


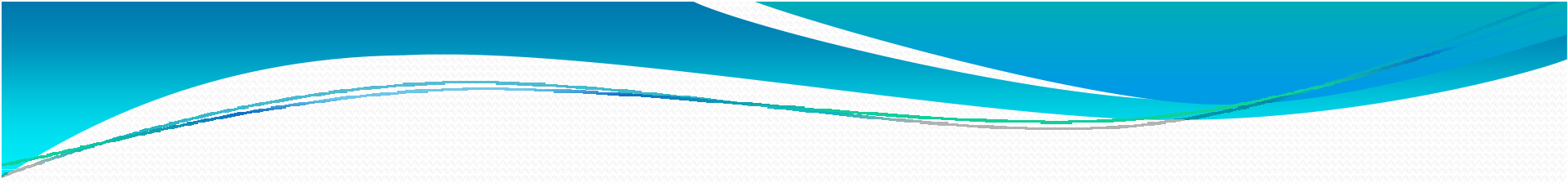


# **Hermitian Matrices, Eigenvalue Multiplicities and Eigenvector Components**



Because of the interlacing inequalities,  $|m_A(\lambda) - m_{A(i)}(\lambda)| \leq 1$ , and all 3 values of  $|m_A(\lambda) - m_{A(i)}(\lambda)|$  are possible. We call the index  $i$  a Parter (resp., downer, neutral) index if  $|m_A(\lambda) - m_{A(i)}(\lambda)| = -1$  (resp., 1, 0). In the event that the graph of  $A$  becomes relevant, recall that  $G(A)$  is the graph on  $n$  vertices in which there is an edge between  $i$  and  $j$  if and only if the  $i, j$  entry of  $A$  is nonzero.

If  $m_A(\lambda) \geq 1$ , denote the corresponding eigenspace by  $E_A(\lambda)$ . If  $m_A(\lambda) = 0$ , then we may, for convenience, adopt the convention that  $E_A(\lambda)$  contains only the zero vector.



## **Theorem:**(PW-theorem)

Let  $T$  be a tree on  $n$  vertices and suppose that  $\lambda \in R$  is such that  $m_A(\lambda) \geq 2$ . Then, there is a vertex  $i$  of  $T$  such that  $m_{A(i)}(\lambda) = m_A(\lambda) + 1$ .

For trees, a useful characterization of when a vertex is Parter was demonstrated.

## **Theorem:**

Let  $T$  be a tree, and  $v$  a vertex of  $T$ ,  $m_A(v)(\lambda) = m_A(\lambda) + 1$  if and only if there is a downer branch at  $v$  for  $\lambda$ .

## Definition:

In the event that entry  $i$  of  $x$  is 0 for every  $x \in E_A(\lambda)$ , we say that  $i$  is a null vertex (for  $A$  and  $\lambda$ ); otherwise  $i$  is a nonzero vertex.

When  $i$  is a null vertex, the structure of  $E_A(\lambda)$  imparts a good deal of information about  $E_{A(i)}(\lambda)$ . Suppose, w.l.o.g., that  $n = i$ :

$$\begin{bmatrix} A(n) & a \\ a^* & a_{nn} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Then,  $A(n)x = \lambda x$ . This implies, in particular, that a null vertex is, at least, neutral. The converse is also valid.



## Lemma 1:

For an  $n$ -by- $n$  Hermitian matrix  $A$  and an identified  $\lambda \in R$ , we have the following:

1. If  $i$  is downer, then  $E_A(\lambda) \supset E_{A(i)}(\lambda)$ .
2. If  $i$  is neutral, then  $E_A(\lambda) = E_{A(i)}(\lambda)$ .
3. If  $i$  is Parter, then  $E_A(\lambda) \subset E_{A(i)}(\lambda)$ .

## Proof:

Assume w.l.o.g. that  $i = n$  and  $\lambda = 0$ , and use the block decomposition of  $A$  shown in above.




If  $a$  is a linear combination of the rows of  $A(n)$ , then

$$E_A(0) \supseteq E_{A(n)}(0).$$

If  $a$  is not a linear combination of the rows of  $A(n)$ , then sequentially extending  $A(n)$  by the row  $a$  and then by the column  $(a^* \ a_{nn})$  increases the rank each time. Thus,  $\text{rank}A = \text{rank}A(n) + 2$ , so  $n$  is Parter.

