On combinatorial trades

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- The concept of trade is related to some combinatorial structures.

  Designs, Latin squares and Graphs.

TRADES ON DESIGNS:

History

- Trades on designs have been defined and utilized in connection with designs in the last three decades.

- The idea behind a trade was used in 1916. White, Cole and Cummings used "quadrangular and hexagonal transformations"

- Later other different names were used. "t-pods" by Graver and Jurkat (1973), "fragments" by Gibbons (1976), "null designs" by Graham, Li and Li (1980) and "functions with strength t" by Deza, Frankl and Singhi (1983).

- Finally, Foody and Hedayat used the terminology "trade" in 1977.

SOME DEFINITIONS:

- **Let $[v]$ denotes the set $\{1, 2, \ldots, v\}$ and t, k and λ be three integers s.t. $v \geq k \geq t$ and $\lambda \geq 1$. A t-(v, k, λ) design $([v], \mathcal{B})$ is a collection $\mathcal{B}$ of k-subsets of $[v]$ (called blocks) s.t. every t-subset of $[v]$ appears exactly in λ blocks.**

- **A t-(v, k, λ) design is called Steiner design if $\lambda = 1$. It is called simple design if it has no repeated blocks.**

Example. $(t = 2, v = 7, k = 3, \lambda = 1)$**:**

A Steiner **and** simple 2-$(7, 3, 1)$ **design**

1	**2**	**4**
2	**3**	**5**
3	**4**	**6**
4	**5**	**7**
5	**6**	**1**
6	**7**	**2**
7	**1**	**3**

- **Isomorphism:** Two t-(v, k, λ) **designs** $([v], \mathcal{B})$ and $([v'], \mathcal{B}')$ are called isomorphic if $\exists$ a 1-1 correspondence $f : [v] \longrightarrow [v']$ s.t. $f(\mathcal{B}) = \mathcal{B}'$.

- **Question:** What are the **necessary** and **sufficient** conditions for the **existence** of a design with parameters v, k, t and λ?

- **Necessary conditions:** $b_i = \frac{\lambda\binom{v-i}{t-i}}{\binom{k-i}{t-i}} \qquad 0 \le i \le t$

  b_i is a positive integer and it is the number of blocks which contains every fixed i-subset of $[v]$.

- The sufficiency of the conditions is unsolved in general.

- **Special case:** $t = 2, k = 3, \lambda = 1$ **and** v **arbitrary (Steiner triple system) STS**(v)

- **Theorem:** **(Kirkman** 1847**) There exists a STS**(v) **iff** $v \equiv 1$ or 3 mod 6.

- **A** t-(v, k) trade $T = (T_1, T_2)$ **is a pair of two disjoint collections of** k**-subsets of** $[v]$ **(called** blocks**) such that for every** t**-subset of** $[v]$**, the number of blocks containing this subset is the same in both** T_1 **and** T_2**.**

- **Clearly,** $|T_1| = |T_2|$ **and this common value is called the** volume **of** T **and is denoted by** vol(T)**.**

- **The** spectrum **of a** t-(v, k) trade **is the set of all possible volumes of** t-(v, k) trades **(**t **and** k **are fixed and** v **is free).**

- **A** t-(v, k) trade $T = (T_1, T_2)$ **is called** simple **trade if there are no repeated blocks in** T_1 **or in** T_2**. It is called** Steiner **trade if every** t**-subset of** $[v]$ **appears at most once in** T_1 (T_2)**.**

- **The largest subset of** $[v]$ **which is covered by** T_1 **and** T_2 **is the same and it is called the** foundation **of** T **and is denoted by** found(T).

- **Two trades** $T = (T_1, T_2)$ **and** $T = (T'_1, T'_2)$ **are called** isomorphic **if** $\exists$ **a** 1-1 **correspondence** f : found$(T) \longrightarrow$ found(T') **s.t.** $f(T) = (f(T_1), f(T_2)) = (T'_1, T'_2) = T'$.

A LINEAR ALGEBRAIC APPROACH:

- t-**inclusion** matrix $P = P_{t,k}(v)$

  Columns: k-subsets of $[v]$

  Rows: t-subsets of $[v]$

  Both in lexicographic order

  P is a $(0,1)$-matrix and:

$$P_{A,B} = 1 \iff A \subseteq B.$$

- P is a $\binom{v}{t} \times \binom{v}{k}$ matrix.

