Hermitian Matrices, Eigenvalue Multiplicities and Eigenvector Components

Because of the interlacing inequalities, $|m_A(\lambda) - m_{A(i)}(\lambda)| \leq 1$, and all 3 values of $|m_A(\lambda) - m_{A(i)}(\lambda)|$ are possible. We call the index i a Parter (resp., downer, neutral) index if $|m_A(\lambda) - m_{A(i)}(\lambda)| = -1$ (resp., 1, 0). In the event that the graph of A becomes relevant, recall that G(A) is the graph on n vertices in which there is an edge between i and j if and only if the i, j entry of A is nonzero.

If $m_A(\lambda) \geq 1$, denote the corresponding eigenspace by $E_A(\lambda)$. If $m_A(\lambda) = 0$, then we may, for convenience, adopt the convention that $E_A(\lambda)$ contains only the zero vector.

Theorem:(PW-theorem)

Let T be a tree on n vertices and suppose that $\lambda \in R$ is such that $m_A(\lambda) \geq 2$. Then, there is a vertex i of T such that $m_{A(i)}(\lambda) = m_A(\lambda) + 1$.

For trees, a useful characterization of when a vertex is Parter was demonstrated.

Theorem:

Let T be a tree, and v a vertex of T, $m_A(v)(\lambda) = m_A(\lambda) + 1$ if and only if there is a downer branch at v for λ.

Definition:

In the event that entry i of x is 0 for every $x \in E_A(\lambda)$, we say that i is a null vertex (for A and λ); otherwise i is a nonzero vertex.

When i is a null vertex, the structure of $E_A(\lambda)$ imparts a good deal of information about $E_{A(i)}(\lambda)$. Suppose, w.l.o.g., that $n = i$:

$$\begin{bmatrix} A(n) & a \\ a^\star & a_{nn} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

Then, $A(n)x = \lambda x$. This implies, in particular, that a null vertex is, at least, neutral. The converse is also valid.

Lemma 1:

For an n-by-n Hermitian matrix A and an identified $\lambda \in R$, we have the following:

1. If i is downer, then $E_A(\lambda) \supset E_{A(i)}(\lambda)$.
2. If i is neutral, then $E_A(\lambda) = E_{A(i)}(\lambda)$.
3. If i is Parter, then $E_A(\lambda) \subset E_{A(i)}(\lambda)$.

Proof:

Assume w.l.o.g. that $i = n$ and $\lambda = 0$, and use the block decomposition of A shown in above.

If a is a linear combination of the rows of $A(n)$, then

$$E_A(0) \supseteq E_{A(n)}(0).$$

If a is not a linear combination of the rows of A(n), then sequentially extending A(n) by the row a and then by the column $\begin{pmatrix} a^* & a_{nn} \end{pmatrix}$ increases the rank each time. Thus, $rankA = rankA(n) + 2$, so n is Parter.

Therefore, if n is downer or neutral, $E_A(0) \supseteq E_{A(n)}(0)$. By definition, if n is downer, the containment is strict, and if n is neutral, the containment is actually equality.

Suppose n is Parter. Let X be the maximal subspace of $E_{A(n)}(0)$ that is orthogonal to $\begin{pmatrix} a^* & 0 \end{pmatrix}^*$. Clearly, $X \subseteq E_A(0)$. Since $dim X \geq m_{A(n)}(0) - 1 = m_A(0)$, we have $X = E_A(0)$.

Theorem 1:

Let A be an n-by-n Hermitian matrix. Then, index i is null for A if and only if index i is either Parter or neutral.

Distinguishing Parter and neutral

Lemma 2:

If n is a null vertex, then n is neutral if and only if $E_{A(n)}(\lambda)$ is orthogonal to a.

Proof:

By Lemma 1, $E_A(\lambda) \supseteq E_{A(n)}(\lambda)$. In fact, $E_A(\lambda)$ is precisely the maximal subspace of $E_{A(n)}(\lambda)$ that is orthogonal to $\begin{bmatrix} a^* & 0 \end{bmatrix}^*$. Thus, n is neutral if and only if $E_A(\lambda) = E_{A(n)}(\lambda)$ if and only if $E_{A(n)}(\lambda)$ is orthogonal to a.

Lemma 3:

Suppose that the graph of A is a tree and that n is a null vertex for some $\lambda \in R$. The following statements are equivalent.

1. n is neutral.
2. All neighbors of n are null for $A(n)$.

Example:

In fact, if the graph of A is not a tree, then a neutral vertex i may be adjacent to a vertex j that is nonzero for $A(n)$. Consider

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Vertex 3 is neutral for the eigenvalue 0, and vertices 1 and 2 are nonzero for $A(n)$.

Theorem 2:

Let A be an Hermitian matrix whose graph is a tree, and let i be a null vertex for A. Then i is Parter if and only if there is a neighbor j that is nonzero for $A(i)$.

Corollary 1:

Suppose that the graph of A is a tree. Every neighbor of a neutral vertex is a null vertex for A.

Proof:

By the theorem, if i is neutral, then every neighbor of i is null for $A(i)$. Because $E_A(\lambda) = E_{A(i)}(\lambda)$, every vertex that is null for $A(i)$ is also null for A.

Implications

Lemma 4:

Let A be an n-by-n Hermitian matrix. If i is neutral, then $j \neq i$ is downer for A if and only if j is downer for $A(i)$.

Proof:

If i is neutral, then $E_A(\lambda) = E_{A(i)}(\lambda)$, which implies that j is nonzero for A if and only if j is nonzero for $A(i)$.