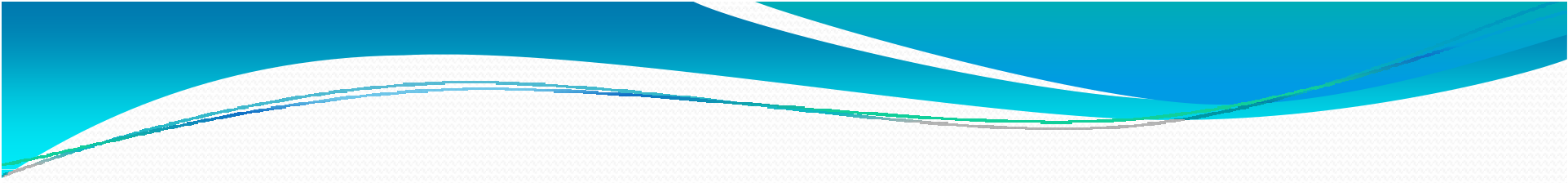
Therefore, if  $n$  is downer or neutral,  $E_A(0) \supseteq E_{A(n)}(0)$ . By definition, if  $n$  is downer, the containment is strict, and if  $n$  is neutral, the containment is actually equality.



Suppose  $n$  is Parter. Let  $X$  be the maximal subspace of  $E_{A(n)}(0)$  that is orthogonal to  $(a^* \ 0)^*$ . Clearly,  $X \subseteq E_A(0)$ . Since  $\dim X \geq m_{A(n)}(0) - 1 = m_A(0)$ , we have  $X = E_A(0)$ .

## **Theorem 1:**

Let  $A$  be an  $n$ -by- $n$  Hermitian matrix. Then, index  $i$  is null for  $A$  if and only if index  $i$  is either Parter or neutral.



# Distinguishing Parter and neutral

## Lemma 2:

If  $n$  is a null vertex, then  $n$  is neutral if and only if  $E_{A(n)}(\lambda)$  is orthogonal to  $a$ .

## Proof:

By Lemma 1,  $E_A(\lambda) \supseteq E_{A(n)}(\lambda)$ . In fact,  $E_A(\lambda)$  is precisely the maximal subspace of  $E_{A(n)}(\lambda)$  that is orthogonal to  $[a^* \ 0]^*$ . Thus,  $n$  is neutral if and only if  $E_A(\lambda) = E_{A(n)}(\lambda)$  if and only if  $E_{A(n)}(\lambda)$  is orthogonal to  $a$ .






### **Lemma 3:**

Suppose that the graph of  $A$  is a tree and that  $n$  is a null vertex for some  $\lambda \in R$ . The following statements are equivalent.

1.  $n$  is neutral.
2. All neighbors of  $n$  are null for  $A(n)$ .

### **Example:**

In fact, if the graph of  $A$  is not a tree, then a neutral vertex  $i$  may be adjacent to a vertex  $j$  that is nonzero for  $A(n)$ . Consider


$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Vertex 3 is neutral for the eigenvalue 0, and vertices 1 and 2 are nonzero for  $A(n)$ .

## Theorem 2:

Let  $A$  be an Hermitian matrix whose graph is a tree, and let  $i$  be a null vertex for  $A$ . Then  $i$  is Parter if and only if there is a neighbor  $j$  that is nonzero for  $A(i)$ .



## Corollary 1:

Suppose that the graph of  $A$  is a tree. Every neighbor of a neutral vertex is a null vertex for  $A$ .

### Proof:

By the theorem, if  $i$  is neutral, then every neighbor of  $i$  is null for  $A(i)$ . Because  $E_A(\lambda) = E_{A(i)}(\lambda)$ , every vertex that is null for  $A(i)$  is also null for  $A$ .



# Implications

## Lemma 4:

Let  $A$  be an  $n$ -by- $n$  Hermitian matrix. If  $i$  is neutral, then  $j \neq i$  is downer for  $A$  if and only if  $j$  is downer for  $A(i)$ .

## Proof:

If  $i$  is neutral, then  $E_A(\lambda) = E_{A(i)}(\lambda)$ , which implies that  $j$  is nonzero for  $A$  if and only if  $j$  is nonzero for  $A(i)$ .