- **Any t-(v, k, λ) design D can be thought of a solution to the equation**

$$P\mathbf{F} = \lambda\bar{\mathbf{1}}, \tag{1}$$

where $P = P_{t,k}(v)$, and

$\bar{\mathbf{1}}$: all components equal to 1 vector.

$\mathbf{F}$: frequency vector, i.e. $\mathbf{F}(A)$ is the number of times that D contains the k-set A.

- **Also, any** t-(v,k) trade T **can be thought of a solution to the equation**

$$P\mathbf{T} = \bar{\mathbf{0}}, \tag{2}$$

where $P = P_{t,k}(v)$**, and**

$\bar{\mathbf{0}}$**: all components equal to** 0 **vector.**

$\mathbf{T}$**: frequency vector, i.e.**

$$\mathbf{T}(A) = \begin{cases} p & \text{if } A \in T_1 \text{ (} p \text{ times),} \\ -q & \text{if } A \in T_2 \text{ (} q \text{ times),} \\ 0 & \text{otherwise} \end{cases}$$

- Conversely, if $\mathbf{T}$ is a vector in the null space of P with integer components, then $\mathbf{T}$ determines a t-(v,k) trade $T = T_1 - T_2$:
  T_1: positive components of $\mathbf{T}$ and
  T_2: negative components of $\mathbf{T}$.

- Thus, there is a one-to-one correspondence between the null space of P over the ring Z and the set of all t-(v,k) trades T.

- The set of all t-(v,k) trades forms a Z-module.

SOME PROPERTIES:

- **The foundation size of every** t-(v,k) trade T **is at least** $t+k+1$**, i.e.** $|\text{found}(T)| \geq t+k+1$.
- **The volume of every** t-(v,k) trade T **is at least** 2^t**, i.e.** $|\text{vol}(T)| \geq 2^t$.
- t-(v,k) trade T **is called** basic **if** $|\text{vol}(T)| = 2^t$ **and it is called** minimal **if** $|\text{vol}(T)| = 2^t$ **and** $|\text{found}(T)| = t+k+1$.

- **Every basic** t-(v,k) trade T **is associated to a polynomial** $(S_1 - S_2)(S_3 - S_4)\cdots(S_{2t+1} - S_{2t+2})S_{2t+3}$ **s.t.** $|S_{2i-1}| = |S_{2i}|\ ,\ 1 \leq i \leq t+1\ ,$ $S_i \subseteq [v]$ and $\Sigma_{i=1}^{t+2}|S_{2i-1}| = k.$

- **The** minimal t-(v,k) trade **exists and it is unique up to isomorphism.**

- **Every** t-(v,k) trade T **is a linear combination of** minimal t-(v,k) trades**.**

SOME RESULTS ABOUT SPECTRUM:

- **There exists a** 2-$(v,3)$ trade **of volume** s **iff** $s \neq 1,2,3,5$.
- t-(v,k) trades **of volume** 2^t+1 **do not exist.**
- **For every** $s \in (2^t, 2^t+2^{t-1})$ **there does not exist any** t-(v,k) trade **of volume** s.

TWO CONJECTURES OF MAHMOODIAN AND SOLTANKHAH:

- **The numbers $s_i = 2^{t+1} - 2^{t+1-i}$, $0 \leq i \leq t+1$ are called critical points.**

1) For each $s_i = 2^{t+1} - 2^{(t+1)-i}$, $1 \leq i \leq t+1$, there exists a t-(v, k) trade of volume s_i.

2) For any $s \in (s_i, s_{i+1})$, $1 \leq i \leq t$, there does not exist a t-(v, k) trade of volume s.

- **The first conjecture was proved by** Gray **and** Ramsay (1998).

- **The second conjecture was proved for** Steiner trades **(in two part):**

  1) $k > t + 1$ **by** Hoorfar **and** Khosrovshahi (2005).

  2) $k = t+1$ **by** Asgari **and** Soltankhah (2007).

TWO LOWER BOUNDS:

- For any $s \geq (2t-1)2^t$ there exists a t-(v,k) trade of volume s (Gray and Ramsay 1998).

