On the Dynamic Chromatic Number of Graphs

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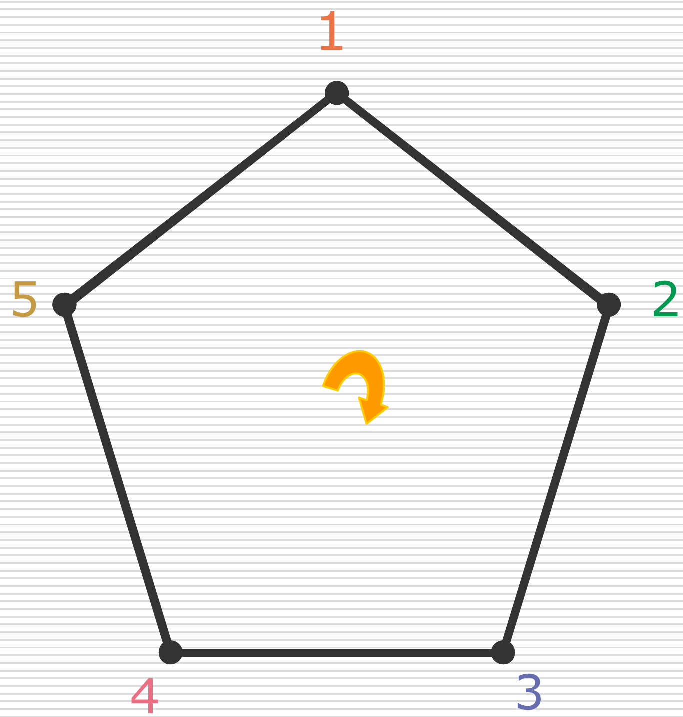
1. Introduction

Let G be a graph. A vertex coloring of G is a function c: $V(G) \to L$ *, where L is a set with this property: if* $u, v \in V(G)$ *are adjacent, then c(u) and c(v) are different.*

A ***vertex k-coloring*** *is a proper vertex coloring with |L|=k.*

A proper vertex k-coloring of a graph G is called ***dynamic*** *if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. The smallest integer k such that G has a k-dynamic coloring is called the* ***dynamic chromatic number of G*** *and denoted by* $\chi_2(G)$.

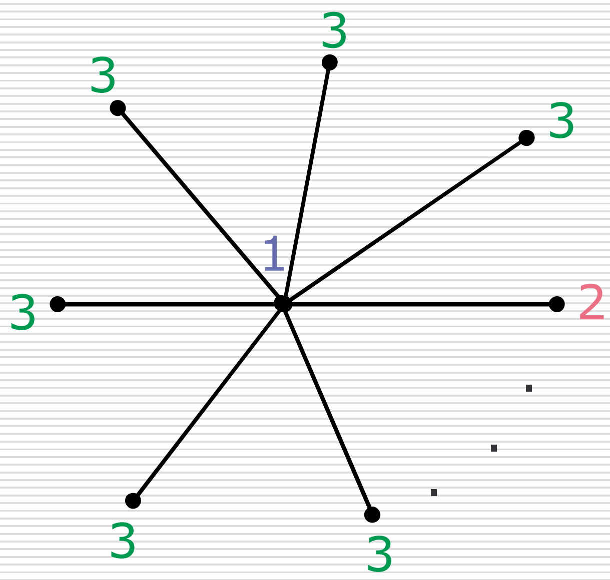
Some Examples



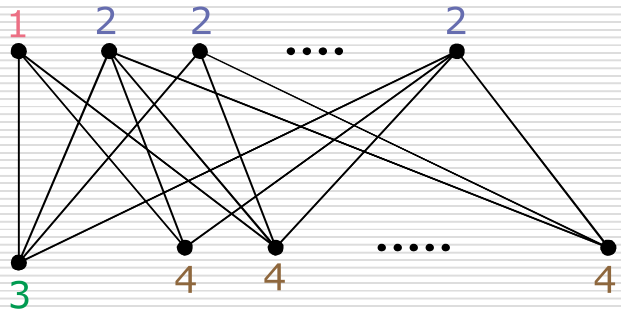
$$\chi_2(C_5) = 5$$

$$\chi_2(K_{1,1}) = 2$$

$$\chi_2(K_{1,m}) = 3$$



$$\chi_2(K_{m,n}) = 4 \qquad m, n \geq 2$$



Some Theorems

Theorem 1. *Let G be a connected, non-trivial Graph. Then* $\chi_2(K_2) = 2$ *and otherwise* $\chi_2(G) \geq 3$.

Brook's Theorem.

For every graph G, $\chi(G) \le \Delta(G) + 1.$

Theorem 2. *If* $\Delta(G) \le 3$, *then* $\chi_2(G) \le 4$, *with the only exception that* $G = C_5$, *in which case* $\chi_2(G) = 5$.