## Lemma 5:

Let  $A$  be an  $n$ -by- $n$  Hermitian matrix. If  $i$  is Parter and  $j$  is downer (for  $A$  and  $\lambda$ ), then  $j$  is also downer for  $A(i)$  and  $\lambda$ .




# Vertex classification

## Proposition 1:

Let  $A$  be an  $n$ -by- $n$  Hermitian matrix. If  $m_A(\lambda) = m$ , then  $A$  has at least  $m$  downer vertices.

## Proof:

Assume  $m \geq 1$ . Because  $\dim E_A(\lambda) = m$ , there is some vector in  $E_A(\lambda)$  that has at least  $m$  nonzero entries. These entries identify downer vertices.



## Proposition 2:

Suppose that the graph of  $A$  is connected. If  $m_A(\lambda) = m \geq 1$ , then  $A$  has at least  $m + 1$  downer vertices.

### Proof:

By Proposition 1,  $A$  has at least  $m$  nonzero vertices. Suppose  $A$  has exactly  $m$  nonzero vertices. Then  $E_A(\lambda)$  is spanned by vectors  $e_{i_1}, \dots, e_{i_m}$ , where  $e_j$  is the  $j$ th standard basis vector for  $C^n$ . Since  $(A - \lambda I)e_j = 0$  implies the  $j$ th column of  $A - \lambda I$  is zero, the graph of  $A$  is not connected.

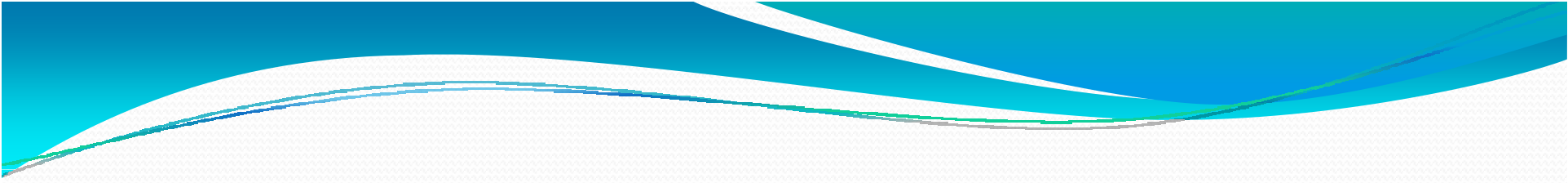


## Example:

A star is a graph that is a tree and has exactly one vertex of degree  $> 1$ . If the graph of  $A$  is the star on  $n$  vertices, and every diagonal entry of  $A$  is  $\lambda$ , then  $m_A(\lambda) = n - 2$ . Also, the central vertex is Parter, and every pendant vertex is a downer vertex, so  $A$  has exactly  $m_A(\lambda) + 1$  downer vertices. Therefore, Proposition 2 is the strongest statement that can be made for all connected graphs.

## Proposition 3:

Suppose that the graph of  $A$  is a tree, and let  $i$  be a neutral vertex. Then every neighbor of  $i$  is either Parter or neutral for  $A$ .



# Classification of vertex pairs

## Proposition 4:

Let  $A$  be an  $n$ -by- $n$  Hermitian matrix, and let  $i$  and  $j$  be distinct indices. We have the following three statements.

1. If  $i$  and  $j$  are Parter, then  $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) \in \{-2, 0\}$ .
2. If  $i$  and  $j$  are neutral, then  $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) \in \{-1, 0\}$ .
3. If  $i$  is neutral and  $j$  is downer, then  $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 1$ .



## Proof:

1. Clearly, if  $i$  and  $j$  are Parter vertices, then

$$-2 \leq m_A(\lambda) - m_{A(\{i,j\})}(\lambda) \leq 0.$$

Suppose that the difference is  $-1$ , for the sake of contradiction. Assuming w.l.o.g. that our eigenvalue  $\lambda$  equals 0 and that  $i = n - 1$  and  $j = n$ , we write

$$\begin{bmatrix} A(\{n-1, n\}) & a_{1,n-1} & a_{1,n} \\ * & a_{n-1,n-1} & a_{n-1,n} \\ * & * & a_{nn} \end{bmatrix}$$

where the entries marked  $*$  are determined by the Hermiticity of  $A$ .

