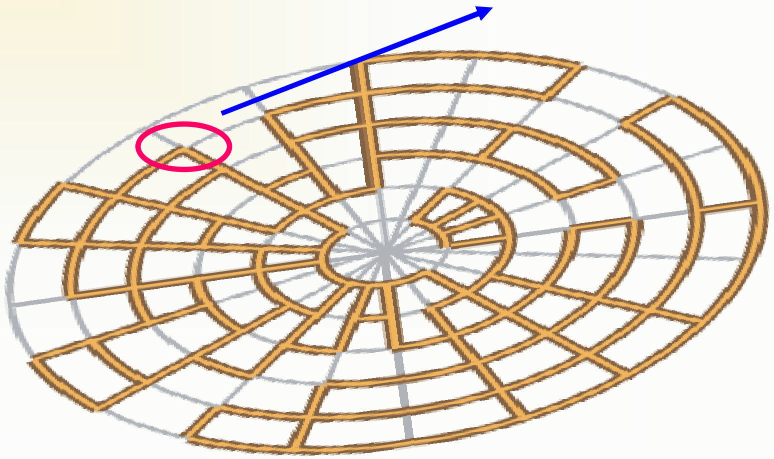
Radial and circular segments

- Given a point *S* on the two dimensional plane, a segment is called ***radial segment*** if it is part of a half line starting at *S*. An arc is called ***circular segment*** if it is part of a circle with center *S*.

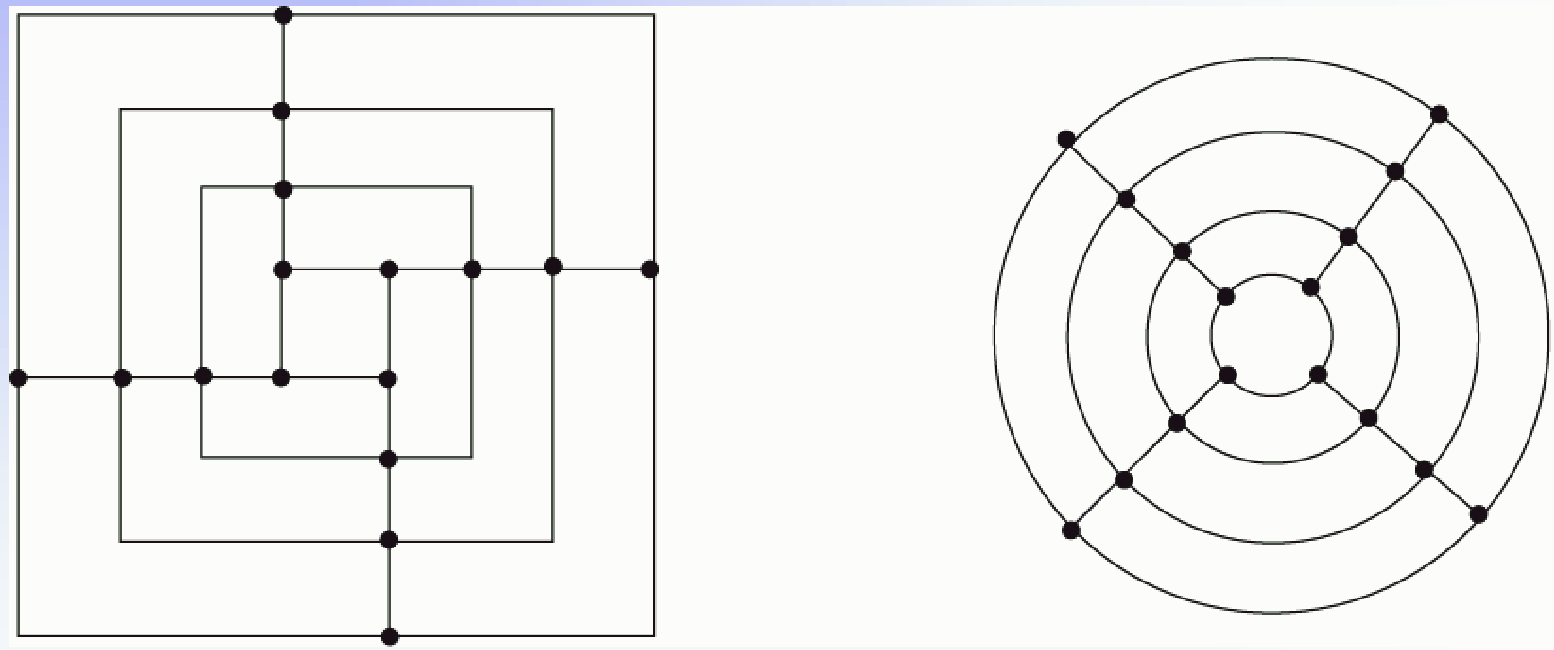


Ortho-radial Drawing

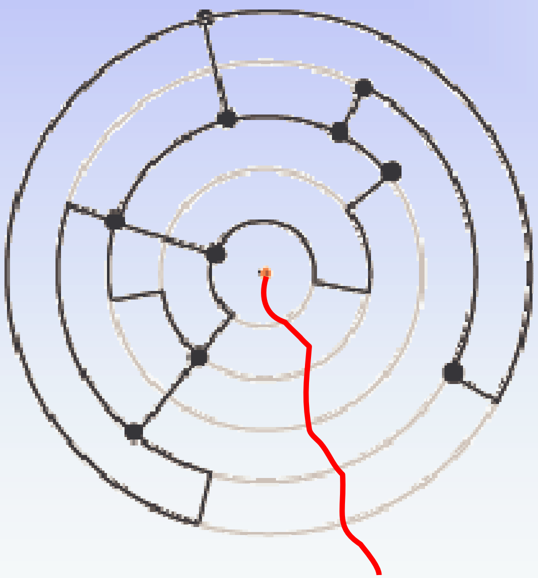
- A planar drawing of a graph *G* is called an *ortho-radial drawing* w.r.t. *S* if each edge is a chain of radial and circular segments.
- In an ortho-radial or orthogonal drawing a place where an edge changes its direction is called a bend.



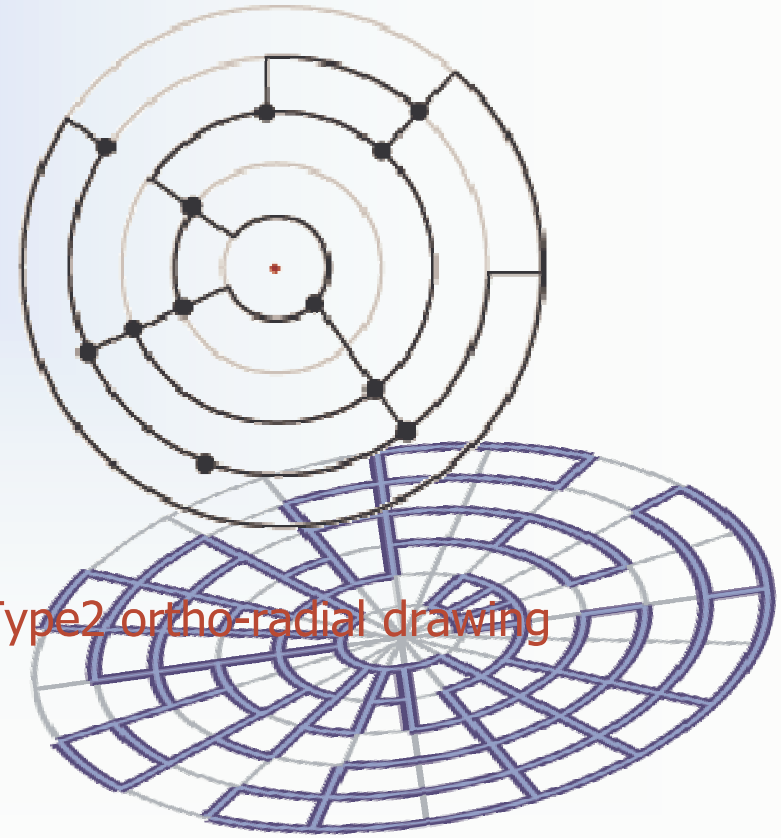
Why Ortho-radial drawings?