- Let $t \geq 3$. For any $s \geq (t-2)2^t + 2^{t-1} + 2$ there exists a t-(v,k) trade of volume s (Hoorfar and Khosrovshahi 2005).

- **The spectrum of Steiner** 2-(v, k) trades **is given by** $N_0 \setminus \{1, 2, \ldots, 2k-3\} \cup \{2k-1, 2k+1, \ldots, 3k-5(4)\}$

  (whichever of $3k-5, 3k-4$ **is odd), except** 15 **for** $k = 7$**.**

TRADES ON LATIN SQUARES:

History

- Trades on Latin squares have been defined in 1979 by Curran and Van Rees.

- Some different names were used such as "exchangeable partial groupoids" by Drápal and Kepka (1983), "disjoint and mutually balanced" (DMB) partial Latin squares by Fu and Fu (1990), "critical partial Latin squares" (CPLS) by Keedwell (1994), "Latin interchange" by Donovan et al (1997) and recently as a "Latin bitrade" by Drápal et al (2003).

SOME DEFINITIONS.

A **Latin square** L of order v is an $v \times v$ array with entries chosen from set $[v]$, in such a way that each element of $[v]$ occurs precisely **once** in each row and **once** in each column of the array.

MORE DEFINITIONS.

Let T_1 and T_2 be two Latin squares of same size. The pair $\{T_1 - (T_1 \cap T_2),\ T_2 - (T_1 \cap T_2)\}$ is called a **Latin bitrade**.

Example.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

&

3	2	1	4
2	1	4	3
1	4	3	2
4	3	2	1

Example Latin bitrade.

1	·	3	·
·	3	·	1
3	·	1	·
·	1	·	3

&

3	·	1	·
·	1	·	3
1	·	3	·
·	3	·	1

SOME NOTATIONS:

- Let $[v] := \{1, 2, \ldots, v\}$ and
  $[v]^k$: the set of all ordered k-tuples of $[v]$,
  $[v]^k := \{(x_1, \ldots, x_k) \mid x_i \in [v],\ i = 1, \ldots, k\}$.

- Also, $[v]_I^t := \{(u_1, \ldots, u_t)_I \mid u_i \in [v],\ i \in [t]\}$,
  I is an element of $[k]^t$ with components in increasing order $(t < k)$
  Note that for each I, $|[v]_I^t| = v^t$.

- **For** $(x_1, \ldots, x_k) \in [v]^k$ **and** $(u_1, \ldots, u_t)_I \in [v]_I^t$, **where** $I = (i_1, \ldots, i_t)$ **and** $i_1 < \cdots < i_t$, **we say**

$$\boxed{(u_1, \ldots, u_t)_I \in (x_1, \ldots, x_k) \iff u_j = x_{i_j}, \quad j = 1, \ldots, t.}$$

- **Example.** **For** $v = k = 3$, $t = 2$:

$$(3,1)_{(1,3)} \in (3,2,1)$$

- t-**inclusion matrix** M(t-(v,k)):
  Columns: the elements of $[v]^k$
  (in lexicographic order)
  Rows: the elements of $\cup_I [v]_I^t$.
  M **is a** $(0,1)$**-matrix and:**

$$\boxed{\mathsf{M}_{(u_1,\ldots,u_t)_I,(x_1,\ldots,x_k)} = 1 \iff (u_1,\ldots,u_t)_I \in (x_1,\ldots,x_k).}$$

- M **is a** $\binom{k}{t}v^t \times v^k$ **matrix.**

SOME DEFINITIONS.