Theorem 3. *If* $\Delta(G) \ge 3$, *then* $\chi_2(G) \le \Delta(G) + 1$.

Theorem 4. *If* $\alpha(G)$ *be the length of the longest cycle in G, then* $\chi(G) \leq \alpha(G)$.

Also we have, $\chi_2(G) \leq \alpha(G)$.

Theorem 5. *A well-known upper bound for* $\chi(G)$ *in terms of the length* $l(G)$ *of the longest path in* G *is* $\chi(G) \leq l(G)+1$.

Also, for any graph G, $\chi_2(G) \leq l(G)+1$.

2. The Dynamic Chromatic Number of Bipartite Graphs

First we show that the dynamic chromatic number of a bipartite graph can be sufficiently large.

Theorem 6. *For any k-regular bipartite graph G,* $k \geq 3$ *, if* $n \leq 2^k$*, then* $\chi_2(G) \leq 4$.

By Montgomery (2001)

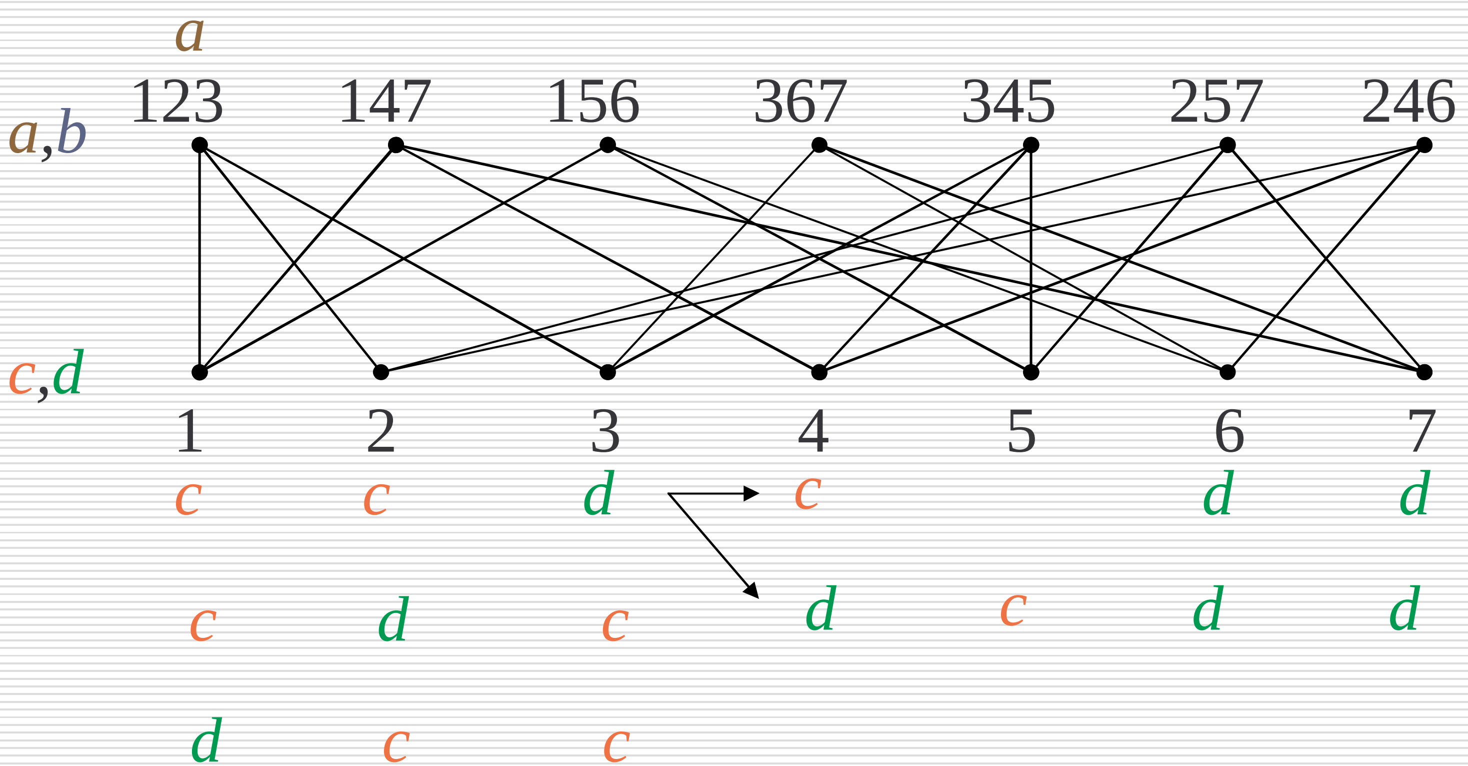
Theorem 7. *Let G be a k-regular bipartite graph where $k \geq 4$. Then there is a 4-dynamic coloring of G, using 2 colors in each part.*

Proof by Alon.