Different Types of Ortho-radial drawings



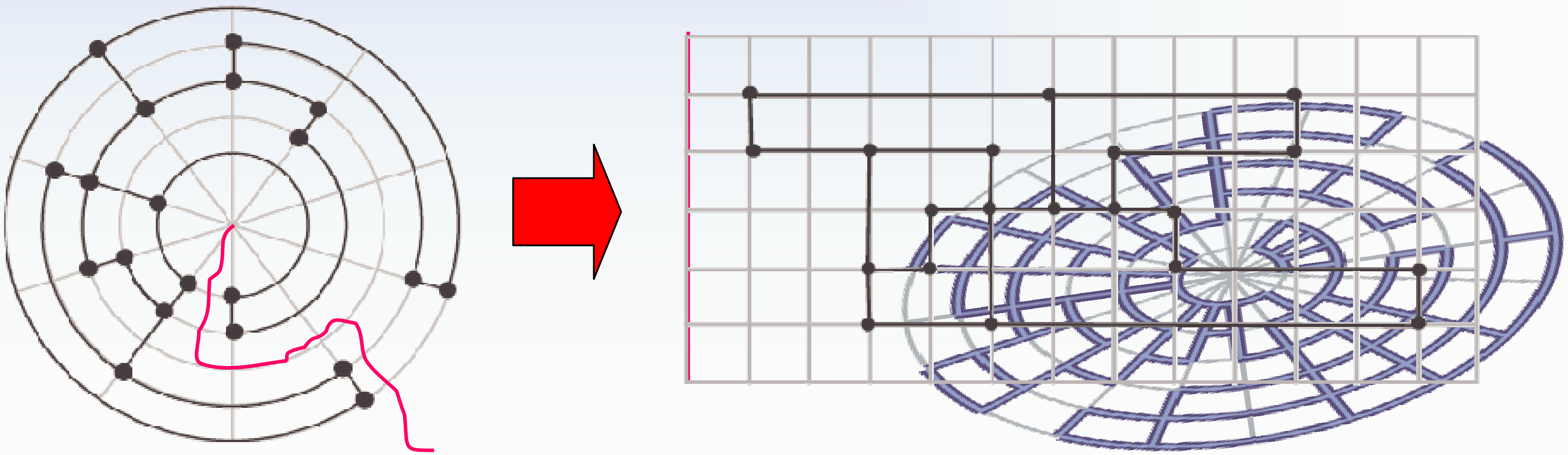
Type1 ortho-radial drawing



Type2 ortho-radial drawing

Type1 Ortho-radial Drawings and Orthogonal Drawings

- **Theorem 1:** Every type-1 ortho-radial drawing of a graph *G* can be transformed into an orthogonal drawing in the plane in such a way that each vertical segment becomes a radial segment and each horizontal segment becomes a circular segment, and vice versa.



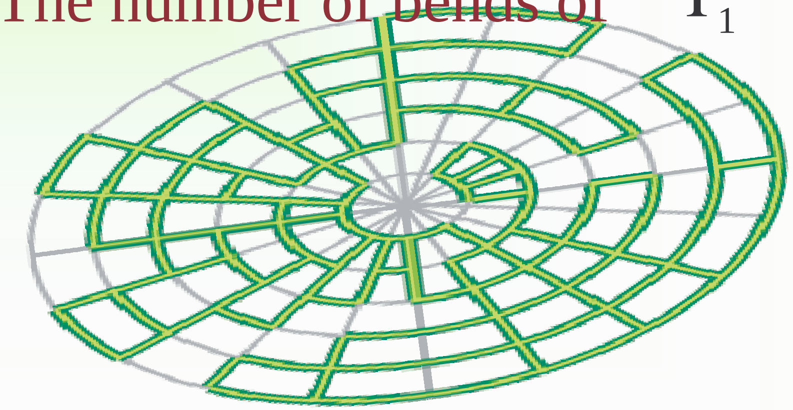
Number of bends

Corollary: Given a graph *G:*

Γ_1 : an orthogonal drawing of graph G with minimum number of bends

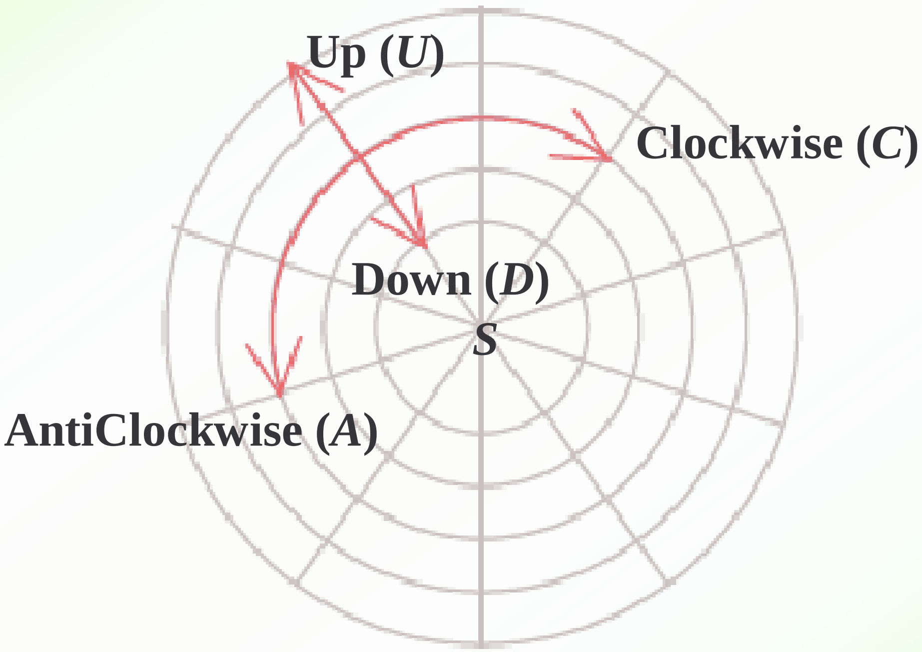
Γ_2 :an ortho-radial drawing of graph G with minimum number of bends

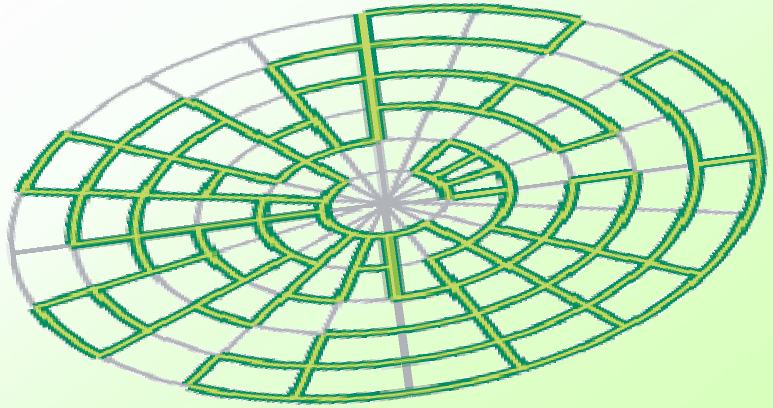
The number of bends of Γ_2 $\leq$ The number of bends of Γ_1



Directions

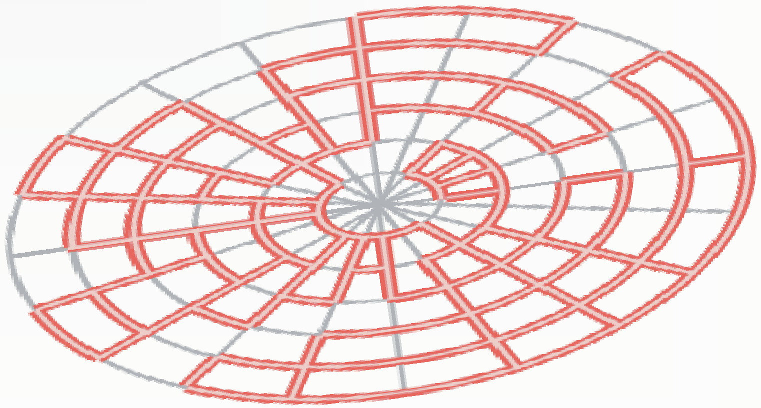
In an ortho-radial grid w. r. t. point S, we define four directions.





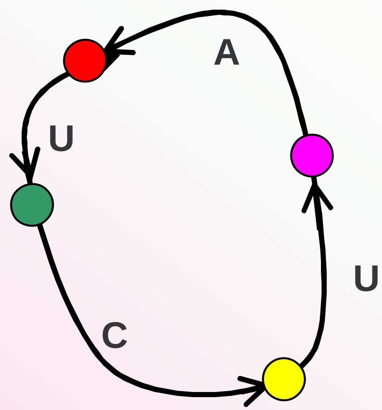
C-shape

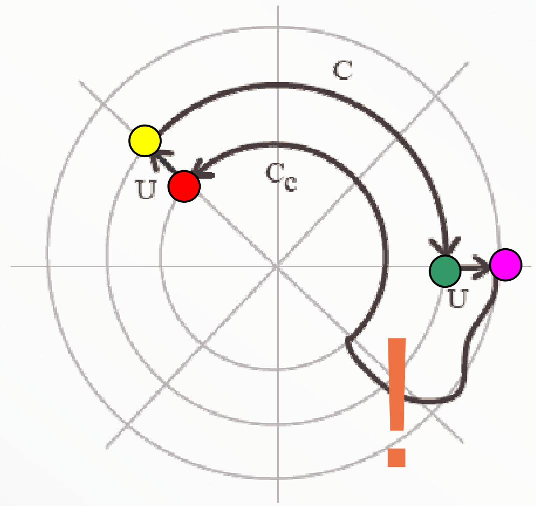
- A C-shape is a labeling of the darts of *G* with labels in the set {C, A, D, U} such that
  - » The two darts of each edge have opposite labels
  - » No two darts exiting a vertex have the same label.