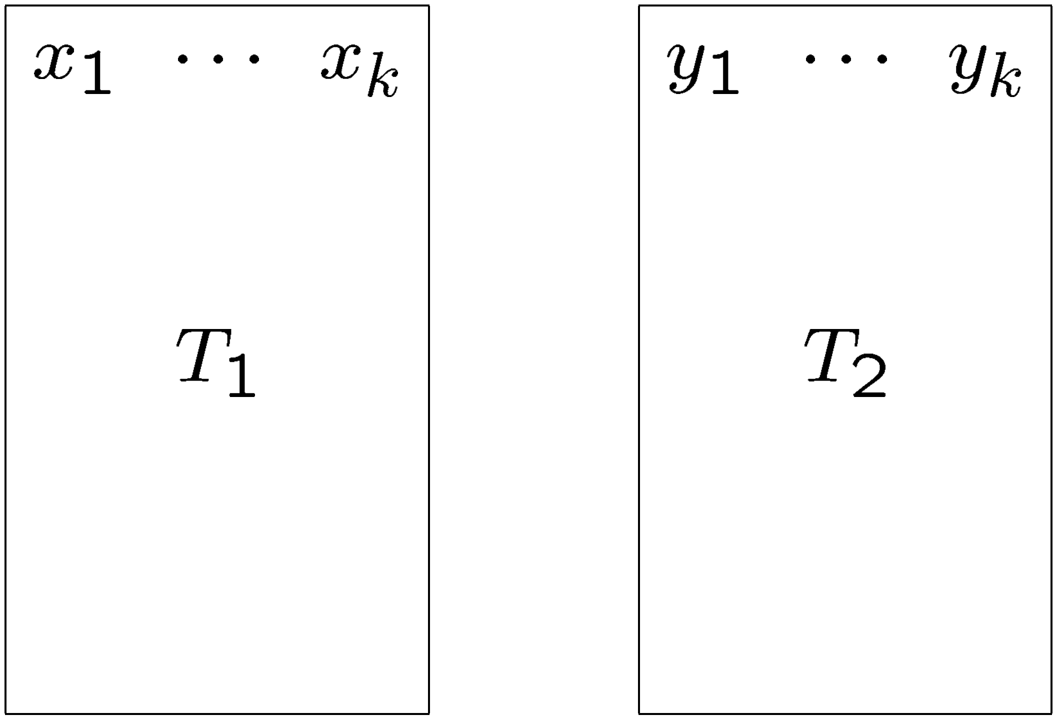
- **An** Orthogonal array **(**$OA_t(v,k,\lambda)$**) on a** v**-set** $[v]$**, is a collection of ordered** k**-tuples of** $[v]$ **such that for each element** I **of** $[k]^t$ **with components in increasing order, every element of** $[v]_I^t$ **belongs to exactly** λ **elements of the collection.**

- A pair $T = (T_1, T_2)$ of two disjoint collections of the elements of $[v]^k$ is called a t-(v,k) Latin trade if for each I and every element $(u_1, \ldots, u_t)_I$ of $[v]_I^t$, the number of elements containing $(u_1, \ldots, u_t)_I$ is the same in both T_1 and T_2. Clearly, $|T_1| = |T_2|$ and this common value is called the volume of T. t-(v,k) Latin trade T also, is denoted by $T = T_1 - T_2$

- **Another definition: A t-(v,k) Latin trade is**

$$(T_1, T_2) = (A_1 - A_2, A_2 - A_1),$$

where A_1 and A_2 are two Orthogonal arrays with the same parameters.



- Spectrum of t-(v,k) Latin trades, $S(t,k)$: The set of all numbers s, such that for each s, $\exists$ a t-(v,k) Latin trade of volume s.

- Let $T = (T_1, T_2)$ be a t-(v,k) Latin trade, $j \in [k]$, and $a \in V$. Take $\{(x_1, \ldots, x_k) \in T_i | \; x_j = a\}$, for $i = 1, 2$. Delete a from the j^{th} co-ordinate in all of them. Then we obtain a $(t-1)$-$(v, k-1)$ Latin trades $T^* = (T_1^*, T_2^*)$ which is called a level trade of T in the direction of j.

Example. ($j = 3$, $a = 3$):

T_1	3	3	2	2	2	1	1	2	2	1	1	3	3	2	2
	3	2	3	2	1	2	1	2	1	2	1	3	2	3	2
	3	3	3	3	3	3	3	2	2	2	2	1	1	1	1
	2	3	3	1	2	2	1	2	1	1	2	3	2	2	3

T_2	3	3	2	2	2	1	1	2	2	1	1	3	3	2	2
	3	2	3	2	1	2	1	2	1	2	1	3	2	3	2
	3	3	3	3	3	3	3	2	2	2	2	1	1	1	1
	3	2	2	3	1	1	2	1	2	2	1	2	3	3	2

A 2-(3,3) Latin trades of volume 7:

	3	3	2	2	2	1	1
T_1^*	3	2	3	2	1	2	1
	2	3	3	1	2	2	1

	3	3	2	2	2	1	1
T_2^*	3	2	3	2	1	2	1
	3	2	2	3	1	1	2

1_2	2_1	$\cdot$
2_1	1_3	3_2
$\cdot$	3_2	2_3

- **Any $\mathrm{OA}_t(v,k,\lambda)$ can be thought of a solution to the equation**

$$\mathrm{M}\mathbf{F} = \lambda\bar{\mathbf{1}}, \tag{3}$$

where $\mathrm{M} = \mathrm{M}(t\text{-}(v,k))$, and

$\bar{\mathbf{1}}$: all components equal to 1 vector.