In the following, it is shown that there are some 3-regular bipartite graphs with no 4-dynamic coloring using two colors in each part.

Consider the following example:

Let F={123,147,156,367,345,257,246} be the lines of a Fano Plane. We define a 3-regular bipartite graph of order 14 named G by two parts A and B. The Part A contains all lines of F as its vertices and the Part B contains numbers 1,2,3,4,5,6,7 as its vertices. We join $i \in B$ *to the vertex xyz of A, if and only if* $i \in \{x, y, z\}$.



a
123 147 156 367 345 257 246
a,b
c,d
1 2 3 4 5 6 7
c c d c d d
c d c d c d d
d c c

***Conjecture*: (Montgomery 2001)**

For every regular graph G, $\chi_2(G) - \chi(G) \leq 2$.

The previous theorem shows that conjecture is true for bipartite regular graphs.

Theorem 8. *Let G be a bipartite graph with parts A and B, and suppose that the length of each cycle of G is divisible by 4. Then one can color the vertices of A by two colors, and those of B by two colors so that every vertex of degree at least 2 has neighbors of two colors.*

Sketch of proof . *We show that B has a 2-coloring so that each vertex of A of degree at least 2 has neighbors of both colors, the result for coloring A is symmetric…*

Lemma 9. *Let G be a bipartite graph such that for every u,v* $\in V(G)$, $|N(u) \cap N(v)| \neq 1$. *Then* $\chi_2(G) \leq 4$.

3. On the Dynamic Chromatic Number of Graphs Whose the Length of Each Cycle Are Divisible by L.

Lemma 10. *Let G be a graph and $l \geq 3$ be a natural number. If the length of each cycle of G is divisible by l, then for any $e \in E(G)$, there exists $e' \in E(G)$ such that $\{e, e'\}$ is an edge cut for G.*

Theorem 11. *If G is a graph such that the length of every cycle of G is divisible by 3, then* $\chi_2(G) \leq 3$. *Moreover, if one of the components of G is neither* K_1 *nor* K_2, *then* $\chi_2(G) = 3$.

Sketch of proof. *By induction on $|V(G)|$, we prove the theorem.*

If $\delta(G)=1$, and $v \in V(G)$ is a pendant vertex, then by induction hypothesis $\chi_2(G \setminus v) \leq 3$, and so $\chi_2(G) \leq 3$.

Thus we can assume that $\delta(G) \geq 2$.

Now, three cases can be considered:

Case1. The graph G has two adjacent vertices of degree 2.

Case 2. There are two adjacent vertices of degree at least 3, say u and v which e=uv.

Suppose that {e,e'} is the edge-cut of G.

First suppose that e and e' are not incident.

Now, suppose that e and e' are incident.

Case 3. All vertices of degree at least 3 and all vertices of degree 2 form two independent sets.

Theorem 12. *Let G be a graph whose none of the components is isomorphic to* C_5. *If p is an odd prime and the length of each cycle of G is divisible by p, then* $\chi_2(G) \leq 4$.

Sketch of proof. *The proof is based on induction on n=|V(G)|.* *If one of the components of G is a cycle, then by induction and noting that* $\chi_2(C_n) \le 4$, *for* $n \ne 5$ *then we have* $\chi_2(G) \le 4$.

Hence suppose that no component of G is a cycle.

If n= 1 or $\delta(G) = 1$, *then the result is obvious.*

Thus we assume that $\delta(G) \ge 2$.

Three cases can be considered:

Case 1. There are two adjacent vertices of degree 2.

Case 2. There are two adjacent vertices of degree at least 3.

First assume that e and e' are not incident.

Next suppose that e and e' are incident.

Case 3. In this case we can assume that all vertices of degree at least 3 form an independent set.

Conjecture:

Let G be a graph with $\delta(G)\geq 2$. *If* $l \neq 4$ *is a natural number and the length of every cycle of G is divisible by l, then G has at least two adjacent vertices of degree 2.*

Bondy, 1998

With the exception of K_1 *and* K_2 *every simple graph having at most two vertices of degree less than three contains two cycles whose lengths differ by one or two.*

Corollary 13. *Let G be a graph and* $l \geq 3$ *be a natural number. If the length of every cycle of G is divisible by l and G has no component isomorphic to* C_5, *then* $\chi_2(G) \leq 4$.

Proof. *Apply Lemma 10 and Theorem 12.*

Theorem 14. *Let G be a graph and* $l \geq 2$ *be a natural number. If the length of every cycle of G is divisible by l, then* $\chi(G) \leq 3$.

Proof. *If* $2|\,l$, *then G is bipartite and* $\chi(G) \leq 2$. *Otherwise, by Lemma 10,G has an edge cut of size 2. Now, by induction on The vertices, it is not hard to see that* $\chi(G) \leq 3$.

Thanks for your attention