Drawable C-shape

- A C-shape of a graph *G* is called drawable, if there is an ortho-radial drawing of *G* where each dart is drawn in the same direction as its label.



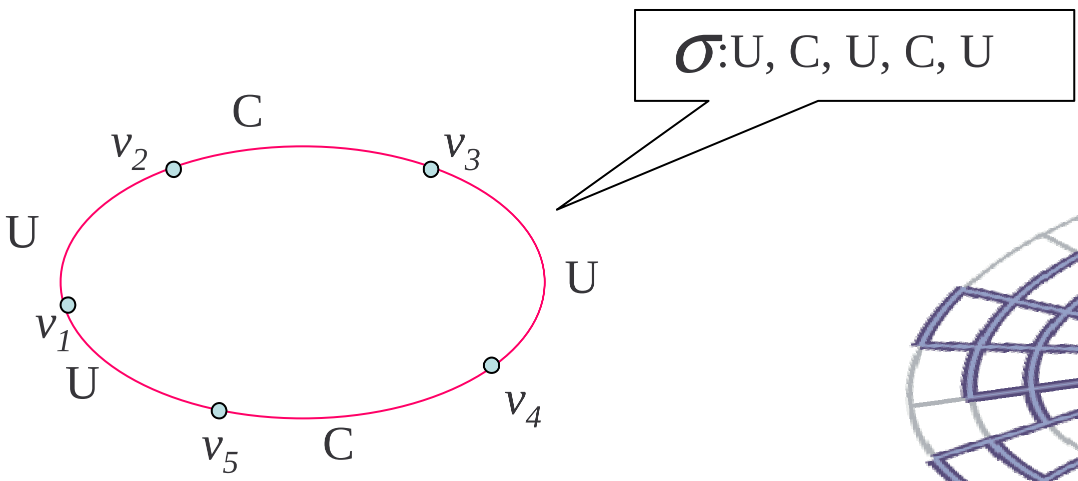


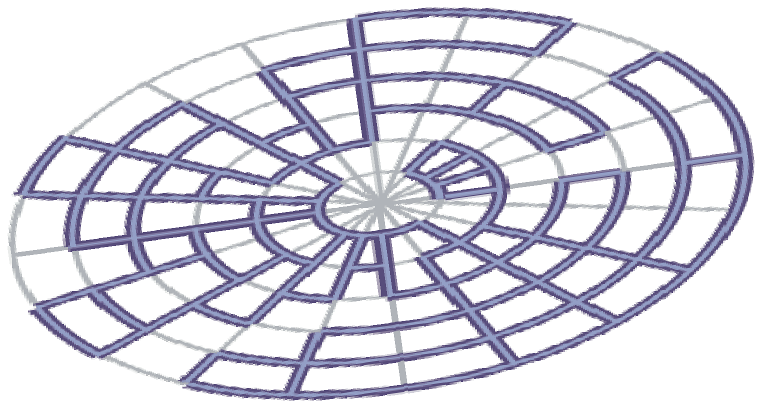
Embeddings induced by C-shapes

- Let G be a graph with C-shape γ .The labels γ of define a cyclic order of the darts leaving each vertex; that is, they define an embedding of the graph on a surface.
- This is a planar embedding if the number of faces obeys Euler's formula.

C-shape cycle

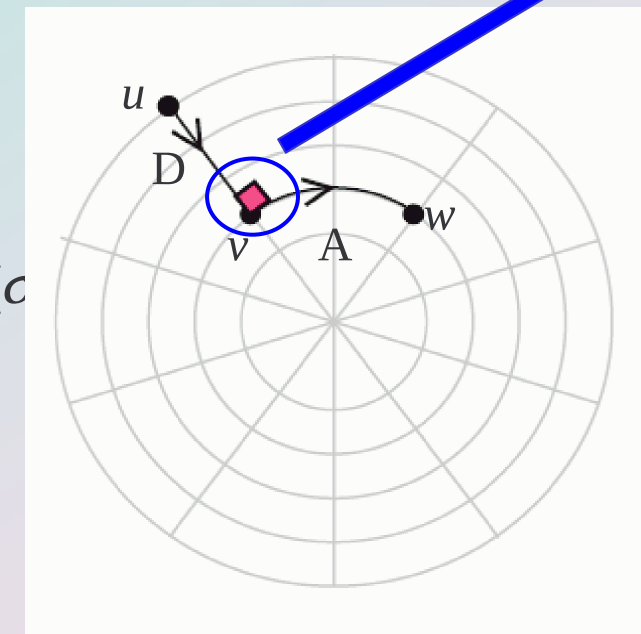
- Let C: $v_1, v_2, \ldots, v_{n=}v_1$ be a cycle with C-shape γ. And $\sigma: \sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be the labels of darts (v_1, v_2), (v_2, v_3), …, (v_{n-1}, v_n). We call σ a C-shape cycle.



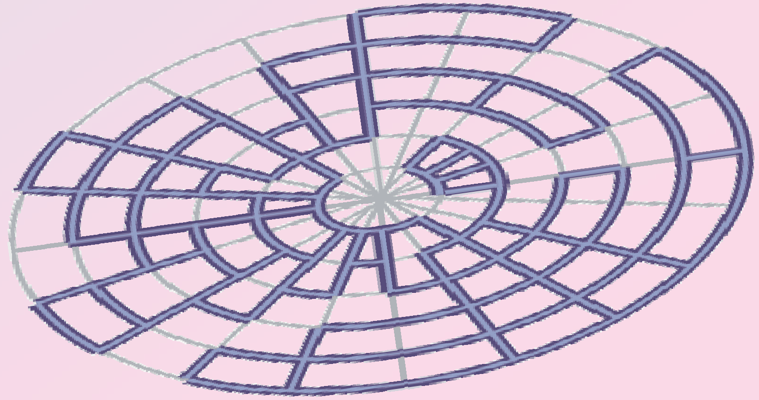


Turn and Rotation

- Let $e=(u,v)$ and $e'=(v,w)$ be two consecutive darts with labels σ_1 and σ_2. $\theta(e,e')$ is the angle on the left when we move from u to w.



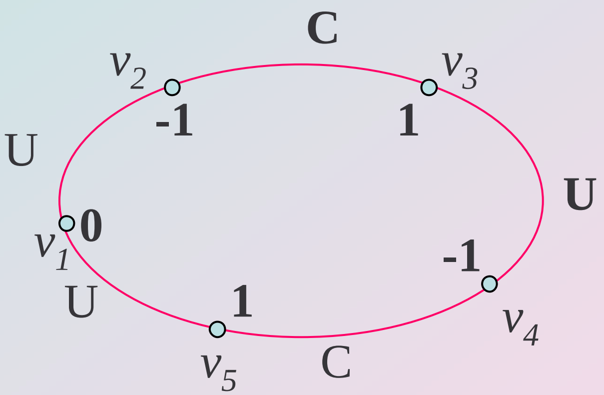
$turn(\sigma$ … $\frac{3\pi}{2}$



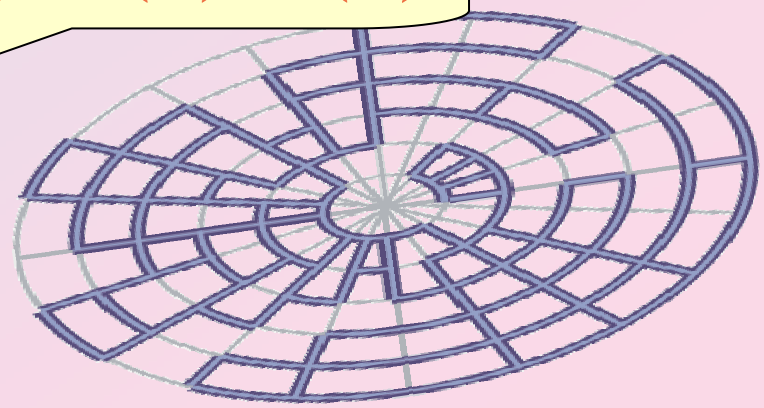
Turn and Rotation

If $\sigma:\sigma_1,\sigma_2,\ldots,\sigma_{n-1}$ is a C-shape cycle, the rotation of σ , *rot(σ)*, is defined as follows;

$$rot(\sigma)=\sum_{i=1}^{n-1} turn(\sigma_i,\sigma_{i+1})+turn(\sigma_n,\sigma_1)$$



$$rot(\sigma)=1+(-1)+1+(-1)$$



Some results on P-shapes

- A P-shape cycles σ is drawable iff $rot(\sigma) = \pm 4$.