$\mathbf{F}$: frequency vector, i.e. $\mathbf{F}(\mathbf{x})$ is the number of times that OA contains the ordered k-tuple $\mathbf{x}$.

Also, any t-(v,k) Latin trade T can be thought of a solution to the equation

$$\mathsf{M}\mathbf{T} = \overline{\mathbf{0}}, \tag{4}$$

where $\mathsf{M} = \mathsf{M}(t\text{-}(v,k))$,
$\overline{\mathbf{0}}$: all components equal to 0 vector, and
$\mathbf{T}$ is a signed integer valued frequency vector i.e. for each $x \in [v]^k$, $T(x)$ is defined as the following:

$$T(x) = \begin{cases} p & \text{if } x \in T_1 \ (p \text{ times}), \\ -q & \text{if } x \in T_2 \ (q \text{ times}), \\ 0 & \text{otherwise} \end{cases}$$

- **Conversely, if $\mathbf{T}$ is a vector in the null space of $\mathbf{M}$ with integer components, then $\mathbf{T}$ determines a t-(v,k) Latin trade T:**
  $T = T_1 - T_2$,
  T_1: **positive components of $\mathbf{T}$ and**
  T_2: **negative components of $\mathbf{T}$.**

- **Thus, there is a one-to-one correspondence between the null space of $\mathbf{M}$ over the ring Z and the set of all t-(v,k) Latin trades T.**

RESULTS:

- For each $s_i = 2^{t+1} - 2^{(t+1)-i}$, $0 \leq i \leq t+1$, there exists a t-(v,k) Latin trade of volume s_i.
- For any $s \in (2^{t+1}-2^{(t+1)-i}, 2^{t+1}-2^{(t+1)-(i+1)})$ $0 \leq i \leq t$, there does not exist any t-$(v,t+1)$ Latin trade of volume s.
- Let $t \geq 3$. For any $s \geq (t-2)2^t + 2^{t-1} + 2$ there exists a t-$(v,t+1)$ *Latin trade* of volume s.

- **Let a and b be two non negative integer and $a + b < t + 1$. Then there exists a** t-(v, k) Latin trade **of volume** $s = 3 \times 2^t - (2^a + 2^b + 1)$.

- **Let a be a non negative integer and $a < t+1$. Then there exists a** t-(v, k) Latin trade **of volume** $s = 3 \times 2^t - (2^a + 1)$.

RELATION BETWEEN t-TRADES AND t-LATIN TRADES:

- **Theorem.** By using any t-(v, k) *Latin trade* of volume s, we can obtain a t-(v, k) *trade* of the same volume.

  Sketch of proof. Let $T = (T_1, T_2)$ be a t-(v, k) Latin trade of volume s and for every $j \in [k]$, l_j be the number of non-trivial level trades in the direction of j.

Assume $A_1 = [l_1]$ and $A_j = [(\Sigma_{i=1}^{j-1} l_i)+1, \Sigma_{i=1}^{j} l_i]$, $2 \leq j \leq k$ and $v' = | \cup_{j=1}^{k} A_j|$, we have:

$A_i \cap A_j = \emptyset, \quad \forall i \neq j \in [k]$,

$|A_j| = l_j$, $\forall j \in [k]$,

$\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ is a partition of the set $[v']$.

$\exists$ 1-1 correspondence between A_j and the set of all symbols of the j^{th} column of T_1 and T_2.

By using this bijection, we obtain $T^* = (T_1^*, T_2^*)$ which is isomorphic to T.

Now components of each element of $T_1^*(T_2^*)$ are pair-wise distinct.