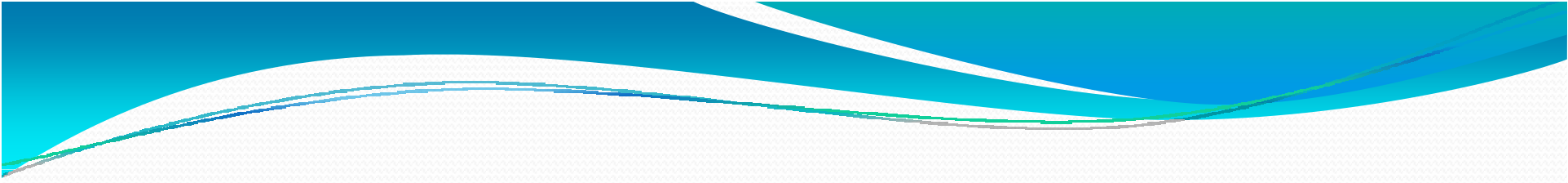
By our assumption that  $m_A(\lambda) - m_{A(\{n-1,n\})}(\lambda) = -1$ , it follows that  $n-1$  is neutral for  $A(n)$  and that  $n$  is neutral for  $A(n-1)$ , and therefore  $a_{1,n-1}$  and  $a_{1,n}$  are linear combinations of the columns of  $A(\{n-1, n\})$ . Hence,

$$\text{rank} A \leq \text{rank} \begin{bmatrix} A(\{n-1, n\}) & a_{1,n-1} & a_{1,n} \end{bmatrix} + 2 = \text{rank} A(\{n-1, n\}) + 2,$$

so that

$$m_A(0) = n - \text{rank}(A) \geq (n-2) - \text{rank} A(\{n-1, n\}) = m_{A(\{n-1,n\})}(0),$$

contradicting the assumption that  $m_A(\lambda) - m_{A(\{n-1,n\})}(\lambda) = -1$ .

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2. By Lemma 4, if  $i$  and  $j$  are neutral, then  $j$  is Parter or neutral for  $A(i)$ .
  3. By the same lemma, if  $i$  is neutral and  $j$  is downer, then  $j$  is downer for  $A(i)$ .

## Proposition 5:

Suppose that the graph of  $A$  is a tree, and let  $i$  and  $j$  be neighbors. We have the following two statements.

1. If  $i$  and  $j$  are neutral, then  $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 0$ .
2. If  $i$  and  $j$  are downer, then  $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 1$ .



## Proof:

1. By Proposition 4, if  $i$  and  $j$  are neutral, then

$$m_A(\lambda) - m_{A(\{i,j\})}(\lambda) \in \{-1, 0\}.$$

Suppose  $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -1$ . Then  $j$  is Parter in  $A(i)$ , so  $j$  is adjacent to a vertex  $k$  which is downer for  $A(\{i, j\})$ . But then  $k$  must also be a downer in  $A(j)$  since  $i$  and  $j$  are adjacent. It follows that  $j$  is Parter for  $A$  - a contradiction.

## Example:

We will show that if  $i$  and  $j$  are not adjacent, then the conclusions of Proposition 5 may not hold.

Take  $\lambda = 0$  and let

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$m_B(0) = 1$  and  $m_{B(1)}(0) = m_{B(3)}(0) = 0$ , so the pendant vertices are downer vertices. However, in contrast to claim 2 of Proposition 5,  $m_B(0) - m_{B(\{1,3\})}(0) = 0$ .

$i$	$j$	$m_A(\lambda) - m_{A(\{i,j\})}(\lambda)$
Parter	Parter	-2, 0
Parter	Neutral	-1, 0
Parter	Downer	0
Neutral	Neutral	-1, 0
Neutral	Downer	1
Downer	Downer	0, 1, 2

**Table 1**

$i$	$j$	$m_A(\lambda) - m_{A(\{i,j\})}(\lambda)$
Parter	Parter	-2, 0
Parter	Neutral	-1, 0
Parter	Downer	0
Neutral	Neutral	0
Neutral	Downer	not possible
Downer	Downer	1

**Table 2**