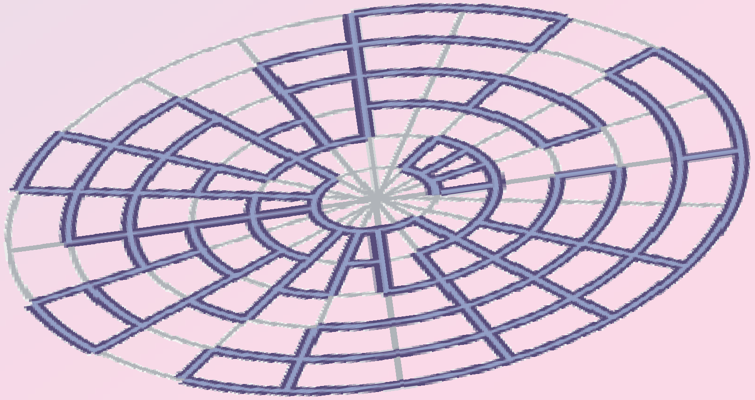
- Theorem (Vijayan and Wigderson 1985):

Let G be a biconnected graph with at least three edges. A P-shape σ of G is drawable iff the embedding induced by σ is a planar embedding and the P-shape of each face is drawable.

Drawable C-shape cycles

- **Theorem2**: A C-shape cycle σ is drawable if and only if one of the following cases happens:

  - $rot(\sigma) = \pm 4$
  - $rot(\sigma) = 0$ and all labels of σ are the same *(C or A)*.
  - $rot(\sigma) = 0$ and σ contains at least one *D* and one *U* label.

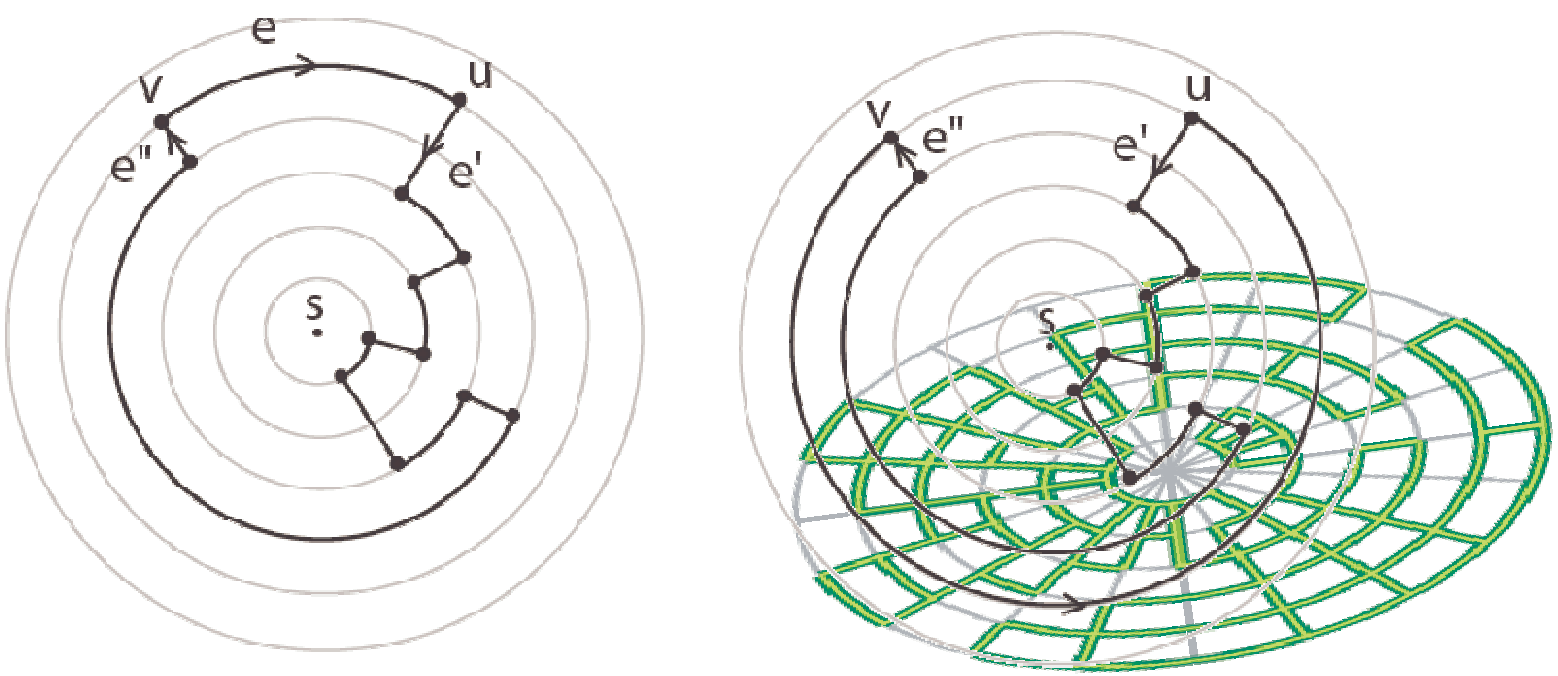


proof of necessity: Let Γ be a drawing of the cycle with C-shape σ.

1- Γ is a type1 ortho-radial drawing ⟹ $rot(\sigma) = \pm 4$

$rot(\sigma') = rot(\sigma) - turn(U,C) - turn(C,D) + turn(U,A) + turn(A,D)$

2- Γ is a type2 ortho-radial drawing ⟹ $rot(\sigma) = 0$



proof of sufficiency:

3- $rot(\sigma)=0$ and σ has at least one U and one down label. ⟹ σ has three consecutive labels U, A, D (U, C, D).

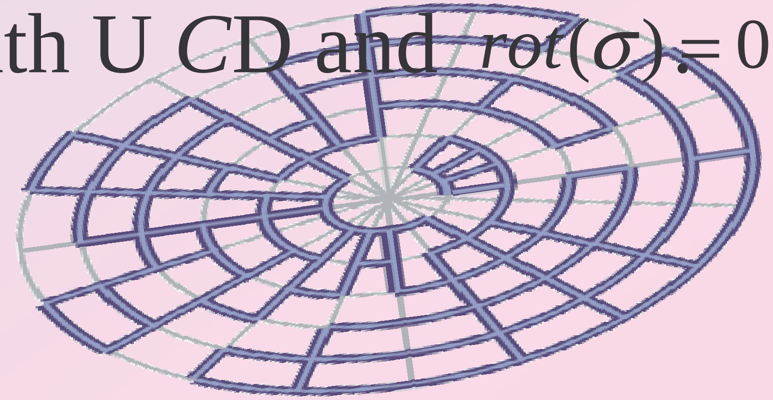
Suppose that $\sigma : \sigma' UAD$. Let $\tau : \sigma' UCD$, Hence:

$$rot(\tau) = rot(\sigma) + turn(U,C) + turn(C,D) - turn(U,A) - turn(A,D) = 4$$ ⟹

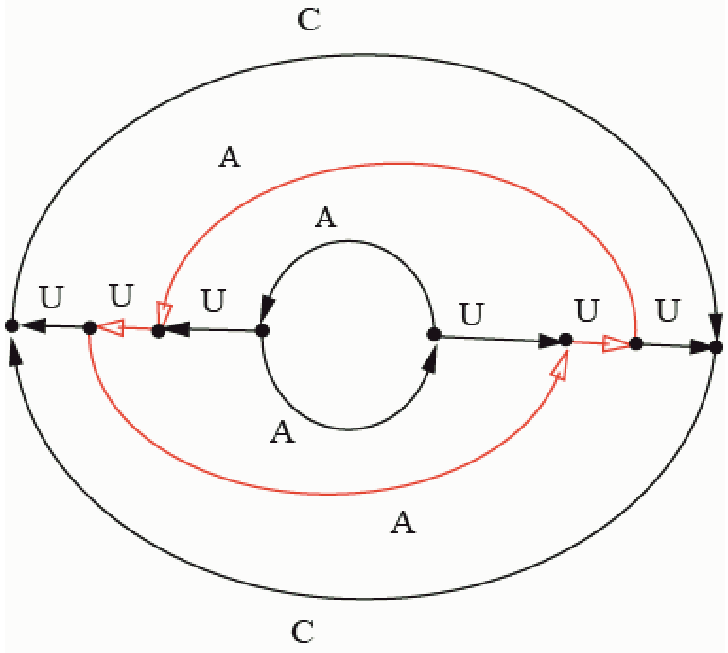
τ has a type1 ortho-radial drawing in which this drawing can easily transform to a drawing of σ.