Lemma 5:

Let A be an n-by-n Hermitian matrix. If i is Parter and j is downer (for A and λ), then j is also downer for $A(i)$ and λ.

Vertex classification

Proposition 1:

Let A be an n-by-n Hermitian matrix. If $m_A(\lambda) = m$, then A has at least m downer vertices.

Proof:

Assume $m \geq 1$. Because $\dim E_A(\lambda) = m$, there is some vector in $E_A(\lambda)$ that has at least m nonzero entries. These entries identify downer vertices.

Proposition 2:

Suppose that the graph of A is connected. If $m_A(\lambda) = m \geq 1$, then A has at least $m + 1$ downer vertices.

Proof:

By Proposition 1, A has at least m nonzero vertices. Suppose A has exactly m nonzero vertices. Then $E_A(\lambda)$ is spanned by vectors $e_{i_1}, \ldots, e_{i_m}$, where e_j is the jth standard basis vector for C^n. Since $(A - \lambda I)e_j = 0$ implies the jth column of $A - \lambda I$ is zero, the graph of A is not connected.

Example:

A star is a graph that is a tree and has exactly one vertex of degree > 1. If the graph of A is the star on n vertices, and every diagonal entry of A is λ, then $m_A(\lambda) = n - 2$. Also, the central vertex is Parter, and every pendant vertex is a downer vertex, so A has exactly $m_A(\lambda) + 1$ downer vertices. Therefore, Proposition 2 is the strongest statement that can be made for all connected graphs.

Proposition 3:

Suppose that the graph of A is a tree, and let i be a neutral vertex. Then every neighbor of i is either Parter or neutral for A.

Classification of vertex pairs

Proposition 4:

Let A be an n-by-n Hermitian matrix, and let i and j be distinct indices. We have the following three statements.

1. If i and j are Parter, then $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) \in \{-2, 0\}$.
2. If i and j are neutral, then $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) \in \{-1, 0\}$.
3. If i is neutral and j is downer, then $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 1$.

Proof:

1. Clearly, if i and j are Parter vertices, then

$$-2 \leq m_A(\lambda) - m_{A(\{i,j\})}(\lambda) \leq 0.$$

Suppose that the difference is -1, for the sake of contradiction. Assuming w.l.o.g. that our eigenvalue λ equals 0 and that $i = n-1$ and $j = n$, we write

$$\begin{bmatrix} A(\{n-1,n\}) & a_{1,n-1} & a_{1,n} \\ * & a_{n-1,n-1} & a_{n-1,n} \\ * & * & a_{nn} \end{bmatrix}$$

where the entries marked $*$ are determined by the Hermicity of A.

By our assumption that $m_A(\lambda) - m_{A(\{n-1,n\})}(\lambda) = -1$, it follows that $n-1$ is neutral for $A(n)$ and that n is neutral for $A(n-1)$, and therefore $a_{1,n-1}$ and $a_{1,n}$ are linear combinations of the columns of $A(\{n-1,n\})$. Hence,

$$rank A \leq rank \begin{bmatrix} A(\{n-1,n\}) & a_{1,n-1} & a_{1,n} \end{bmatrix} + 2 = rank A(\{n-1,n\}) + 2,$$

so that

$$m_A(0) = n - rank(A) \geq (n-2) - rank A(\{n-1,n\}) = m_{A(\{n-1,n\})}(0),$$

contradicting the assumption that $m_A(\lambda) - m_{A(\{n-1,n\})}(\lambda) = -1$.

2. By Lemma 4, if i and j are neutral, then j is Parter or neutral for $A(i)$.

3. By the same lemma, if i is neutral and j is downer, then j is downer for $A(i)$.

Proposition 5:

Suppose that the graph of A is a tree, and let i and j be neighbors. We have the following two statements.

1. If i and j are neutral, then $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 0$.

2. If i and j are downer, then $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = 1$.

Proof:

1. By Proposition 4, if i and j are neutral, then

$$m_A(\lambda) - m_{A(\{i,j\})}(\lambda) \in \{-1, 0\}.$$

Suppose $m_A(\lambda) - m_{A(\{i,j\})}(\lambda) = -1$. Then j is Parter in $A(i)$, so j is adjacent to a vertex k which is downer for $A(\{i,j\})$. But then k must also be a downer in $A(j)$ since i and j are adjacent. It follows that j is Parter for A- a contradiction.

Example:

We will show that if i and j are not adjacent, then the conclusions of Proposition 5 may not hold.

Take $\lambda = 0$ and let

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$m_B(0) = 1$ and $m_{B(1)}(0) = m_{B(3)}(0) = 0$, so the pendant vertices are downer vertices. However, in contrast to claim 2 of Proposition 5, $m_B(0) - m_{B(\{1,3\})}(0) = 0$.

i	j	$m_A(\lambda) - m_{A(\{i,j\})}(\lambda)$
Parter	Parter	$-2, 0$
Parter	Neutral	$-1, 0$
Parter	Downer	0
Neutral	Neutral	$-1, 0$
Neutral	Downer	1
Downer	Downer	$0, 1, 2$

Table 1

i	j	$m_A(\lambda) - m_{A(\{i,j\})}(\lambda)$
Parter	Parter	$-2, 0$
Parter	Neutral	$-1, 0$
Parter	Downer	0
Neutral	Neutral	0
Neutral	Downer	not possible
Downer	Downer	1

Table 2