Take $R_i = \{\{x_1, \ldots, x_k\} | (x_1, \ldots, x_k) \in T_i^*\}$, $i = 1, 2.$

R_1 and R_2 have the following properties:

1) Every element of R_i is a k-subset of $[v']$.

2) Each element of $B \in R_i$ belongs to exactly one element of $\mathcal{A}$.

3) $R_1 \cap R_2 = \emptyset$.

4) $|R_1| = |R_2| = s$.

We show that $R = (R_1, R_2)$ is a t-(v', k) trade.

Consider $H = \{a_1, \ldots, a_t\} \subset [v']$. There are two cases:

Case i. At least two elements of H belong to the one element of $\mathcal{A}$. So by property (2), there is no block of R_1 and R_2 which contains H.

Case $ii.$ Every element of H belongs to exactly one element of $\mathcal{A}$. $\forall\ I \in [k]^t$, the number of elements of T_1^* and T_2^* which contain $L = (a_1, \ldots, a_t)_I$, is the same. Each element of $T_i^*,\ i = 1, 2$ which contains L is corresponded to one block of $R_i,\ i = 1, 2$ which contains H. So the number of blocks of R_1 and R_2 which contain H is the same.

Example. **In the following a 2-$(3,3)$ Latin bitrade $T = (T_1, T_2)$ of volume 7 is given.**

T_1	1	1	2	2	2	3	3
	1	2	1	2	3	2	3
	1	2	2	1	3	3	2

T_2	1	1	2	2	2	3	3
	1	2	1	2	3	2	3
	2	1	1	3	2	2	3

In this Latin bitrade we have $l_1 = l_2 = 3$, so $T^* = (T_1^*, T_2^*)$ is as in the following:

T_1^*	1	1	2	2	2	3	3
	4	5	4	5	6	5	6
	7	8	8	7	9	9	8

T_2^*	1	1	2	2	2	3	3
	4	5	4	5	6	5	6
	8	7	7	9	8	8	9

Take $R_1 = \{147, 158, 248, 257, 269, 359, 368\}$,
and $R_2 = \{148, 157, 247, 259, 268, 358, 369\}$.
We can easily check that $R = (R_1, R_2)$ is a 2-$(9, 3)$ trade of volume 7.

Theorem. If there exists a t-(v, k) trade of volume s, then a t-(v, k) Latin trade of volume $s(k!)$ can be constructed.

Sketch of proof. Begin with a t-(v, k) trade $R = (R_1, R_2)$ of volume s. Take T_i as follows:

$T_i = \{(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) | \{x_1, x_2, \ldots, x_k\} \in R_i$, $\{i_1, \ldots, i_k\} = [k]\}$.

T_1 **and** T_2 **have the following properties:**

1) Components of each ordered k**-tuples of** T_i $(i = 1, 2)$ **are pair-wise distinct.**

2) Each block $B = \{x_1, x_2, \ldots, x_k\} \in R_i$ **gives** $k!$ **elements of** T_i**.**

3) $T_1 \cap T_2 = \emptyset$.

4) $|T_1| = |T_2| = s(k!)$.

We show that $T = (T_1, T_2)$ is a t-(v, k) Latin trade. $\forall\ I \in [k]^t$ **and** $\forall\ H = (a_1, \ldots, a_t)_I \in [v]^t_I$**, there are two cases:**

Case i. At least two components of $(a_1, \ldots, a_t)$ are equal. So by property (1), there is no element of T_1 and T_2 which contains H.

Case ii. Components of $(a_1, \ldots, a_t)$ are pairwise distinct. Let $L = \{a_1, \ldots, a_t\} \subset [v]$. The

number of blocks of R_1 and R_2 which contain L, is the same, say to λ_L. Assume $B \in R_i$, $i = 1, 2$ is one of them, by (b) there exist, exactly, $(k-t)!$ elements of T_i which contain H. So the **number** of elements of T_1 and T_2 which contain H, are equal to $\lambda_L(k - t)!$. So $T = (T_1, T_2)$ is a t-(v, k) Latin trade of volume $s(k!)$.

- **SOME OPEN PROBLEMS:**

- **Determination of basis for** t-(v,k) Latin trades **when** $k > t+1$.

- **Improvement of lower bound for the spectrum of** t-(v,k) Latin trades.

THANK YOU!

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