S-shape cycles and N-shape cycles

- **I-shape cycle**: A drawable C-shape cycle σ, is called an I-shape cycle if all labels of σ are A or σ contains a C-shape p ath U AD and $rot(\sigma)=0$
- **E-shape cycle**: A drawable C-shape cycle σ, is called an E-shape cycle if all labels of σ are C or σ contains a C-shape path U CD and $rot(\sigma)=0$

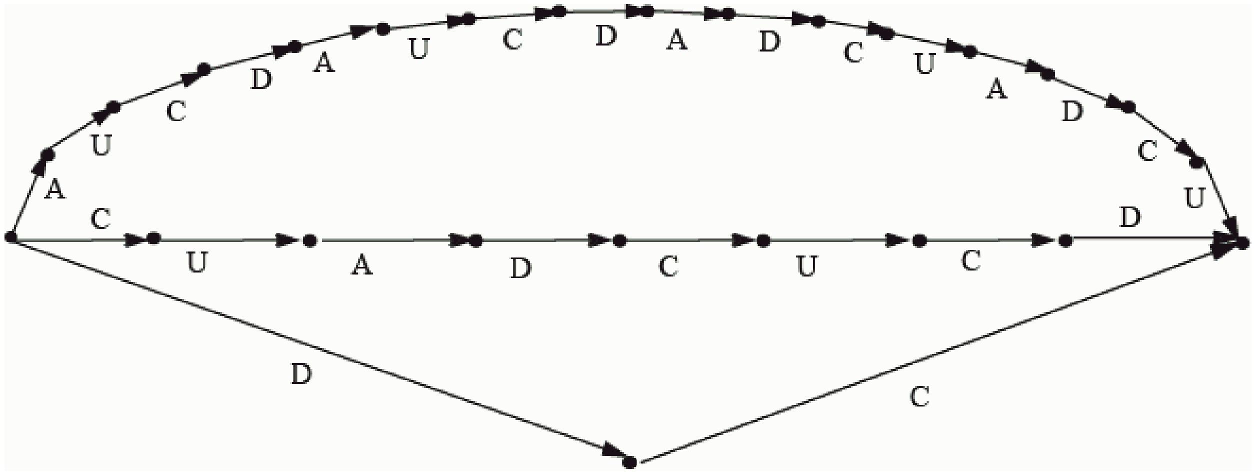


The drawabilty of C-shapes



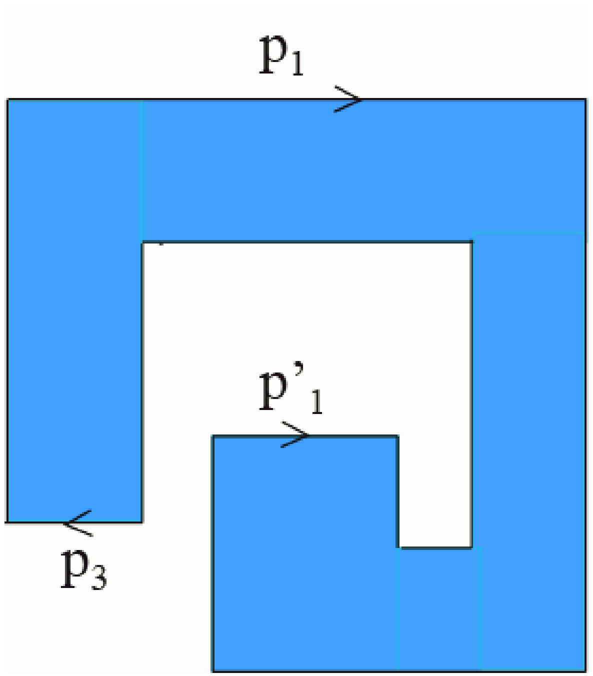
The C-shape of all its faces are drawable but
it itself is not drawable

The drawabilty of C-shapes

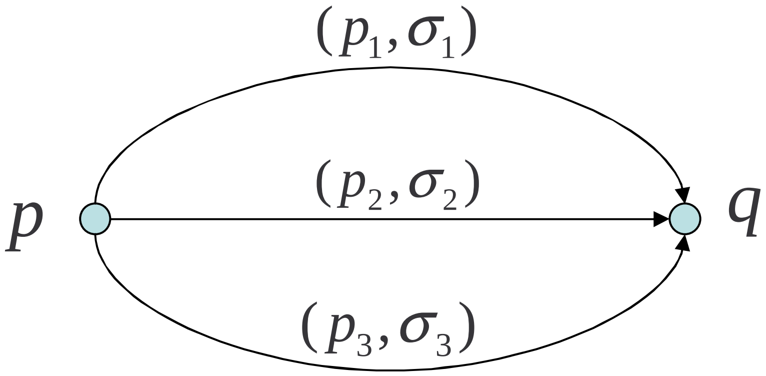


The C-shape of all its cycles are drawable
but it itself is not drawable

Admissible edges in a graph with given shape



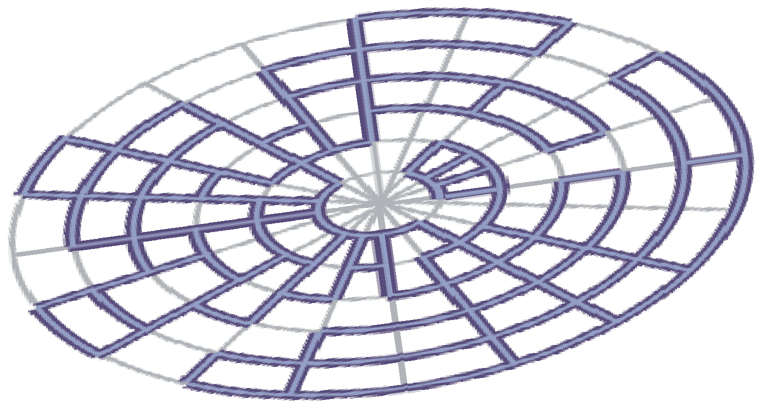
A theta graphs *T* with C-shape τ



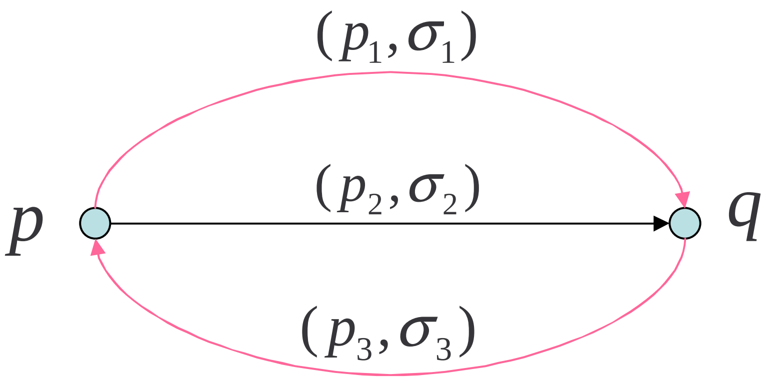
$$\tau_1 : \sigma_1 \overline{\sigma}_3$$

$$\tau_2 : \sigma_2 \overline{\sigma}_1$$

$$\tau_3 : \overline{\sigma}_3 \sigma_2$$



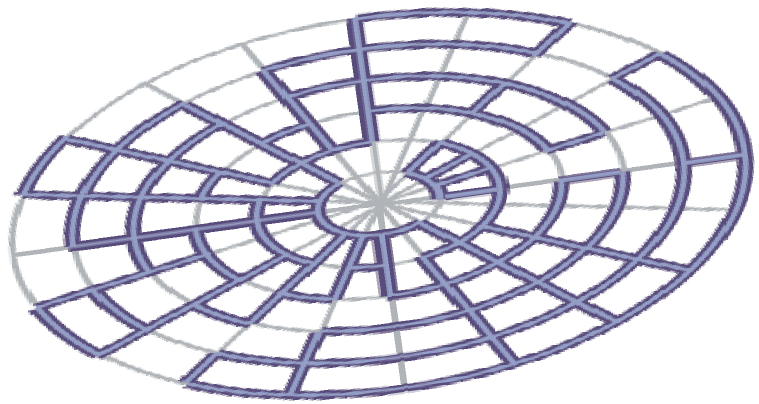
A theta graphs T with C-shape τ



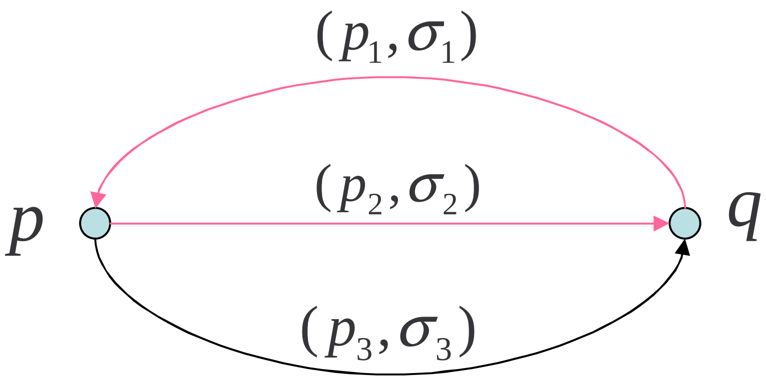
$\tau_1 : \sigma_1 \overline{\sigma}_3$

$\tau_2 : \sigma_2 \overline{\sigma}_1$

$\tau_3 : \overline{\sigma}_3 \sigma_2$



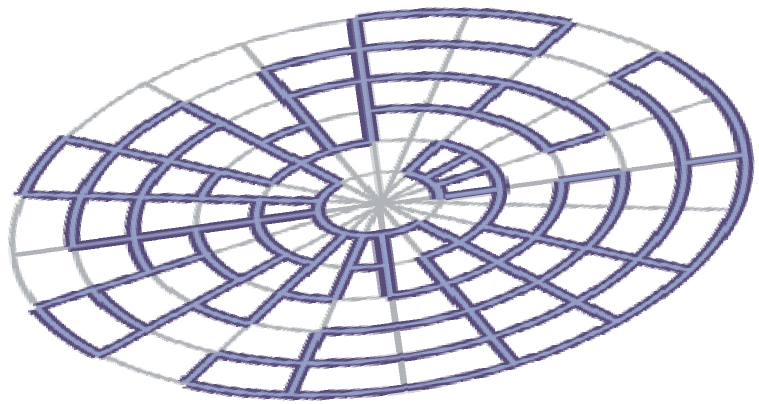
A theta graphs T with C-shape τ



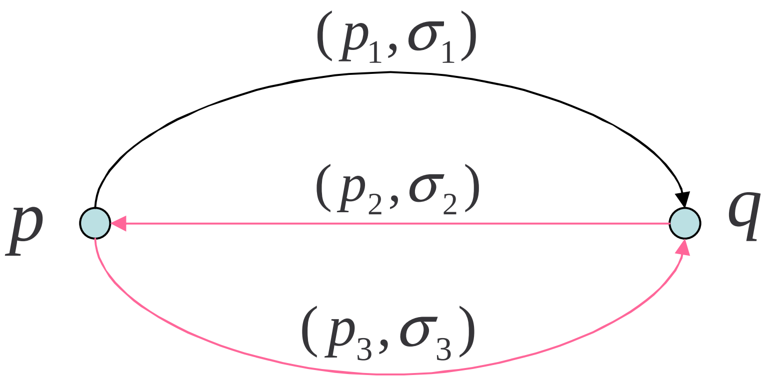
$\tau_1 : \sigma_1 \overline{\sigma}_3$

$\tau_2 : \sigma_2 \overline{\sigma}_1$

$\tau_3 : \overline{\sigma}_3 \sigma_2$



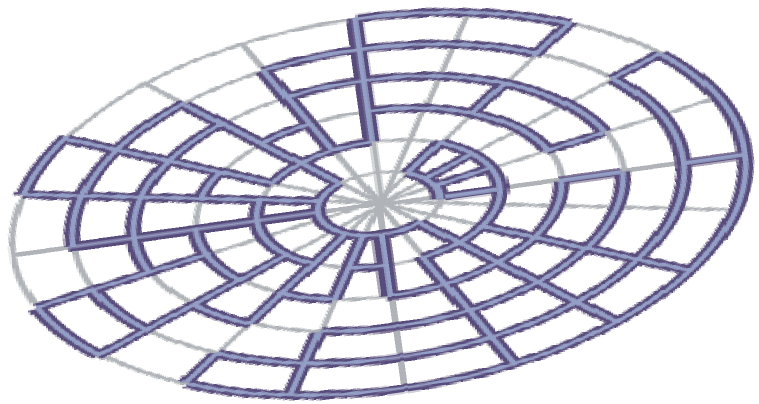
A theta graphs T with C-shape τ



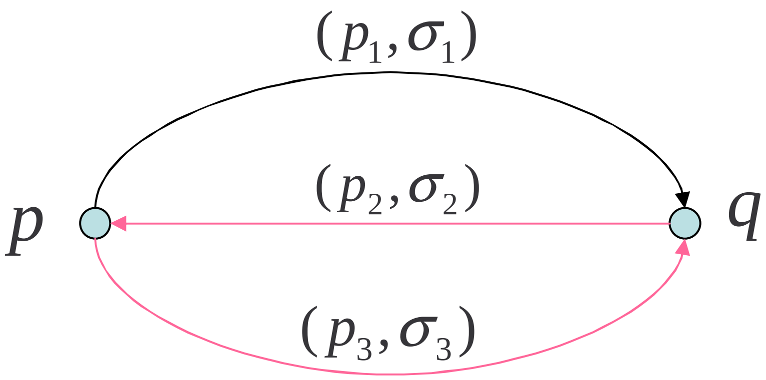
$\tau_1 : \sigma_1 \overline{\sigma}_3$

$\tau_2 : \sigma_2 \overline{\sigma}_1$

$\tau_3 : \sigma_3 \overline{\sigma}_2$



A theta graphs T with C-shape τ



$\tau_1 : \sigma_1 \overline{\sigma}_3$

$\tau_2 : \sigma_2 \overline{\sigma}_1$

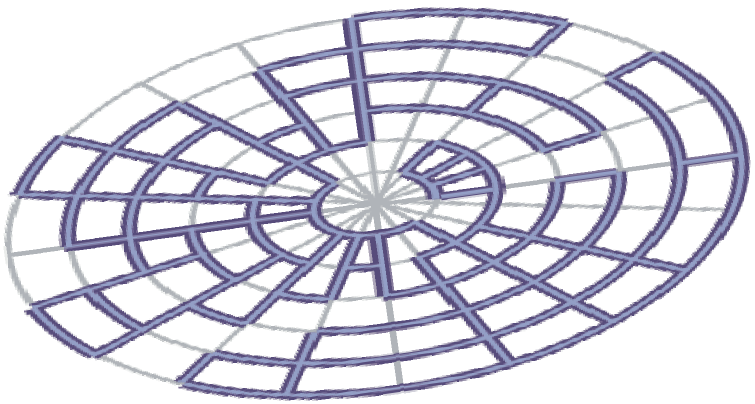
$\tau_3 : \sigma_3 \overline{\sigma}_2$

Suppose that:

$rot(\tau_2) = 4$

τ_1 be an E-shape cycle

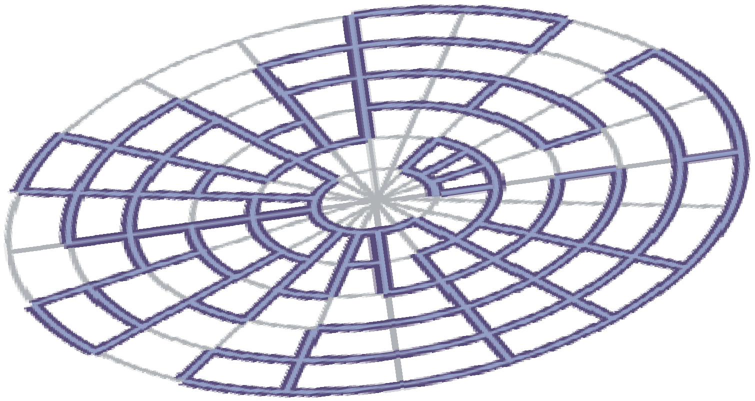
τ_3 be an I-shape cycle



Drawable C-shapes of theta graphs

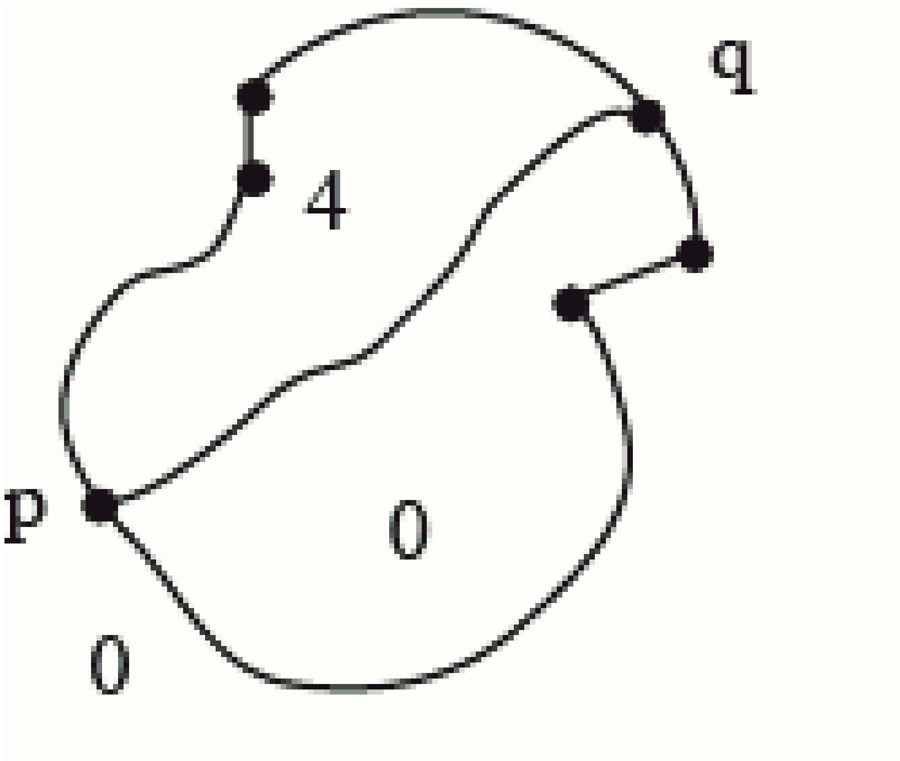
- **Theorem3**: A theta graph T with C-shape τ has a type2 ortho-radial drawing iff one of the following cases happens:

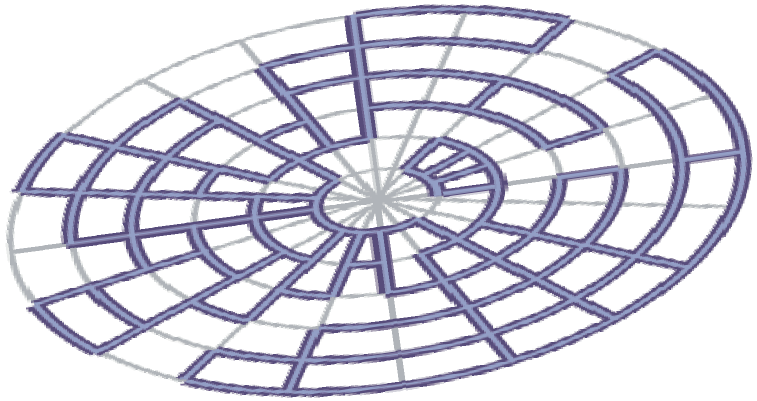
  - . All labels of τ_1 are C.
  - . All labels of τ_2 are A .



Drawable C-shapes of theta graphs

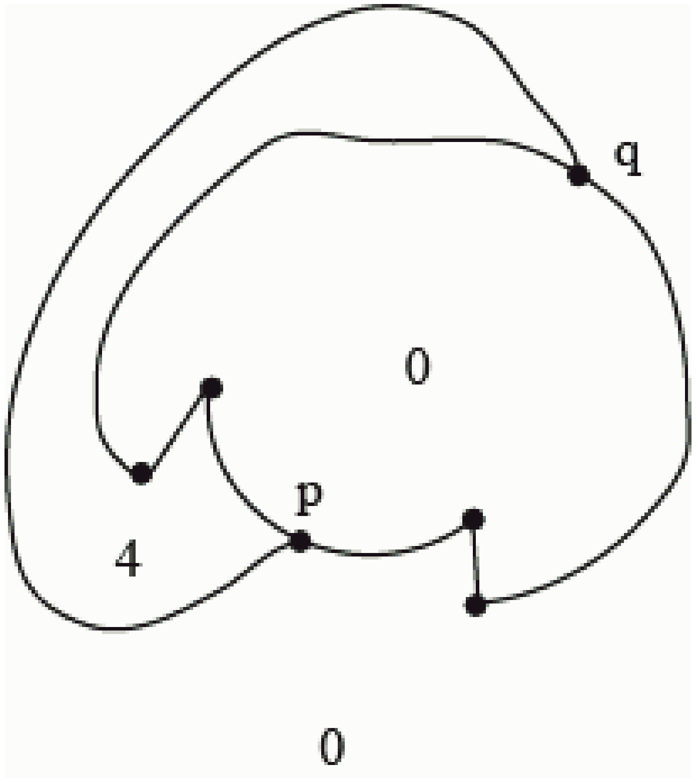
- τ_1 has a C-shape path *UCD* such that at least one of the darts of subpath labeled by *C* is on p_3

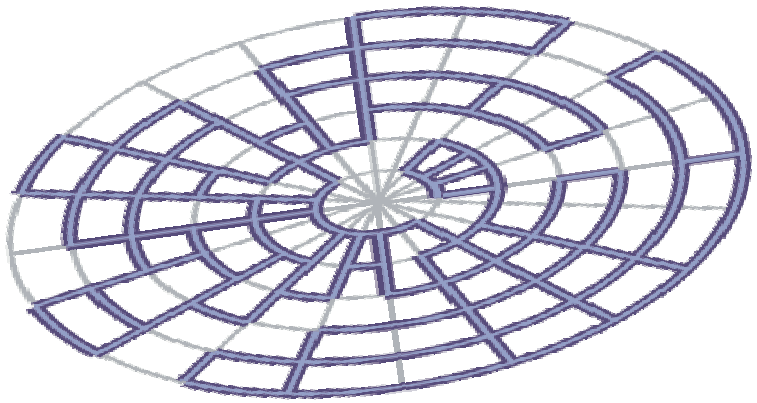




Drawable C-shapes of theta graphs

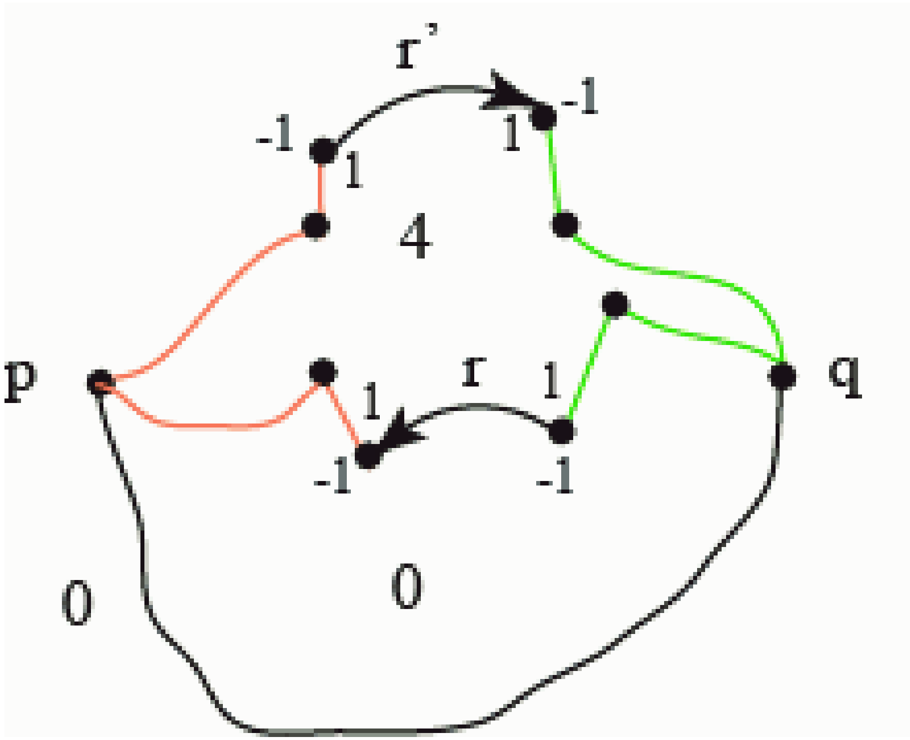
- τ_3 has a C-shape path DAU such that at least one of the darts of subpth labeled by A is on p_3

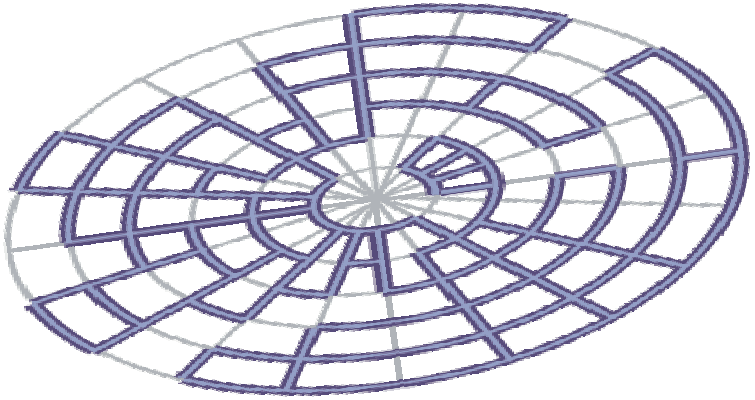




Drawable C-shapes of theta graphs

. σ_2 *has a C-shape path* DCU *and* $\bar{\sigma_1}$ *has a C-shape path* UC_cD *such that* τ_2 *is* $\tau_{21}DCU\tau_{22}UC_cD\tau_{23}$ *for some C-shape paths* τ_{21}, τ_{22} *and* τ_{23}, *and* $rot(D\tau_{23}\tau_{21}U) = rot(D\tau_{22}U) = 0$.

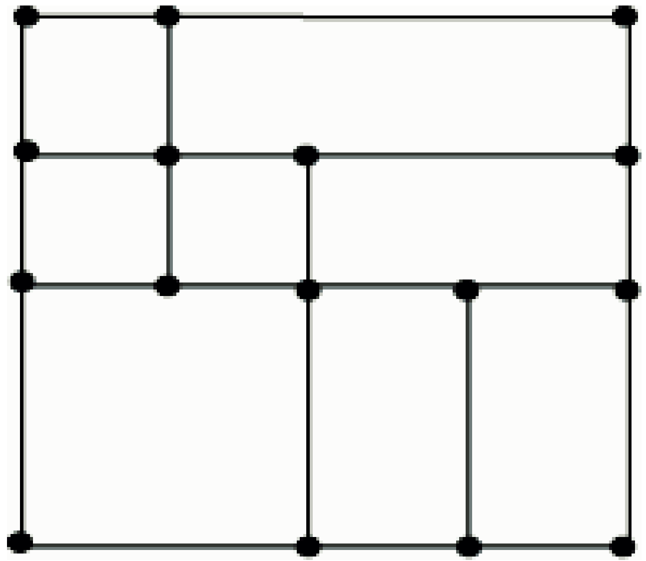




Rectangular Drawing

A rectangular drawing of a plane graph *G* is a planar drawing of *G* where

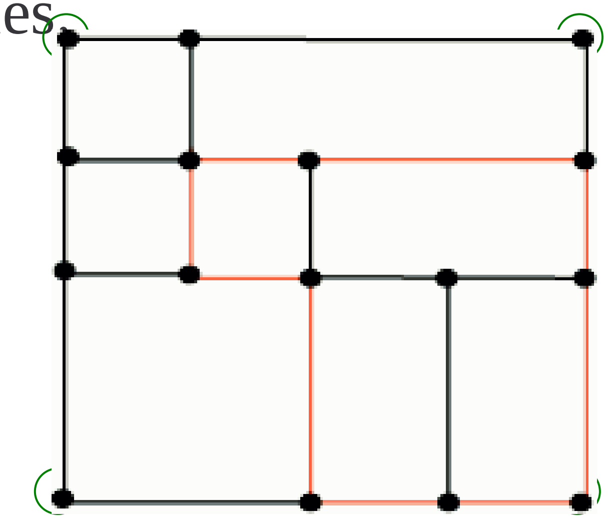
- each vertex is drawn as a point,
- each edge is drawn as a vertical or horizontal segment and
- and each face is rectangular.



Rectangular Drawings

Let *G* be a plane graph with a rectangular drawing, than:

- The degree of each vertex is a most four.
- *G* is biconnected.
- *G* has at least four degree two vertices on the boundary of the external face to form the corners of external face.
- Each cycle of *G* is drawn as a polygon, so it should have at least four 90 degree angles.



Rectangular Drawings

Rectangular drawings are studied in several papers:

- A necessary and sufficient condition is given for cubic plane graphs by Thomasson(1984) to have rectangular drawings.

- Rahman and et al.(1998-2004) have presented linear time algorithms to test graphs with maximum degree 3 to have rectangular drawings and find a rectangular drawing if one exists.

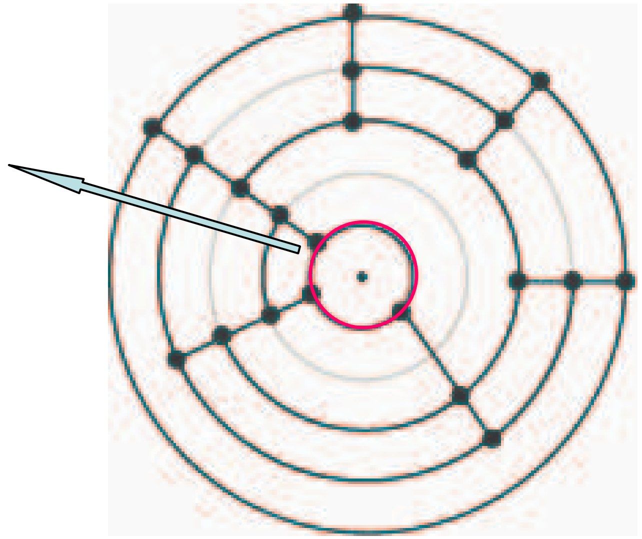
- Mura and et al (2006) found a polynomial time algorithm to test graphs with maximum degree 4 to have rectangular drawings and find a rectangular drawing if one exists.

Rectangular-Radial (RR) Drawing

An RR drawing of a plane graph *G* is a planar drawing of *G* where

- each vertex is drawn as a point,
- each edge is drawn as a circular or radial segment,
- each internal face not containing *S* is rectangular,
- And the external face and the internal face containing the center of the circles are circular.

We call the internal face containing the center of the circles the inner face.

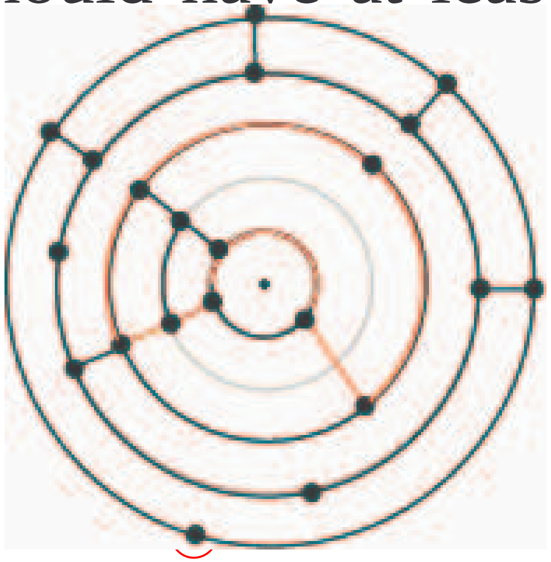


Rectangular drawings and (RR) Drawing

1- Vertices of degree two have no role in an RR drawing.

2- Biconnectivity is not necessary to have RR drawing.

3- Each cycle of *G* not containing the inner face is drawn as a polygon, so should have at least four 90 degree angles.



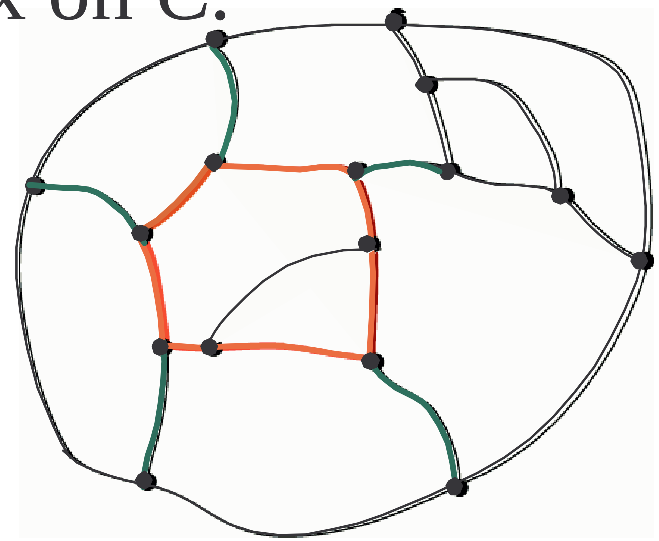
Legs

Let G be a plane graph. For a cycle C of G, an edge e is called a leg of C if:

1- e is not a bridge of G,

2- e is out of C ,and

3- e is incident to one vertex on C.



Rectangular-Radial (RR) Drawing

Theorem: A cubic plane graph G with a prescribed face f has a rectangular drawing if and only if each cycle not containing the inner face has at least four legs.

Thanks for your attention